Solution Sheet n°1

Solution of exercise 1:

1. $$\mathrm{P}(z,x,y)\ :\ \forall t(t\in z\leftrightarrow(t=x\vee t=y)).$$

2. $$\mathrm{PO}(z,x,y)\ :\ \forall t(t\in z\leftrightarrow(\mathrm{P}(t,x,x)\vee\mathrm{P}(t,x,y))).$$

3. $$\forall x\forall y\forall x'\forall y'\big((\exists z(\mathrm{PO}(z,x,y)\wedge\mathrm{PO}(z,x',y')))\leftrightarrow(x=x'\wedge y=y')\big).$$

Proof of the proposition:

($\Longleftarrow$): If $x=x'$ and $y=y'$, then by the axiom of extensionality $\{x\}=\{x'\}$ and $\{x,y\}=\{x',y'\}$. Therefore, again by extensionality, $\langle x,y\rangle=\langle x',y'\rangle$.

($\Longrightarrow$): Suppose on the other hand that $\langle x,y\rangle=\langle x',y'\rangle$.

- If $x=y$: then $\{x,y\}=\{x\}$ and $\langle x',y'\rangle=\langle x,y\rangle=\langle x,x\rangle=\{\{x\}\}$. Therefore $\{x'\}=\{x',y'\}=\{x\}$ and so $x'=y'=x$.
- If $x\neq y$: Firstly, it follows from $\langle x,y\rangle=\langle x',y'\rangle$ that we have either $\{x\}=\{x',y'\}$ or $\{x\}=\{x'\}$. If $\{x\}=\{x',y'\}$ then $x=x'=y'$. Therefore $\langle x,y\rangle=\langle x',y'\rangle=\langle x\rangle=\{\{x\}\}$ and thus $\{x,y\}=\{x\}$. This contradicts $x\neq y$. Thus $\{x\}=\{x'\}$ and therefore $x=x'$.

  Secondly, it follows from $\langle x,y\rangle=\langle x',y'\rangle$ that we have either $\{x,y\}=\{x',y'\}$ or $\{x,y\}=\{x'\}$. If $\{x,y\}=\{x'\}$, then $x=y=x'$ contradict the fact that $x\neq y$. Thus $\{x,y\}=\{x',y'\}$ and therefore $y=x'$ or $y=y'$. If $y=x'$, as we have already seen that $x=x'$, we obtain a contradiction with $x\neq y$. Therefore, $y=y'$.

Solution of exercise 2:

Let us show that the proposed axiom schema implies the axiom schema of replacement. Let $\varphi(x,y,w_1,\dots,w_n)$ be a formula of set theory, A a set and $p_1,\dots,p_n$ some sets. Suppose that φ is functional and total on A, i.e. for $x\in A$ there exists a unique y such that $\varphi(x,y,p_1,\dots,p_n)$. Let us define

$$\psi(x,y,w_1,\dots,w_n,A)\ :\ (x\in A\wedge\varphi(x,y,w_1,\dots,w_n))\vee(\neg x\in A\wedge\neg\exists z\ z\in y).$$

This new formula satisfies the premise of the proposed axiom schema, namely, ψ is functional and total everywhere and in particular functional and partial everywhere. The conclusion of the proposed axiom schema assures us the conclusion of the instance of the axiom of replacement that we considered.

On the other hand, let $\varphi(x,y,w_1,\dots,w_n)$ be a formula of set theory and $p_1,\dots,p_n$ some sets. Let us show, using the axiom schema of replacement, the instance Repl_φ of the proposed axiom schema. Suppose $\varphi(x,y,p_1,\dots,p_n)$ is functional and partial everywhere. Let A be a set. Let us define then

$$\begin{aligned}\psi(x,y,w_1,\dots,w_n)\ :\ &(\exists z\ \varphi(x,z,w_1,\dots,w_n)\wedge\varphi(x,y,w_1,\dots,w_n))\\&\vee(\neg\exists z\ \varphi(x,z,w_1,\dots,w_n)\wedge\neg\exists z\ z\in y).\end{aligned}$$

This new formula is functional and total everywhere. In particular, it is functional and total on A. Thus, the axiom schema of replacement gives the existence of a set B containing the image of ψ on A. This set B satisfies thus the conclusion of the instance of the proposed axiom schema.

Solution of exercise 3:

1. The map $q \mapsto [q]$ of $\mathbb{N}$ towards $\mathcal{P}_{\text{fin}}$ is a bijection, its inverse is given by $F \mapsto \sum_{n\in F} 2^n$ where we agree that the empty sum is equal to 0. This follows from the fact that each natural number is uniquely determined by its binary expansion. By definition of E, we have for all $p, q \in \mathbb{N}$ that pEq iff $p \in [q]$ iff $p = \sum_{n\in[p]} 2^n \in [q]$ iff $[p]\epsilon[q]$.

2. The fact that (N, E) satisfies the axiom of extensionality is equivalent to the injectivity of the map from the previous point. Indeed, one must verify that for all $p, q \in \mathbb{N}$, $[p] = [q]$ implies $p = q$.

3. **Foundation** Let p be a natural not E-empty number, i.e. $[p] \neq \emptyset$, i.e. non zero. Let r be the minimum of the set $[p]$. Also notice that for m and n natural numbers, the relation mEn implies $m < 2^m \leq n$. Thus by minimality of r in $[q]$, no E-element of r can belong to E-elements of p.

   Comprehension Let $\varphi(x, w_1, \ldots, w_n)$ be a formula of set theory, $p \in \mathbb{N}$ and $r_1, \ldots, r_n \in \mathbb{N}$. The set $[p]$ of E-elements of p is finite, as is the set $F = \{k \in [p] \mid (\mathbb{N}, E) \models \varphi(k, r_1, \ldots, r_n)\}$. The natural number we are looking for is thus $\sum_{k\in F} 2^k$.

   Pair Let $p, q \in \mathbb{N}$. If $p \neq q$, then the natural number $r = 2^p + 2^q$ is such that $[r] = \{p, q\}$, i.e. its E-elements are exactly p and q. If $p = q$, one must choose $r = 2^p$.

   Union Let $p \in \mathbb{N}$. The set of natural numbers $U = \bigcup_{r\in[p]}[r]$ is finite since it is a finite union of finite sets and it is such that $q = \sum_{n\in U} 2^n$ is the E-union of p. That is to say that for all $n \in \mathbb{N}$, there exists r such that rEp and nEr iff there exists $r \in [p]$ with $n \in [r]$ iff $n \in U = [q]$.

   Replacement Let $\varphi(x, y, w_1, \ldots, w_n)$ a formula of set theory, $p \in \mathbb{N}$ and $r_1, \ldots, r_n \in \mathbb{N}$. Suppose that φ is functional on $[p]$, i.e. for each q E-element of p there exists a unique natural number m_q such that $(\mathbb{N}, E) \models \varphi(q, m_q, r_1, \ldots, r_n)$. For $I = \{m_q \mid q \in [p]\}$, the natural number we are looking for is $s = \sum_{q\in I} 2^q$, i.e. its E-elements are the images by φ of the E-elements of p.

   Power Let $p \in \mathbb{N}$. Notice that the interpretation of the E-inclusion of two natural numbers n and m corresponds exactly to the inclusion $[n] \subseteq [m]$ in the usual sense. Thus for the (finite) set $\mathcal{P}([p])$ of the subsets (in the usual sense) of $[p]$, we consider the set $Q = \{\sum_{n\in F} 2^n \mid F \in \mathcal{P}([p])\}$. Then $q = \sum_{n\in Q} 2^n$ is the natural number of the E-subsets of p.

   Choice Let p be a natural number. We show the existence of an E-well ordering of p. Let us consider the set $R = \{\langle m, n\rangle \in [p] \times [p] \mid m < n\}$ where $<$ denotes the usual strict order on the natural numbers. The set R is a well order on the set $[p]$. For all distinct natural numbers m, n, the ordered pair $\langle m, n\rangle = \{\{m\}, \{m, n\}\}$ is given by the natural number $q_{m,n} = 2^{2^m} + 2^{(2^m+2^n)}$. Therefore, the natural number $\sum_{(m,n)\in R} 2^{q_{m,n}}$ is a E-well order on p.

   Finally, let us show that $(\mathbb{N}, E)$ does not satisfy the axiom of infinity. Let $p \in \mathbb{N}$. By the previous part we know that «$\{p\}$» is interpreted in $(\mathbb{N}, E)$ as the natural number 2^p. In the same way, «$\{p, \{p\}\}$» is interpreted as the natural number $2^p + 2^{2^p}$. Therefore, «$s(p) = p \cup \{p\} = \bigcup\{p, \{p\}\}$» is interpreted in $(\mathbb{N}, E)$ as the natural number $\sum_{n\in U} 2^n = p + 2^p$, where $U = [p] \cup [2^p] = [p] \cup \{p\}$. Therefore, $s^{n+1}(p) = s^n(p) + 2^{s^n(p)}$ for all $n \in \mathbb{N}$ and for m, n distinct, $s^m(p) \neq s^n(p)$. The natural number i of which the axiom of infinity requires the existence must therefore necessarily have an infinite number of E-elements, i.e. be such that $[i]$ is infinite. Such a natural number does not exists.