Solution Sheet n°3

Solution of exercise 1:

1. the natural order on $\mathbb{N}$;
2. the natural order on the positive integers and 0 above everything else;
3. for two distinct integers i, j:

$$i \prec j \text{ iff } \begin{cases} i, j \text{ even and } i < j, \text{ or} \\ i, j \text{ odd and } i < j, \text{ or} \\ i \text{ even and } j \text{ odd;} \end{cases}$$

4. for two distinct integers i, j such that $i \equiv n \mod 3$ and $j \equiv m \mod 3$:

$$i \prec j \text{ iff } \begin{cases} n = m \text{ and } i < j, \text{ or} \\ n < m; \end{cases}$$

5. for two distinct integers i, j:

$$i \prec j \text{ iff } \begin{cases} i, j > 16 \text{ and } i, j \text{ even and } i < j, \text{ or} \\ i, j > 16 \text{ and } i, j \text{ odd and } i < j, \text{ or} \\ i, j > 16 \text{ and } i \text{ odd and } j \text{ pair, or} \\ i, j \leq 16 \text{ and } i < j, \text{ or} \\ j \leq 16 \text{ and } i > 16; \end{cases}$$

6. using a bijection[1] between $\mathbb{N}^2$ and $\mathbb{N}$ it is enough to define an order on $\mathbb{N}^2$. The lexicographic order

$$(i, j) \prec (k, l) \quad \text{iff} \quad i < k \text{ or } (i = k \text{ and } j < l)$$

works;

7. it is enough to separate the integers in two copies, both of which infinite (for example: even and odd numbers), and then order the first as in 6. and the second as in 4., then place the second after of the first.

Solution of exercise 2:

1. $3 + \omega = \sup\{3 + n \mid n \in \omega\} = \omega$.
2. $\omega + 3$ cannot be simplified further.
3. $\omega+15+\omega+9+3+\omega = \omega+(15+\omega)+(12+\omega) = \omega+\omega+\omega = \omega\cdot(1+1+1) = \omega\cdot 3$.
4. $\omega \cdot 3$ cannot be simplified further.

[1]For example: $g_2(n, m) = \frac{(n+m)\cdot(n+m+1)}{2} + m$

5. $3 \cdot \omega = \sup\{3 \cdot n \mid n \in \omega\} = \omega$.

6. $(\omega \cdot 3) \cdot (\omega \cdot 5) = \omega \cdot (3 \cdot \omega) \cdot 5 = \omega^2 \cdot 5$.

7. $\omega^2 \cdot \omega = \omega \cdot \omega \cdot \omega = \omega^3$.

8. $\omega \cdot \omega^2 = \omega \cdot \omega \cdot \omega = \omega^3$.

9. $(\omega + 3) \cdot 4 = \omega + 3 + \omega + 3 + \omega + 3 + \omega + 3 = \omega \cdot 4 + 3$.

10. $4 \cdot (\omega + 3) = 4 \cdot \omega + 4 \cdot 3 = \omega + 12$

11. $(\omega + 3) \cdot \omega = \sup\{(\omega + 3) \cdot n \mid n \in \omega\} = \sup\{\omega \cdot n + 3 \mid n \in \omega\} = \sup\{\omega \cdot (n+1) \mid n \in \omega\} = \omega^2$. The second to last equality, namely, $\bigcup\{\omega \cdot n + 3 \mid n \in \omega\} = \bigcup\{\omega \cdot (n+1) \mid n \in \omega\}$, follows from the fact that for all $n \in \omega$, on one hand $\omega \cdot n + 3 < \omega \cdot (n+1)$ and therefore $\omega \cdot n + 3 \subseteq \omega \cdot (n+1)$ and on the other $\omega \cdot (n+1) < \omega \cdot (n+1) + 3$ and therefore $\omega \cdot (n+1) \subseteq \omega \cdot (n+1) + 3$.

12. $\omega \cdot (\omega + 3) = \omega \cdot \omega + \omega \cdot 3 = \omega^2 + \omega \cdot 3$.

13. $10 \cdot \omega \cdot 7 \cdot 3 \cdot \omega = \omega^2$ by associativity of the ordinal multiplication.

14. $\omega^3 \cdot \omega^2 \cdot 9 \cdot \omega + 7 \cdot \omega^4 + 3 \cdot (\omega + 2) = \omega^6 + \omega^4 + \omega + 6$.

15. $2 \cdot \omega^3 \cdot 3 + \omega^6 + (\omega + 3) \cdot 12 = \omega^3 \cdot 3 + \omega^6 + \omega \cdot 12 + 3$.

Solution of exercise 3:

In order to distinguish the two notions of ordinal addition we are dealing with, we write $\alpha \oplus \beta$ for the unique ordinal γ which is isomorphic to the well order $(\alpha \times \{0\} \cup \beta \times \{1\}, <)$ where $(\gamma, i) < (\eta, j)$ iff $i < j$ or $i = j$ and $\gamma < \eta$. We remark first of all that for all ordinals α, $\alpha + 1 = s(\alpha + 0) = s(\alpha) = \alpha \cup \{\alpha\} = \text{type}(\alpha \times \{0\} \cup 1 \times \{1\}) = \alpha \oplus 1$ by the isomorphism $f : \alpha \times \{0\} \cup 1 \times \{1\} \to \alpha \cup \{\alpha\}$ given by $(\beta, 0) \mapsto \beta$ for $\beta \in \alpha$ and $(0, 1) \mapsto \alpha$. Let α be an ordinal. We show that for each ordinal β, $\alpha + \beta = \alpha \oplus \beta$ by induction on β.

1. We have $\alpha \oplus 0 = \text{type}(\alpha \times \{0\} \cup \emptyset \times \{1\}) = \text{type}(\alpha \times \{0\}) = \alpha = \alpha + 0$.

2. Suppose that $\alpha \oplus \beta = \alpha + \beta$. We have $\alpha + s(\beta) = s(\alpha + \beta) = (\alpha + \beta) + 1 = (\alpha \oplus \beta) + 1 = (\alpha \oplus \beta) \oplus 1 = \alpha \oplus (\beta \oplus 1) = \alpha \oplus s(\beta)$.

3. Suppose now that λ is a limit ordinal and that for all $\xi < \lambda$ we have $\alpha + \xi = \alpha \oplus \xi$. For all $\xi < \lambda$, we write $f_\xi : \alpha \oplus \xi \to \alpha + \xi$ for the isomorphism. Let us observe that for all $\xi < \lambda$ the domain $(\alpha \times \{0\} \cup \xi \times \{1\})$ of f_ξ is included in $(\alpha \times \{0\} \cup \lambda \times \{1\})$. Moreover, for all $\xi < \lambda$ the codomain $\alpha + \xi$ of f_ξ is included in $\alpha + \lambda = \sup_{\zeta < \lambda} \alpha + \zeta = \bigcup_{\zeta < \lambda} \alpha + \zeta$. We can therefore consider each f_ξ as a subset of $(\alpha \times \{0\} \cup \lambda \times \{1\}) \times \alpha + \lambda$. Moreover for all $\xi < \zeta < \lambda$ we have that the restriction $f_\zeta \restriction (\alpha \oplus \xi)$ is an isomorphism of $\alpha \oplus \xi$ on a proper initial segment of $\alpha + \zeta$, i.e. an ordinal γ which belongs to $\alpha + \zeta$. Also, since there exists a unique ordinal which is isomorphic to $\alpha \oplus \xi$, namely $\alpha + \xi$, by the induction hypothesis, and there exists a unique isomorphism of $\alpha \oplus \xi$ on $\alpha + \xi$, we have $f_\zeta \restriction (\alpha \oplus \xi) = f_\xi$. It follows that $f = \bigcup_{\xi < \lambda} f_\xi$ and a function $f : \alpha \oplus \lambda \to \alpha + \lambda$. It is furthermore an isomorphism since each f_ξ is.

Solution of exercise 4:

1. *If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

   By induction on γ.

   (a) If $\gamma = 0$, then for all ordinals β, $\beta \not< \gamma$ and the implication is therefore verified.

   (b) Let γ be successor, i.e. $\gamma = S(\gamma')$ for a certain ordinal γ'. If $\beta < \gamma$, then $\beta \leq \gamma'$, so:

   i. if $\beta = \gamma'$, then $\alpha + \beta = \alpha + \gamma' < S(\alpha + \gamma') = \alpha + S(\gamma') = \alpha + \gamma$;

   ii. if $\beta < \gamma'$, then by the induction hypothesis we have $\alpha + \beta < \alpha + \gamma' < S(\alpha + \gamma') = \alpha + S(\gamma') = \alpha + \gamma$.

   (c) If γ is limit and $\beta < \gamma$ then for all $\beta < \gamma' < \gamma$ we have $\alpha + \beta < \alpha + \gamma'$. Thus,

$$\alpha+\beta < \bigcup_{\beta<\gamma'<\gamma} \alpha+\gamma' = \sup\{\alpha+\gamma' \mid \beta < \gamma' < \gamma\} = \sup\{\alpha+\gamma' \mid \gamma' < \gamma\} = \alpha+\gamma.$$

2. *If $\alpha < \beta$, then there exists a unique ordinal δ such that $\alpha + \delta = \beta$.*

   The class of ordinals θ such that $\alpha + \theta > \beta$ is not empty since by example $\beta+1$ belongs to it. It thus admits a minimal element δ'. We show that δ' is successor. If it were limit, we would have $\alpha + \delta' = \sup\{\alpha + \xi \mid \xi < \delta'\} \leq \beta$ since $\alpha + \xi \leq \beta$ for all $\xi < \delta'$ by minimality of δ' with respect to verifying $\alpha + \delta' > \beta$. Now $\alpha + \delta' \leq \beta$ contradicts the fact that $\alpha + \delta' > \beta$. Thus, δ' is successor and therefore $\delta' = S(\delta)$ for a certain ordinal δ. It must be $\alpha + \delta \leq \beta$ and since $\alpha + \delta + 1 > \beta$ we have $\alpha + \delta = \beta$.

   The uniqueness is obtained by simplification. Indeed, it follows from point 1. that $\alpha + \delta = \alpha + \delta'$ implies $\delta = \delta'$.

3. *If $\alpha \neq 0$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.* By induction on γ:

   (a) if $\gamma = 0$, then $\beta \not< \gamma$ and the implication is therefore verified;

   (b) if $\gamma = S(\gamma')$, then $\beta < \gamma$ is equivalent to $\beta \leq \gamma'$;

   i. if $\beta = \gamma'$, then $\alpha \cdot \beta = \alpha \cdot \gamma' < \alpha \cdot \gamma' + \alpha = \alpha \cdot S(\gamma') = \alpha \cdot \gamma$;

   ii. if $\beta < \gamma'$, then $\alpha \cdot \beta < \alpha \cdot \gamma'$ by the induction hypothesis, and we conclude by remarking that $\alpha \cdot \gamma = (\alpha \cdot \gamma') + \alpha$ and using point 1.;

   (c) if γ is limit and $\beta < \gamma$, we have by the induction hypothesis that for all $\beta < \gamma' < \gamma$ $\alpha \cdot \beta < \alpha \cdot \gamma'$. Thus,

$$\alpha \cdot \beta < \sup\{\alpha \cdot \gamma' \mid \beta < \gamma' < \gamma\} = \sup\{\alpha \cdot \gamma' \mid \gamma' < \gamma\} = \alpha \cdot \gamma.$$

4. *Euclidean Division: if α is an ordinal and $\xi > 0$, then there exist two unique ordinals θ (the quotient) and ρ (the reminder) such that $\rho < \xi$ and $\alpha = \xi \cdot \theta + \rho$.*

   Since $\xi > 0$, there exists at least an ordinal θ_0 such that $\xi \cdot \theta_0 > \alpha$ (for example $\theta_0 = \alpha + 1$). The class of ordinals θ such that $\xi \cdot \theta > \alpha$ admits therefore a minimum θ'. The ordinal θ' is successor since if it were limit, then we would have $\xi\cdot\theta' = \sup\{\xi\cdot\zeta \mid \zeta < \theta'\} \leq \alpha$ which is a contradiction. There exists therefore an ordinal θ such that $S(\theta) = \theta'$. By minimality of θ' we have that $\xi\cdot\theta \leq \alpha$. If equality holds, then we have $\rho = 0$. If not we have $\xi\cdot\theta < \alpha$ and by point 2. there exists a unique ordinal ρ such that $\xi\cdot\theta + \delta = \alpha$. Moreover, $\xi\cdot\theta + \rho = \alpha < \xi\cdot(\theta+1) = \xi\cdot\theta + \theta$ from which it follows that $\rho < \theta$ (thus it follows from 1. that $\alpha + \beta < \alpha + \beta'$ implies $\beta < \beta'$).

For uniqueness, let $\theta, \theta', \rho, \rho'$ be such that $\alpha = \xi\cdot\theta + \rho = \xi\cdot\theta' + \rho'$ with $\rho, \rho' < \xi$. Suppose towards contradiction, and without loss on generality, that $\theta < \theta'$. By points 1. and 3., we have

$$\alpha = \xi\cdot\theta + \rho < \xi\cdot\theta + \xi = \xi\cdot(\theta+1) \leq \xi\cdot\theta' \leq \alpha,$$

a contradiction. Thus, $\theta = \theta'$ and by simplification it follows that $\rho = \rho'$.

5. $\alpha\cdot(\beta+\gamma) = \alpha\cdot\beta + \alpha\cdot\gamma$. By induction on γ:

   (a) if $\gamma = 0$: $\alpha\cdot(\beta+0) = \alpha\cdot\beta = \alpha\cdot\beta + \alpha\cdot 0$;

   (b) if $\gamma = \delta+1$: $$\begin{aligned}\alpha\cdot\big(\beta+(\delta+1)\big) &= \alpha\cdot\big((\beta+\delta)+1\big)\\ &= \alpha\cdot(\beta+\delta)+\alpha\\ &= (\alpha\cdot\beta+\alpha\cdot\delta)+\alpha\\ &= \alpha\cdot\beta+(\alpha\cdot\delta+\alpha)\\ &= \alpha\cdot\beta+\big(\alpha\cdot(\delta+1)\big);\end{aligned}$$

   (c) if γ is limit: $$\begin{aligned}\alpha\cdot(\beta+\gamma) &= \sup_{\xi<\gamma}\alpha\cdot(\beta+\xi)\\ &= \sup_{\xi<\gamma}\alpha\cdot\beta+\alpha\cdot\xi\\ &= \alpha\cdot\beta+\sup_{\xi<\gamma}\alpha\cdot\xi\\ &= \alpha\cdot\beta+\alpha\cdot\gamma;\end{aligned}$$

6. If $\alpha > 1$ and $\beta < \gamma$, then $\alpha^\beta < \alpha^\gamma$. By induction on γ:

   (a) if $\gamma = 0$, then the implication is true;

   (b) if $\gamma = \delta+1$: $$\begin{aligned}\alpha^\gamma &= \alpha^{\delta+1}\\ &= \alpha^\delta\cdot\alpha\\ &\geq \alpha^\beta\cdot\alpha\\ &> \alpha^\beta;\end{aligned}$$

   (c) if γ is limit: $$\begin{aligned}\alpha^\gamma &= \sup_{\delta<\gamma}\alpha^\delta\\ &= \sup_{\delta<\gamma}\alpha^{\delta+1}\\ &> \alpha^\beta;\end{aligned}$$

7. $\alpha^{\beta+\gamma} = \alpha^\beta\cdot\alpha^\gamma$. By induction on γ:

   (a) if $\gamma = 0$: $\alpha^{\beta+0} = \alpha^\beta = \alpha^\beta\cdot\alpha^0$;

   (b) if $\gamma = \delta+1$: $$\begin{aligned}\alpha^{\beta+(\delta+1)} &= \alpha^{(\beta+\delta)+1}\\ &= \alpha^{\beta+\delta}\cdot\alpha\\ &= (\alpha^\beta\cdot\alpha^\delta)\cdot\alpha\\ &= \alpha^\beta\cdot(\alpha^\delta\cdot\alpha)\\ &= \alpha^\beta\cdot\alpha^{\delta+1};\end{aligned}$$

(c) if γ is limit: $\alpha^{\beta+\gamma} = \sup_{\delta<\gamma} \alpha^{\beta+\delta}$
$= \sup_{\delta<\gamma} \alpha^{\beta} \cdot \alpha^{\delta}$
$= \alpha^{\beta} \cdot \sup_{\delta<\gamma} \alpha^{\delta}$
$= \alpha^{\beta} \cdot \alpha^{\gamma}$;

8. $(\alpha^{\beta})^{\gamma} = \alpha^{\beta\cdot\gamma}$. By induction on γ:

   (a) if $\gamma = 0$: $(\alpha^{\beta})^{0} = 1 = \alpha^{0} = \alpha^{\beta\cdot 0}$;

   (b) if $\gamma = \delta + 1$:
   $$\begin{aligned}(\alpha^{\beta})^{\delta+1} &= (\alpha^{\beta})^{\delta} \cdot \alpha^{\beta} \\ &= \alpha^{\beta\cdot\delta} \cdot \alpha^{\beta} \\ &= \alpha^{\beta\cdot\delta+\beta} \\ &= \alpha^{(\beta\cdot\delta)+1} \\ &= \alpha^{\beta\cdot(\delta+1)};\end{aligned}$$

   (c) if γ is limit:
   $$\begin{aligned}(\alpha^{\beta})^{\gamma} &= \sup_{\delta<\gamma} (\alpha^{\beta})^{\delta} \\ &= \sup_{\delta<\gamma} \alpha^{\beta\cdot\delta} \\ &= \alpha^{\sup_{\delta<\gamma}(\beta\cdot\delta)} \\ &= \alpha^{\beta\cdot\gamma};\end{aligned}$$

Solution of exercise 5: The existence is proved by induction on α.

1. If $\alpha = 1$, then $\alpha = \omega^{0}$.
2. If $\alpha > 1$, we further distinguish:
   (a) if there exists $\beta \leq \alpha$ such that $\omega^{\beta} = \alpha$, then it gives us the result;
   (b) if not we consider the smallest $\beta \leq \alpha$ such that $\omega^{\beta} > \alpha$. Notice that β cannot be limit, therefore $\beta = \gamma + 1$. We now check whether there exists a strictly positive integer n such that $\omega^{\gamma} \cdot n = \alpha$:
      i. if it is the case, it gives us the result;
      ii. if not, consider the smallest integer[2] n such that $\omega^{\gamma} \cdot n > \alpha$. Notice that $n > 1$, therefore $n = n_0 + 1$ for a strictly positive integer n_0.
      We then consider (by **Exercice 4** 2.) the unique ordinal δ such that $\omega^{\gamma} \cdot m + \delta = \alpha$. Notice that $\delta < \omega^{\gamma}$ leads to $\delta < \alpha$. By the induction hypothesis δ admits a Cantor's normal form:
      $$\delta = \omega^{\beta_1} \cdot n_1 + \ldots + \omega^{\beta_k} \cdot n_k$$
      which furthermore verifies $\gamma > \beta_1$. It then suffices to let $\gamma = \beta_0$ to obtain:
      $$\alpha = \omega^{\beta_0} \cdot n_0 + \omega^{\beta_1} \cdot n_1 + \ldots + \omega^{\beta_k} \cdot n_k.$$

Uniqueness is proved easily by remarking that for all Cantor's normal forms we have:
$$\omega^{\beta_1} \cdot n_1 + \ldots + \omega^{\beta_k} \cdot n_k < \omega^{\beta_1} \cdot (n_1 + 1).$$
Thus we deduce that two different normal forms "compute" two different ordinals.

[2] It necessarily exists since $\sup_{n\in\mathbb{N}} \omega^{\gamma} \cdot n = \omega^{\beta} > \alpha$.