Solution Sheet n°13

Solution of exercise 1:

1. *Proof.*

   (a) $\mathcal{G} \in \mathscr{F}$.
   $\mathcal{G} = \mathit{fix}_{\mathscr{G}}(\emptyset) = \{\pi \in \mathscr{G} \mid \forall x \in \emptyset \ \breve{\pi}(x) = x\}$.

   (b) If $\mathcal{H} \in \mathscr{F}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $\mathcal{K} \in \mathscr{F}$.
   This is by the very definition of the fact $\mathscr{F}$ is generated by $\{\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{\mathit{fin}}(\mathbb{A})\}$.

   (c) If $\mathcal{H} \in \mathscr{F}$ and $\mathcal{K} \in \mathscr{F}$, then $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$.
   Because $\mathit{fix}_{\mathscr{G}}(F_0) \cap \ldots \cap \mathit{fix}_{\mathscr{G}}(F_n) = \mathit{fix}_{\mathscr{G}}(F_0 \cup \ldots \cup F_n)$.

   (d) If $\mathcal{H} \in \mathscr{F}$, then $\pi \circ \mathcal{H} \circ \pi^{-1} \in \mathscr{F}$.
   Assume $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathcal{H}$. For every $\mathbb{a} \in F$, and $\rho \in \mathit{fix}_{\mathscr{G}}(F)$, it holds that $\pi \circ \rho \circ \pi^{-1}(\pi(\mathbb{a})) = \pi \circ \rho(\mathbb{a}) = \pi(\mathbb{a})$, so $\pi \circ \mathit{fix}_{\mathscr{G}}(F) \circ \pi^{-1} = \mathit{fix}_{\mathscr{G}}(\pi[F])$. But $\pi \circ \mathit{fix}_{\mathscr{G}}(F) \circ \pi^{-1} \subseteq \pi \circ \mathcal{H} \circ \pi^{-1}$, so we are done.

   (e) For each atom $\mathbb{a} \in \mathbb{A}$, $\{\pi \in \mathcal{G} \mid \pi(\mathbb{a}) = \mathbb{a}\} \in \mathscr{F}$.
   We have $\{\pi \in \mathcal{G} \mid \pi(\mathbb{a}) = \mathbb{a}\} = \mathit{fix}_{\mathscr{G}}(\{\mathbb{a}\})$.

   □

2. *Proof.*

   - If $S \cap (\mathbb{A} \setminus F) = \emptyset$, then $S \subseteq F$.
   - If $S \cap (\mathbb{A} \setminus F) \neq \emptyset$, we show that $S \supseteq (\mathbb{A} \setminus F)$. We fix some $\mathbb{a} \in S \cap (\mathbb{A} \setminus F)$ and consider any $\mathbb{b} \in (\mathbb{A} \setminus F)$ such that $\mathbb{b} \neq \mathbb{a}$. The permutation $\pi_{\mathbb{a} \leftrightarrow \mathbb{b}}$ which exchanges $\mathbb{a}$ and $\mathbb{b}$, and is the identity everywhere else, belongs to $\mathit{fix}_{\mathscr{G}}(F)$. Now, $\breve{\pi}_{\mathbb{a} \leftrightarrow \mathbb{b}}(S) = S$ implies $\mathbb{a} \in S \iff \breve{\pi}_{\mathbb{a} \leftrightarrow \mathbb{b}}(\mathbb{a}) \in S$, which shows that $\mathbb{b}$ belongs to S. Thus $(\mathbb{A} \setminus S) \subseteq F$.

   □

3. *Proof.* Towards a contradiction, we assume that inside $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ there exists $f : \aleph_0 \xrightarrow{1-1} \mathbb{A}$. Then we consider the set

   $$S = \{f(2n) \in \mathbb{A} \mid n \in \omega\}$$

   S belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ since $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ satisfies **ZFA**; thus, $\mathit{sym}_{\mathscr{G}}(S) \in \mathscr{F}$, hence there exists some finite $F \subseteq \mathbb{A}$ such that $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathit{sym}_{\mathscr{G}}(S)$.

   By previous exercise, S is either finite or co-finite, a contradiction.

   □

4. *Proof.* Towards a contradiction, we assume that inside $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ there exists $f : \aleph_0 \xrightarrow{1-1} \mathcal{P}(\mathbb{A})$. Since f belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$, there exists some finite $F_f \subseteq \mathbb{A}$ such that

$$\mathit{fix}_{\mathscr{G}}(F_f) \subseteq \mathit{sym}_{\mathscr{G}}(f)\,.$$

By Point 2. any $S \subseteq \mathbb{A}$ that satisfies $\mathit{fix}_{\mathscr{G}}(F_f) \subseteq \mathit{sym}_{\mathscr{G}}(S)$ satisfies also either $S \subseteq F_f$ or $(\mathbb{A} \smallsetminus S) \subseteq F_f$. Since F_f is finite, at most finitely many sets S can satisfy either $S \subseteq F_f$ or $(\mathbb{A} \smallsetminus S) \subseteq F_f$. So, take any $n \in \omega$ such that $f(n) \subseteq \mathbb{A}$ satisfies

$$\mathit{fix}_{\mathscr{G}}(F_f) \nsubseteq \mathit{sym}_{\mathscr{G}}\big(f(n)\big).$$

Take any $\pi \in \mathit{fix}_{\mathscr{G}}(F_f) \smallsetminus \mathit{sym}_{\mathscr{G}}\big(f(n)\big)$ in order to have both

$$\breve{\pi}\,(f) = f \quad \text{and} \quad \breve{\pi}\big(f(n)\big) \neq f(n).$$

Since n belongs to the kernel, $\breve{\pi}\,(n) = n$ holds, which leads to $f\big(\breve{\pi}\,(n)\,\big) = f(n)$.

By construction,

$$\begin{aligned}\breve{\pi}\,(f) &= \breve{\pi}\Big(\big\{\big(k, f(k)\big) \mid k \in \omega\big\}\Big)\\ &= \Big\{\Big(\breve{\pi}\,(k)\,, \breve{\pi}\big(f(k)\big)\Big) \mid k \in \omega\Big\}\\ &= \Big\{\Big(k, \breve{\pi}\big(f(k)\big)\Big) \mid k \in \omega\Big\}.\end{aligned}$$

So that, in particular, we have

$$\breve{\pi}\,(f)\,(n) = \breve{\pi}\big(f(n)\big),$$

which contradicts $\breve{\pi}\big(f(n)\big) \neq f(n)$.

□

Solution of exercise 2:

1. *Proof.* For every permutation $\pi \in \mathscr{G}$ and every $(\mathbb{a}, \mathbb{b}) \in \mathbb{A} \times \mathbb{A}$ we have

$$\begin{aligned}(\mathbb{a}, \mathbb{b}) \in <_{\mathbf{M}} &\iff \mathbb{a} <_{\mathbf{M}} \mathbb{b}\\ &\iff \pi(\mathbb{a}) <_{\mathbf{M}} \pi(\mathbb{b})\\ &\iff \big(\pi(\mathbb{a}), \pi(\mathbb{b})\big) \in <_{\mathbf{M}}\\ &\iff \breve{\pi}\big(\mathbb{a}, \mathbb{b}\big) \in <_{\mathbf{M}}\,.\end{aligned}$$

This shows $\breve{\pi}\big(\,<_{\mathbf{M}}\,\big) =<_{\mathbf{M}}$ holds for every $\pi \in \mathscr{G}$, hence $\mathit{sym}_{\mathscr{G}}\big(\,<_{\mathbf{M}}\,\big) = \mathscr{G} \in \mathscr{F}$, thus $<_{\mathbf{M}} \in \boldsymbol{\mathcal{M}}$. □

2. (a) *Proof.* Given an permutation $\pi \in \mathit{fix}_{\mathscr{G}}(F \cap F')$ there exists permutations $\rho_1, \ldots, \rho_k \in \mathit{fix}_{\mathscr{G}}(F)$ and $\rho'_1, \ldots, \rho'_k \in \mathit{fix}_{\mathscr{G}}(F')$ — for some k large enough — such that $\rho_1 \circ \rho'_1 \circ \rho_2 \circ \rho'_2 \circ \ldots \rho_k \circ \rho'_k = \pi$. Instead of

giving a very tedious proof, we illustrate this fact by an example: assume $F = \{\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3, \mathbb{a}_4\}$ and $F' = \{\mathbb{a}_1, \mathbb{b}_2, \mathbb{a}_4\}$ with $F \cap F' = \{\mathbb{a}_1, \mathbb{a}_4\}$ and

$$\mathbb{a}_1 <_{\mathbf{M}} \mathbb{a}_2 <_{\mathbf{M}} \mathbb{b}_2 <_{\mathbf{M}} \mathbb{a}_3 <_{\mathbf{M}} \mathbb{a}_4$$

Assume π satisfies $\mathbb{a}_2 <_{\mathbf{M}} \pi(\mathbb{a}_2) <_{\mathbf{M}} \mathbb{b}_2 <_{\mathbf{M}} \pi(\mathbb{b}_2) <_{\mathbf{M}} \pi(\mathbb{a}_3) <_{\mathbf{M}} \mathbb{a}_3$, then take:

i. ρ' defined by
   - on $]-\infty, \mathbb{a}_2]$, $\rho' = \pi$
   - on $]\mathbb{a}_2, \mathbb{b}_2[$, $\rho' = \theta$ for some (any) order isomorphism between $]\mathbb{a}_2, \mathbb{b}_2[$ and $]\pi(\mathbb{a}_2), \mathbb{b}_2[$
   - $\rho'(\mathbb{b}_2) = \mathbb{b}_2$
   - on $]\mathbb{b}_2, \mathbb{a}_3[$, $\rho' = \delta$ for some (any) order isomorphism between $]\mathbb{b}_2, \mathbb{a}_3[$ and $]\mathbb{b}_2, \pi(\mathbb{a}_3)[$
   - $\rho'(\mathbb{a}_3) = \pi(\mathbb{a}_3)$
   - on $]\mathbb{a}_3, +\infty]$, $\rho' = \pi$

ii. ρ defined by
   - on $]-\infty, \pi(\mathbb{a}_2)]$, $\rho = id$
   - on $]\pi(\mathbb{a}_2), \mathbb{b}_2[$, $\rho = \pi \circ \theta^{-1}$
   - $\rho(\mathbb{b}_2) = \pi(\mathbb{b}_2)$
   - on $]\mathbb{b}_2, \mathbb{a}_3[$, $\rho = \pi \circ \delta^{-1}$
   - on $]\mathbb{a}_3, +\infty]$, $\rho = id$

   Notice that $\rho' \in \mathit{fix}_{\mathscr{G}}(F')$ and $\rho \in \mathit{fix}_{\mathscr{G}}(F)$ and $\rho \circ \rho' = \pi$.

□

(b) *Proof.* Take any $F \in \mathcal{P}_{fin}(\mathbb{A})$ such that $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathit{sym}_{\mathscr{G}}(x)$ and consider

$$E = \bigcap \{ F' \subseteq F \mid \mathit{fix}_{\mathscr{G}}(F') \subseteq \mathit{sym}_{\mathscr{G}}(x) \}.$$

Clearly, $\mathit{fix}_{\mathscr{G}}(E) \subseteq \mathit{sym}_{\mathscr{G}}(x)$ and E is $\subseteq$-minimal. □

(c) *Proof.* For any $\pi \in \mathscr{G}$ we have $\breve{\pi}(x, E) = (\breve{\pi}(x), \breve{\pi}(E))$. Moreover, $\mathit{fix}_{\mathscr{G}}(\breve{\pi}(E)) = \pi \circ \mathit{fix}_{\mathscr{G}}(E) \circ \pi^{-1}$ and $\mathit{sym}_{\mathscr{G}}(\breve{\pi}(x)) = \pi \circ \mathit{sym}_{\mathscr{G}}(x) \circ \pi^{-1}$. So, if E is the $\subseteq$-least support of x, then $\breve{\pi}(E)$ is the $\subseteq$-least support of $\breve{\pi}(x)$. Therefore, we have shown

$$\mathit{sym}_{\mathscr{G}}\Big(\{(x, E) \in \boldsymbol{\mathcal{M}} \times \mathcal{P}_{fin}(\mathbb{A}) \mid E \text{ is least support of } x\}\Big) = \mathscr{G} \in \mathscr{F}.$$

□

3. *Proof.* Assume $F = \{\mathbb{a}_1, \dots, \mathbb{a}_n\}$ with $\mathbb{a}_1 <_{\mathbf{M}} \dots <_{\mathbf{M}} \mathbb{a}_n$ and F is a support of S. We have for every $\mathbb{b} \in S$:

   (a) if $\mathbb{b} <_{\mathbf{M}} \mathbb{a}_1$, then $\{\mathbb{c} \in \mathbb{A} \mid \mathbb{c} <_{\mathbf{M}} \mathbb{a}_1\} \subseteq S$ holds since for any $\mathbb{c} <_{\mathbf{M}} \mathbb{a}_1$ there exists some mapping $\pi \in \mathit{fix}_{\mathscr{G}}(F)$ which satisfies $\pi(\mathbb{b}) = \mathbb{c}$. So, we have

   $$\mathbb{b} \in S \Longrightarrow \pi(\mathbb{b}) \in \breve{\pi}(S) \Longrightarrow \mathbb{c} \in \breve{\pi}(S) = S.$$

   (b) if $\mathbb{a}_n <_{\mathbf{M}} \mathbb{b}$, then $\{\mathbb{c} \in \mathbb{A} \mid \mathbb{a}_n <_{\mathbf{M}} \mathbb{c}\} \subseteq S$ since for any $\mathbb{a}_n <_{\mathbf{M}} \mathbb{c}$ there exists some mapping $\pi \in \mathit{fix}_{\mathscr{G}}(F)$ which satisfies $\pi(\mathbb{b}) = \mathbb{c}$. So, we have

   $$\mathbb{b} \in S \Longrightarrow \pi(\mathbb{b}) \in \breve{\pi}(S) \Longrightarrow \mathbb{c} \in \breve{\pi}(S) = S.$$

(c) if $\mathbb{a}_i <_{\mathbf{M}} \mathbb{b} <_{\mathbf{M}} \mathbb{a}_{i+1}$ then $\{\mathbb{c} \in \mathbb{A} \mid \mathbb{a}_i <_{\mathbf{M}} \mathbb{c} <_{\mathbf{M}} \mathbb{a}_{i+1}\} \subseteq S$ since for any $\mathbb{a}_i <_{\mathbf{M}} \mathbb{c} <_{\mathbf{M}} \mathbb{a}_{i+1}$ there exists some mapping $\pi \in \mathit{fix}_{\mathscr{G}}(F)$ which satisfies $\pi(\mathbb{b}) = \mathbb{c}$. So, we have

$$\mathbb{b} \in S \Longrightarrow \pi(\mathbb{b}) \in \breve{\pi}(S) \Longrightarrow \mathbb{c} \in \breve{\pi}(S) = S.$$

So, there are exactly $n+1$ such intervals, each of them either entirely belongs to S or is disjoint from S. There are also n atoms in F, each of which may or may not belong to S. So, there are as many sets of the form S as there are mappings from $n+1+n$ into $\{0,1\}$ which makes a total of 2^{2n+1} different subsets of $\mathbb{A}$.

□

Solution of exercise 3:

1. *Proof.* we write

   - I_0 for $]-\infty, \mathbb{a}_1[= \{\mathbb{b} \in \mathbb{A} \mid \mathbb{b} <_{\mathbf{M}} \mathbb{a}_1\}$
   - I_k for $]\mathbb{a}_k, \mathbb{a}_{k+1}[= \{\mathbb{b} \in \mathbb{A} \mid \mathbb{a}_k <_{\mathbf{M}} \mathbb{b} <_{\mathbf{M}} \mathbb{a}_{k+1}\}$ (any $1 \leq k < n$)
   - I_n for $]\mathbb{a}_n, +\infty[= \{\mathbb{b} \in \mathbb{A} \mid \mathbb{a}_1 <_{\mathbf{M}} \mathbb{b}\}$

   We map every sequence $\chi \in {}^{2n+1}2$ to $S_\chi \subseteq \mathbb{A}$ defined by

   $$S_\chi = \bigcup \Big(\{I_k \subseteq \mathbb{A} \mid 0 \leq k \leq n \wedge \chi(2k) = 1\} \cup \{a_k \in \mathbb{A} \mid 1 \leq k \leq n \wedge \chi(2k-1) = 1\}\Big)$$

   so that $\{S_\chi \mid \chi \in {}^{2n+1}2\}$ is the set of all subsets of $\mathbb{A}$ which have $F = \{\mathbb{a}_1, \ldots, \mathbb{a}_n\}$ as support.

   □

2. *Proof.* We equip $2^{<\omega}$ with the lexicographic ordering $<_{lex.}$ defined by

   $$\chi <_{lex.} \chi' \iff \exists i \ \Big(\chi(i) = 0 \ \wedge \ \chi(i) = 1 \ \wedge \ \forall j < i \ \chi(j) = \chi'(j)\Big).$$

   For every sequence $\chi \in 2^{<\omega}$ we write $\overset{\leftrightsquigarrow}{\chi}$ for the sequence of same length as χ that satisfies $\chi(n) = 1 - \overset{\leftrightsquigarrow}{\chi}(n)$ (any integer $n < lh(\chi)$). We define a mapping $g : 2^{<\omega} \to 2^{<\omega}$ by $g(\emptyset) = \emptyset$ and for χ a non-empty sequence,

   $$\begin{aligned} g(\chi) &= \chi \quad \text{if} \quad \chi(0) = 0 \\ &= \overset{\leftrightsquigarrow}{\chi} \quad \text{if} \quad \chi(0) = 1 \end{aligned}$$

   So, $g(\chi)$ is the one among χ and its dual $\overset{\leftrightsquigarrow}{\chi}$ which starts with a 0.

   For every integer n and every $\chi \in 2^n$ we write $\chi 0$ for the sequence in 2^{n+1} which satisfies $\chi 0 \restriction n = \chi$ and $\chi 0(n) = 0$.

   We define an ordering $\prec_n$ on ${}^{2n+1}2$ by

   $$\chi \prec_n \chi' \iff g(\chi 0) <_{lex.} g(\chi' 0).$$

and denote by

$$h : 2^{2n+1} \xrightarrow{onto} {}^{2n+1}2$$
$$i \mapsto \chi_{(i,n)}$$

the enumeration of ${}^{2n+1}2$ along $\prec_n$. i.e., we have

$$\chi_{(0,n)} \prec_n \chi_{(1,n)} \prec_n \cdots \prec_n \chi_{(2^{2n+1}-1,n)}.$$

We finally define the surjection by

$$f : \mathcal{P}_{fin}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A})$$
$$F \neq \emptyset \mapsto S_{\chi_{(|F|,|F|)}}$$
$$\emptyset \mapsto \emptyset.$$

So, if the cardinality of F is n, then $\chi_{(|F|,|F|)}$ is the n^{th} mapping — with regard to the ordering $\prec_n$ — of the form $\chi : 2n+1 \to \{0,1\}$.

This mapping belongs to the *Mostowski model* $\boldsymbol{\mathcal{M}}$, essentially because, as a permutation model, it satisfies **ZFA** .

It remains to show that f is onto. For this purpose, take any $S \in \mathcal{P}(\mathbb{A}) \setminus \emptyset$ with E the $\subseteq$-least support of S and $|E| = n$. By construction, there exists some integer $i < 2^{2n+1}$ such that $S_{\chi_{(i,n)}} = S$. The way the ordering $\prec_n$ is defined guarantees $i \geq n$: this is because E being the $\subseteq$-least support of S, there are at least n many 1's in the sequence $\chi_{(i,n)}$ (by construction, $\chi_{(i,n)}(2j-1) = 1$ holds for all $1 \leq i \leq n$). So, if $i = n$, then we are done. Otherwise, it is tedious but straightforward to see that E can be extended into a set $F \supseteq E$ which satisfies $|F| = i$ and $S_{\chi_{(i,i)}} = S_{\chi_{(i,n)}}$, which gives the result.

□