Solution Sheet n°2

Solution of exercise 1:

1. The following formula formalises the definition of natural number

$$\varphi_{Nat}(x) \;:\; \Big(\mathtt{On}(x) \wedge \forall y \,\big(y \in S(x) \to (y = 0 \vee \exists z\ y = S(z))\big)\Big).$$

   where $\mathtt{On}(x)$ is the formula stating that x is a ordinal and $S(z)$ denotes the successor of z. By the axiom of Infinity, there exists a set A closed under successors, i.e. such that $0 \in A \wedge \forall x \in A \;\; S(x) \in A$. Let us show that A contains all the natural numbers. Suppose towards contradiction that there is a natural number n which does not belong to A. Then $n \in S(n)$ but $n \notin A$, therefore the set $C = \{k \in S(n) \mid k \notin A\}$ is non empty. Since $S(n)$ is well ordered by $\in$, C admits a minimal element m. Since $0 \in A$, $m \neq 0$. Moreover $m \neq S(y)$ for all y since if not there would be $y \in A$ with $S(y) \notin A$. Therefore $S(n)$ is not a natural number and therefore n is not a natural number.

   Thus A contains all the natural numbers. By comprehension, we can therefore form the set $\{x \in A \mid \varphi_{Nat}(x)\} = \mathbb{N}$.

2. Let us extend the langage by introducing: $\mathbb{N}$, 0, S, $\cup$, $\cap$, $\subseteq$, $\mathcal{P}$, $\emptyset$ and for all sets a and b we write

   - $a \times b$ for the cartesian product of a and of b,
   - (a, b) for the set $\{a, \{a, b\}\}$,
   - b^a for the set of functions from a to b.

   The recursion principal can thus be stated as:

$$\forall w_1 \ldots \forall w_n \;\; \Big(\Big[\phi(\emptyset/x) \;\wedge\; \forall y \in \mathbb{N}\Big(\phi(y/x) \longrightarrow \phi\big(S(y)/x\big)\Big)\Big] \;\longrightarrow\; \forall y \in \mathbb{N} \;\; \phi(y/x)\Big)$$

   i.e.

$$\forall w_1 \ldots \forall w_n \;\Big[\Big(\Big[\phi(\emptyset/x) \wedge \forall y \Big[\varphi_{Nat}(y/x) \longrightarrow \Big(\phi(y/x) \longrightarrow \phi(S(y)/x)\Big)\Big]\Big]\Big) \longrightarrow \forall y \;\Big(\varphi_{Nat}(y/x) \longrightarrow \;\phi(y/x)\Big)\Big].$$

   This formula is verified, since if not it suffices to consider the smallest ordinal which does not verify it to obtain a contradiction.

   We say that the image of an element is well defined if it exists and is unique. A function of $f \in b^a$ is well defined if the image of all elements of a is well defined. Thus, $f \in b^{\mathbb{N}}$ is well defined if

   (a) it is defined on 0 and
   (b) if it is well defined on a natural number n, then it is well defined on its successor.

   Let us define then $+ \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ by:

   (a) $+(0) = \{(x, x) \mid x \in \mathbb{N}\}$
   (b) $+\big(S(n)\big) = \big\{\big(x, S(y)\big) \mid (x, y) \in +(n)\big\}$

   Let $+(x)(y) := x + y$

   Similarly we define $\cdot \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ by letting:

   (a) $\cdot(0) = \{(x, 0) \mid x \in \mathbb{N}\}$
   (b) $\cdot\big(S(n)\big) = \big\{\big(x, (y + x)\big) \mid (x, y) \in +(n)\big\}$

Let $\cdot(x)(y) := x \cdot y$. Verify that these operations are commutative, associative and admit a identity element.

3. Let $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. The relation $\sim$ is:

   (a) reflexive;
   (b) symmetric;
   (c) transitive, since if $(n_0, m_0) \sim (n_1, m_1) \sim (n_2, m_2)$ then
      - $n_0 + m_1 = n_1 + m_0$ and $n_1 + m_2 = n_2 + m_1$,
      - from which it follows that $(n_0 + m_2) + (n_1 + m_1) = (n_2 + m_0) + (n_1 + m_1)$,
      - and thus $n_0 + m_2 = n_2 + m_2$.

   By the comprehension schema and the powerset axiom, the quotient set $\mathbb{N}^2/\sim \ := \mathbb{Z}$ is well defined. If $(n_0, m_0) \sim (n_0', m_0')$ and $(n_1, m_1) \sim (n_1', m_1')$ we have then, by definition, $(n_0{+}n_1, m_0{+}m_1) \sim (n_0'{+}n_1', m_0'{+}n_1')$, from which we obtain an addition on $\mathbb{Z}$. If $(n, m) \in \mathbb{N}^2$ we write $[n, m]$ for its equivalence class in $\mathbb{Z}$. $\mathbb{N}$ embeds in $\mathbb{Z}$ by $x \longrightarrow [x, 0]$. By an abuse of notation we write $-x := \ [0, x]$. Then $(\mathbb{Z}, +)$ is a group and $(\mathbb{Z}, +, \cdot)$ is a ring.

4. By defining on $\mathbb{N} \times \big(\mathbb{Z} \setminus \{[0, 0]\}\big)$ the relation:

$$(p, [q_0, q_1]) \equiv (p', [q_0', q_1']) \text{ iff } [p \cdot q_0', p \cdot q_1'] = [p' \cdot q_0, p' \cdot q_1]$$

   and proceeding as above we obtain $(\mathbb{Q}, +, \cdot)$.

5. Let
$$\mathbb{R} := \ \{X \in \mathcal{P}(\mathbb{Q}) \mid X \neq \emptyset \ \wedge \ X \neq \mathbb{Q} \ \wedge \ \forall x \in X \forall y \in \mathbb{Q} \ \ (y < x \to y \in X)\}.$$

Solution of exercise 2:

1. Let E be a finite set. The proof goes by induction on the number of elements of E.

   If $E = \emptyset$ then the empty function $:= \emptyset$ is a choice function on E.

   If E has $n+1$ elements: $E = \{e_1, \ldots, e_{n+1}\}$ then by the induction hypothesis there exists a choice function f on $\{e_1, \ldots, e_n\}$. We consider the two following cases:

   - if $e_{n+1} = \emptyset$ then f is a choice function on E;
   - if $e_{n+1} \neq \emptyset$ then let $a \in e_{n+1}$ and the function $f' := f \cup \{(e_{n+1}, a)\}$ is a choice function on E.

2. $(AC) \Rightarrow (DC)$: Let R be a binary relation on a set E verifying $\forall x \exists y \ (x, y) \in R$ and by (AC) let f be a choice function on $\mathcal{P}(E)$.

   Let us then consider the sequence $(x_n)_{n \in \mathbb{N}}$ of elements of E recursively defined by:

   - $x_0 = f(E)$;
   - $x_{n+1} = f(\{x \in E \mid (x_n, x) \in R\})$.

   By hypothesis on R, the set $\{x \in E \mid (x_n, x) \in R\}$ is never empty, therefore $(x_n, x_{n+1}) \in R$ is true for all $n \in \mathbb{N}$.

3. $(DC) \Rightarrow (CC)$: let E be a countable set and $(e_i)_{i \in \mathbb{N}}$ an enumeration of its elements.

   Let C be the set of all the choice functions on the finite subsets of E of the form $\{e_0, \ldots, e_n\}$. Notice that by 1. C is non empty. Let R be a binary relation on C defined by

$$(f, g) \in R \text{ iff } \Big[dom(f) \subsetneq dom(g) \text{ and } g \restriction dom(f) = f\Big].$$

   We show that for each function f in C there exists f' in C such that $(f, f') \in R$. Indeed, let f be a choice function on $\{e_0, \ldots, e_n\}$, we have a choice function f' on $\{e_0, \ldots, e_n, e_{n+1}\}$ as defined in 1. Thus there exists a sequence of choice functions $(f_i)_{i\in\mathbb{N}}$ verifying $(f_i, f_{i+1}) \in R$ for each natural number i. Consider then the function $f = \bigcup_{i\in\mathbb{N}} (f_i)_{i\in\mathbb{N}}$. This is indeed a function since the f_i's are coherent. Moreover, since $dom(f_i) \subsetneq dom(f_{i+1})$ for each $i \in \mathbb{N}$ and E is countable , then $dom(f) = E$. Thus f is a choice function on E.