Solution Sheet n°12

Solution of exercise 1:

1. The result is trivial when A is empty, for B must be empty as well. So, we assume A and B are non-empty. Since $A \overset{\text{onto}}{\lesssim} B$, take any $g : B \xrightarrow{onto} A$ and form $\{g^{-1}(a) \mid a \in A\}$ which is a non-empty set of non-empty sets. By **AC**, one obtains a choice function c which for each $a \in A$ provides a *unique* $c(a) \in B$ such that $c(a) \in g^{-1}(a)$. By construction, $c : A \xrightarrow{1-1} B$ witnesses that $A \overset{1-1}{\lesssim} B$.

2. This is immediate from the previous result and the Cantor Schroeder Bernstein Theorem.

3. ($\Longleftarrow$) Given any family $(A_i)_{i\in I}$ of non-empty disjoint sets, we obtain a choice function $f : I \to \bigcup_{i\in I} A_i$ by letting $g : \bigcup_{i\in I} A_i \xrightarrow{onto} I$ be defined as $g(a) = i$ iff $a \in A_i$ and $f : I \xrightarrow{1-1} \bigcup_{i\in I} A_i$ be any function such that $g \circ f = id$ — which guarantees that $f(i) \in A_i$ holds for every $i \in I$.

   ($\Longrightarrow$) The result is trivial when A is empty, for B must be empty as well. So, we assume A and B are non-empty. Since $g : B \xrightarrow{onto} A$, form $\{g^{-1}(a) \mid a \in A\}$ which is a non-empty set of non-empty sets. By **AC**, one obtains a choice function f which for each $a \in A$ provides a unique $f(a) \in g^{-1}(a)$. By construction, $f : A \xrightarrow{1-1} B$ and $g \circ f = id$ both hold.

4. Assume $f : A \xrightarrow{1-1} B$, then take any element $a' \in A$ and define $g : B \xrightarrow{onto} A$ by $g(x) = a'$ if $x \notin f[A]$, and $g(x) = a$ if $f(a) = x$. The fact that f is 1-1 guarantees that g is onto.

5. Given $f : A \xrightarrow{1-1} B$, define $g : \mathcal{P}(A) \xrightarrow{1-1} \mathcal{P}(B)$ by $g(C) = f[C]$.

Solution of exercise 2:
We consider the following set:

$$\mathcal{W} = \{(B, <_B) \subseteq A \times (A \times A) \mid (B, <_B) \textit{ is a well-ordering}\}.$$

Notice that this set is non-empty since the empty ordering $(\emptyset,\emptyset)$ belongs to $\mathcal{W}$. We then consider the functional $\mathbf{F} : \mathcal{W} \to \mathbf{On}$ defined by

$$\mathbf{F}\big((B, <_B)\big) = \text{ the unique ordinal } \beta \text{ s.t. } (\beta, \in_\beta) \simeq (B, <_B).$$

We set

$$\alpha = \sup\Big\{\mathbf{F}\big((B, <_B)\big) + 1 \mid (B, <_B) \in \mathcal{W}\Big\}.$$

It turns out that $\alpha \overset{1-1}{\not\lesssim} A$ holds; for otherwise if we let $f : \alpha \xrightarrow{1-1} A$ and set

$$B = f[\alpha] \text{ and } <_B = \big\{\big(f(\gamma), f(\delta)\big) \mid \gamma < \delta < \alpha\big\},$$

we then obtain $(B, <_B) \in \mathcal{W}$, hence $\alpha \in \mathbf{F}[\mathcal{W}]$, contradicting $\alpha > \mathbf{F}\big((B, <_B)\big)$.

Solution of exercise 3:

1. $\mathbb{R} \simeq {}^\omega\omega \simeq {}^\omega 2$: By Cantor Schroeder Bernstein Theorem, we only need to show $\mathbb{R} \overset{1-1}{\lesssim} {}^\omega\omega \overset{1-1}{\lesssim} {}^\omega 2 \overset{1-1}{\lesssim} \mathbb{R}$.

   $\mathbb{R} \overset{1-1}{\lesssim} {}^\omega\omega$**:** assume every real r is written in base 10 as

   - in case $0 \leq r$: $r = +\, e_0\, e_1\, e_2 \ldots e_k, d_0\, d_1\, d_2\, d_3 \ldots d_n\, d_{n+1}\, d_{n+2} \ldots\ldots$
   - in case $r < 0$: $r = -\, e_0\, e_1\, e_2 \ldots e_k, d_0\, d_1\, d_2\, d_3 \ldots d_n\, d_{n+1}\, d_{n+2} \ldots\ldots$

   where

   (a) k is finite,

   (b) for each $i \leq k$ and each $j \in \mathbb{N}$, $e_i, d_j \in \{0,1,2,3,4,5,6,7,8,9\}$

   (c) $e_0 = 0 \Longrightarrow k = 0$,

   (d) $\langle d_j \mid j \in \mathbb{N}\rangle$ satisfies $\forall j\, \exists j' > j\ d_j \neq 9$. i.e., it is not ultimately constant with value 9. This means for instance that the real $0{,}23999999999999\ldots$ is rather represented by $+0{,}24000000000000\ldots$ and the integer -3 by $-3{,}00000000000\ldots$.

   We describe the following mapping $f : \mathbb{R} \xrightarrow{1-1} {}^\omega\omega$ by

   - If $r = +\, e_0\, e_1\, e_2 \ldots e_k, d_0\, d_1\, d_2\, d_3 \ldots d_n\, d_{n+1}\, d_{n+2} \ldots\ldots$, then
     $$f(r) = \langle 8{,}1 + e_0{,}1 + e_1, \ldots, 1 + e_k{,}0{,}1 + d_0, 1 + d_1, \ldots, 1 + d_n, 1 + d_{n+1}, \ldots\rangle\,.$$
   - If $r = +{,}e_0{,}e_1{,}e_2 \ldots e_k{,}{,}d_0, d_1, d_2, d_3 \ldots d_n, d_{n+1}, d_{n+2} \ldots\ldots$, then
     $$f(r) = \langle 9{,}1 + e_0{,}1 + e_1, \ldots, 1 + e_k{,}0{,}1 + d_0, 1 + d_1, \ldots, 1 + d_n, 1 + d_{n+1}, \ldots\rangle\,.$$

   ${}^\omega\omega \overset{1-1}{\lesssim} {}^\omega 2$**:** we define $g : {}^\omega\omega \xrightarrow{1-1} {}^\omega 2$ by $g\big(\langle a_i \mid i \in \omega\rangle\big) = 1\underbrace{0\ldots0}_{a_0}1\underbrace{0\ldots0}_{a_1}1\underbrace{0\ldots0}_{a_2}1\ldots$

   ${}^\omega 2 \overset{1-1}{\lesssim} \mathbb{R}$**:** we define $h : {}^\omega 2 \xrightarrow{1-1} \mathbb{R}$ by $g\big(\langle a_i \mid i \in \omega\rangle\big) = 0{,}a_0\, a_1\, a_2\ \ldots\ a_n\, a_{n+1}\ \ldots\ldots$

2. ${}^\omega\mathbb{R} \simeq {}^\omega({}^\omega\omega) \simeq {}^\omega({}^\omega 2)$: it is enough to show that whenever $A \overset{1-1}{\lesssim} B$ holds for non-empty sets A and B, then ${}^\omega A \overset{1-1}{\lesssim} {}^\omega B$ holds as well. So, given any $f : A \xrightarrow{1-1} B$, define $h : {}^\omega A \xrightarrow{1-1} {}^\omega B$ by
   $$h\big(\langle a_i \mid i \in \omega\rangle\big) = \langle f(a_i) \mid i \in \omega\rangle\,.$$

3. ${}^\omega 2 \simeq {}^\omega({}^\omega 2)$: ${}^\omega 2 \overset{1-1}{\lesssim} {}^\omega({}^\omega 2)$ is obvious. We show ${}^\omega({}^\omega 2) \overset{1-1}{\lesssim} {}^\omega 2$ by providing $f : {}^\omega({}^\omega 2) \xrightarrow{1-1} {}^\omega 2$ defined by $f\Big(\Big\langle \langle a_{i,j} \mid j < \omega\rangle \mid i < \omega\Big\rangle\Big) = \langle b_k \mid k < \omega\rangle$ where $b_k = a_{i,j}$ iff $k = \frac{(i+j)(i+j+1)}{2} + i$.

   (The mapping $(i,j) \mapsto \frac{(i+j)(i+j+1)}{2} + i$ is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, but any other bijection would work as well.)

Solution of exercise 4:

1. We construct $f : {}^{\omega}2 \xrightarrow{onto} \omega_1$.

   (a) we define a mapping $\ulcorner\quad\urcorner : \mathbb{N} \times \mathbb{N} \xrightarrow{1-1} \mathbb{N}$ by $\ulcorner n,m \urcorner = 2^{n+1} \cdot 3^{m+1}$.

   (b) For each $s = \langle a_i \mid i \in \omega \rangle \in {}^{\omega}2$ we set

      - if $\exists i \, \forall j \geq i \; a_j = 0$, then $f(s) = i$ for the least such i;
      - if $\forall i \, \exists j \geq i \; a_j = 1$, then
         - if $\exists i \, \forall n \, \forall m \; (a_i = 1 \wedge \ulcorner n,m \urcorner \neq i)$, then $f(s) = 0$
         - if $\forall i \, \exists n \, \exists m \; (a_i = 1 \longrightarrow \ulcorner n,m \urcorner = i)$, then
            * if $\Big(\mathbb{N}, \{(n,m) \mid a_{\ulcorner n,m \urcorner} = 1\}\Big)$ is not a well-ordering, then $f(s) = 0$;
            * if $\Big(\mathbb{N}, \{(n,m) \mid a_{\ulcorner n,m \urcorner} = 1\}\Big)$ is a well-ordering, then $f(s) = \alpha$ where α is the unique ordinal isomorphic to $\Big(\mathbb{N}, \{(n,m) \mid a_{\ulcorner n,m \urcorner} = 1\}\Big)$. Notice that $\alpha \in \omega_1$ since α is countable.

            To show that f is onto, it is enough to show that for every infinite countable ordinal α there exists some $s \in {}^{\omega}2$ such that $f(s) = \alpha$. For this, notice that α being countable, any bijection $h : \mathbb{N} \xleftrightarrow{bij.} \alpha$ induces a well-ordering on $\mathbb{N}$ of type α. Namely, $(\mathbb{N}, <_{\alpha})$ where $<_{\alpha} = \{(n,m) \mid h(n) < h(m)\}$. By construction, $s = \langle a_i \mid i \in \omega \rangle \in {}^{\omega}2$ defined by $a_i = 1$ iff there exists $(n,m) \in <_{\alpha}$ such that $\ulcorner n,m \urcorner = i$.

2. We construct $f : {}^{\omega}2 \xrightarrow{onto} {}^{\omega}2 \cup \omega_1$. From the previous case, we are granted with a mapping $f' : {}^{\omega}2 \xrightarrow{onto} \omega_1$. Given any $s = \langle a_i \mid i \in \omega \rangle \in {}^{\omega}2$ we define $f(s)$ as follows:

   - if $a_0 = 0$, then $f(s) = \langle a_{i+1} \mid i \in \omega \rangle$;
   - if $a_0 = 1$, then $f(s) = f'\big(\langle a_{i+1} \mid i \in \omega \rangle\big)$.

Solution of exercise 5:

1. Notice first that since $\mathbb{R} \simeq {}^{\omega}\mathbb{R} \simeq {}^{\omega}\omega \simeq {}^{\omega}({}^{\omega}\omega) \simeq {}^{\omega}2 \simeq {}^{\omega}({}^{\omega}2)$ holds, the assumption is equivalent to saying that any of these sets is is a countable union of countable sets. So, we assume that it is the case of ${}^{\omega}({}^{\omega}2)$. i.e., there exists $(G_n)_{n<\omega}$ where for each integer n, G_n is countable and

$$ {}^{\omega}({}^{\omega}2) = \bigcup_{n<\omega} G_n. $$

   Towards a contradiction, we assume that $\omega_1 \overset{1-1}{\lesssim} {}^{\omega}2$ holds, so that there exists some $f : \omega_1 \xrightarrow{1-1} {}^{\omega}2$. We set

$$ H_n = \{s \in {}^{\omega}2 \mid \exists S \in G_n \; \exists k < \omega \; S(k) = s\}. $$

   We notice that, for each integer n, we have $H_n \overset{1-1}{\lesssim} \omega$. Indeed, we take any $g : G_n \xrightarrow{1-1} \omega$ and construct $\mathcal{I} : H_n \xrightarrow{1-1} \omega$ by $\mathcal{I}(s) = \frac{(i+j)(i+j+1)}{2} + i$

where i is the least integer such that there exists $S \in G_n$ with $g(S) = i$ and there exists some $k < \omega$ $S(k) = n$; and j is the least such k.

We then define $h : \omega \to {}^{\omega}2$ so that for each integer n we have

$$h(n) = f(\alpha_n) \quad \text{where} \quad \alpha_n = \min\{\alpha \in \omega_1 \mid f(\alpha) \notin H_n\}.$$

By definition, $h \in {}^{\omega}({}^{\omega}2) = \bigcup_{n<\omega} G_n$, hence for some integer n we have $h \in G_n$. As usual with this kind of diagonal argument, it is enough to consider $h(n)$ to obtain a contradiction for we end up with

- $h(n) \in H_n$ because $h \in G_n$
- $h(n) \notin H_n$ because $h(n) = f(\alpha_n) \notin H_n$.

2. The statements $\mathcal{R} \overset{1-1}{\not\lesssim} \mathbb{R}$ and $\mathbb{R} \overset{1-1}{\lesssim} \mathcal{R}$ are equivalent to the existence of some partition $\mathcal{C}$ of ${}^{\omega}2$ such that ${}^{\omega}2 \overset{1-1}{\lesssim} \mathcal{C}$ and $\mathcal{C} \overset{1-1}{\not\lesssim} {}^{\omega}2$. Indeed, if $\mathbb{R} \lnsim \mathcal{R}$ holds, then take any $f : \mathbb{R} \overset{bij.}{\longleftrightarrow} {}^{\omega}2$ and define $\mathcal{C} = \{f[p] \mid p \in \mathcal{R}\}$. Clearly $\mathcal{C}$ is a partition of ${}^{\omega}2$ that satisfies $\mathcal{R} \simeq \mathcal{C}$, which yields ${}^{\omega}2 \lnsim \mathcal{C}$ since one has ${}^{\omega}2 \simeq \mathbb{R} \lnsim \mathcal{R} \simeq \mathcal{C}$. Similarly, if ${}^{\omega}2 \lnsim \mathcal{C}$ holds, then take any $g : {}^{\omega}2 \overset{bij.}{\longleftrightarrow} \mathbb{R}$ in order to obtain the partition $\mathcal{R} = \{f[p] \mid p \in \mathcal{C}\}$ that satisfies $\mathcal{C} \simeq \mathcal{R}$ which leads to $\mathbb{R} \lnsim \mathcal{R}$ since one has $\mathbb{R} \simeq {}^{\omega}2 \lnsim \mathcal{C} \simeq \mathcal{R}$.

   We prove that there exists some partition $\mathcal{C}$ of ${}^{\omega}2$ such that ${}^{\omega}2 \lnsim \mathcal{C}$. For this, we make use of ${}^{\omega}2 \uplus \omega_1 \overset{onto}{\lesssim} {}^{\omega}2$ holds and take any $f : {}^{\omega}2 \overset{onto}{\longrightarrow} {}^{\omega}2 \uplus \omega_1$ to form the partition

$$\begin{aligned}\mathcal{C} &= \Big\{\{s \in {}^{\omega}2 \mid f(s) = x\} \mid x \in {}^{\omega}2 \uplus \omega_1\Big\} \\ &= \Big\{f^{-1}\big[\{x\}\big] \mid x \in {}^{\omega}2 \uplus \omega_1\Big\}.\end{aligned}$$

   We obtain

   ${}^{\omega}2 \overset{1-1}{\lesssim} \mathcal{C}$**:** The mapping $g : {}^{\omega}2 \longrightarrow \mathcal{C}$ defined by

$$\begin{aligned}g(x) &= \{s \in {}^{\omega}2 \mid f(s) = x\} \\ &= f^{-1}\big[\{x\}\big]\end{aligned}$$

   is obviously $1-1$, hence witnesses that ${}^{\omega}2 \overset{1-1}{\lesssim} \mathcal{C}$ holds.

   $\mathcal{C} \overset{1-1}{\not\lesssim} {}^{\omega}2$**:** Towards a contradiction, we assume $\mathcal{C} \overset{1-1}{\lesssim} {}^{\omega}2$. We notice that $\omega_1 \overset{1-1}{\lesssim} \mathcal{C}$ holds for the following mapping is 1-1: $h : \omega_1 \longrightarrow \mathcal{C}$ defined by

$$\begin{aligned}h(x) &= \{s \in {}^{\omega}2 \mid f(s) = x\} \\ &= f^{-1}\big[\{x\}\big]\end{aligned}$$

Therefore, we have both

$$\omega_1 \overset{1-1}{\lesssim} \mathcal{C} \quad \text{and} \quad \mathcal{C} \overset{1-1}{\lesssim} {}^{\omega}2$$

which leads to $\omega_1 \overset{1-1}{\lesssim} {}^{\omega}2$, contradicting Exercise 5 (1).