Solution Sheet n°11

Solution of exercise 1:

We extend F to $\widetilde{F}$ by adding a finite number of formulae necessary to show the properties of V_α and of the relation "$M \models \varphi$" of satisfaction for sets. Let φ be the conjunction of the formulae of $\widetilde{F}$ and let $\psi \ : \ \exists\alpha\,(V_\alpha \models \varphi)$. Since Γ extends ZF, ZF proves the reflexion theorem (Ex 2 Sheet 10), and $\Gamma \vdash \varphi$, it follows that $\Gamma \vdash \psi$.

We now show that if $F \vdash \psi$, then Γ is inconsistent. Indeed suppose that $F \vdash \psi$, and work in $\widetilde{F}$ to find a contradiction, this will be enough since it implies that Γ is inconsistent.

Since $\widetilde{F} \vdash \psi$, there exists a minimal β such that $V_\beta \models \varphi$. Now, since, by reflexion, $\varphi \vdash \psi$, we have $V_\beta \models \psi$, i.e. there exists $\alpha < \beta$ such that $(V_\alpha \models \varphi)^{V_\beta}$. Now $\alpha < \beta$, so $(\varphi^{V_\alpha})^{V_\beta}$ is equivalent to φ^{V_α}. Therefore, we have obtained that there exists $\alpha < \beta$ with $V_\alpha \models \varphi$, contradicting the minimality of β.

Solution of exercise 2:

1. $\rightarrow$ Suppose that $p \Vdash \neg(\neg\varphi \wedge \neg\psi)$. Then for all $q \leq p$, $q \nVdash \neg\varphi \wedge \neg\psi$, i.e. $q \nVdash \neg\varphi$ or $q \nVdash \neg\psi$. We show that the set of $r \leq p$ such that $r \Vdash \varphi$ or $r \Vdash \psi$ is dense below p. To see this, suppose that $q \leq p$. We have $q \nVdash \neg\varphi$ or $q \nVdash \neg\psi$. If $q \nVdash \neg\varphi$, then there exists $r \leq q$ such that $r \Vdash \varphi$. Similarly if $q \nVdash \neg\psi$, then there exists $r \leq q$ with $r \Vdash \psi$. In both cases we have found $r \leq q$ with $r \Vdash \varphi$ or $r \Vdash \psi$.

   $\leftarrow$ Suppose that $\{q \leq p \mid p \Vdash \varphi \vee p \Vdash \psi\}$ is dense below p and let us show that $p \Vdash \neg(\neg\varphi \wedge \neg\psi)$. Indeed, $p \Vdash \neg(\neg\varphi \wedge \neg\psi)$ if and only if $\{q \mid q \nVdash \neg\varphi \vee q \nVdash \neg\psi\}$ is dense below p. Now, for all $r \leq p$ there exists $q \leq r$ such that $q \Vdash \varphi$ or $q \Vdash \psi$, and therefore $q \nVdash \varphi$ or $q \nVdash \psi$.

2. We have that $\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$. By the previous point, $q \Vdash \neg\varphi \vee \psi$ iff $\{q \mid q \Vdash \neg\varphi \vee q \Vdash \psi\}$ is dense below p. It is enough therefore to show that $\{q \mid q \Vdash \neg\varphi \vee q \Vdash \psi\}$ is dense below p iff $\neg\exists q \leq p\ (q \Vdash \varphi \wedge q \Vdash \neg\psi)$.

   $\rightarrow$ By contraposition, if there exists $q \leq p$ with $q \Vdash \varphi$ and $q \Vdash \neg\psi$, then for all $r \leq q$ we have $r \Vdash \varphi$ and $r \nVdash \psi$, therefore for all $r \leq p$ $r \nVdash \neg\varphi$ and $r \nVdash \psi$. Therefore the set $\{q \mid q \Vdash \neg\varphi \vee q \Vdash \psi\}$ is not dense below p.

   $\leftarrow$ Suppose $\neg\exists q \leq p\ (q \Vdash \varphi \wedge q \Vdash \neg\psi)$, let us show that $q \Vdash \neg\varphi \vee \psi$, this is enough by the previous point. Let $q \leq p$, then we have $q \nVdash \varphi$ or $q \nVdash \neg\psi$, there exists therefore $r \leq q$ such that $r \Vdash \neg\varphi$ or $r \Vdash \psi$.

3. Recall the following Lemma, (Lemma 309 of the Lecture Notes):

   Lemma. *The following are equivalent:*

   (a) $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$*;*

   (b) for all $r \leq p$, $r \Vdash \varphi(\tau_1, \ldots, \tau_n)$*;*

   (c) the set $\{r \in \mathbb{P} \mid r \Vdash \varphi(\tau_1, \ldots, \tau_n)\}$ *is dense below* p*.*

Now, we have that $\forall v \varphi(v) \equiv \neg\exists v \neg\varphi(v)$.

$\rightarrow$ Suppose $p \Vdash \neg\exists v \neg\varphi(v)$ and let $\tau \in V^{\mathbb{P}}$. For all $q \leq p$ the set $\{r \leq p \mid \exists\sigma \in V^{\mathbb{P}}\ r \Vdash \neg\varphi(\sigma)\}$ is not dense below q. Therefore there exists, in particular, $r \leq q$ such that for all $s \leq r$, $s \nVdash \neg\varphi(\tau)$, i.e. there exists $s_\sigma \leq s$ with $s_\sigma \Vdash \varphi(\tau)$. This shows that the set $\{q \mid q \Vdash \varphi(\tau)\}$ is dense below p. By the Lemma above we thus obtain $p \Vdash \varphi(\tau)$ as we wished.

$\leftarrow$ If $p \Vdash \varphi(\tau)$ for all $\tau \in V^{\mathbb{P}}$, then, by the Lemma above, $q \Vdash \varphi(\tau)$ for all $q \leq p$. Thus for all $q \leq p$ and all $\tau \in V^{\mathbb{P}}$, we have that $q \nVdash \neg\varphi(\tau)$ and therefore $\{r \leq q \mid \exists\sigma \in V^{\mathbb{P}}\ r \Vdash \varphi(\sigma)\}$ is empty. It follows that $p \Vdash \neg\exists v \neg\varphi(v)$.

Solution of exercise 3:

1. Notice that the notion of atom is absolute for M. We work in M to show the existence of a $\mathbb{P}$-generic filter: let $p \in \mathbb{P}$ be an atom of $\mathbb{P}$. We define, by comprehension (in M), the set $G_p = \{q \in \mathbb{P} \mid \exists r \in \mathbb{P}(r \leq p \wedge r \leq q)\}$ of the elements of $\mathbb{P}$ which are compatible with p. Clearly G_p is upward closed and moreover if $q, q' \in G_p$ then there exist $r, r' \leq p$ such that $q \geq r$ and $q' \geq r'$. Now, since p is an atom, r and r' are compatible, therefore there exists $s \leq p$ with $s \leq r$ and $s \leq r'$, and, finally, $s \in G_p$ with $s \leq q$ and $s \leq q'$. To see that G_p is $\mathbb{P}$-generic over of M, it is enough to notice that if $D \subseteq P$ is dense, then there exists $r \leq p$ with $r \in D$; but then $r \in G_p$.

2. It is clear that in V, since M is countable and transitive, $\mathbb{P}$ is countable and therefore $\mathcal{P}(\mathbb{P})$ has cardinality $2^{\aleph_0}$. Therefore the set $\mathcal{G}$ of the $\mathbb{P}$-generic filters over of M has cardinality $\leq 2^{\aleph_0}$.

   We now show that $|\mathcal{G}| \geq 2^{\aleph_0}$. To do so, we define an injective application of 2^ω in $\mathcal{G}$. For each finite sequence $s \in 2^{<\omega}$, we define an element p_s of $\mathbb{P}$ such that:

   - $s \subseteq t$ implies $p_s \geq p_t$,
   - $p_{s^\frown 0} \perp p_{s^\frown 1}$,
   - for all $\alpha \in 2^\omega$:
   $$G_\alpha = \left\{q \in \mathbb{P} \,\middle|\, \exists n\ q \geq p_{\alpha\restriction_n}\right\}$$
   is a filter which is $\mathbb{P}$-generic over M.

   Towards this goal we fix (in V) an enumeration (D_n) of the dense subsets of $\mathbb{P}$ which belong to M and then procede by induction on the length of s: choose $p_\emptyset \in D_0$, then for $s \in 2^n$, since p_s is not an atom, there exist $r_0, r_1 \leq p_s$ with $r_0 \perp r_1$. Since D_{n+1} is dense, there exist $p_{s^\frown 0}, p_{s^\frown 1} \in D_{n+1}$ with $p_{s^\frown 0} \leq r_0$ and $p_{s^\frown 1} \leq r_1$. This construction yields the desired application: indeed, the function $\alpha \mapsto G_\alpha$ is injective.

Solution of exercise 4: Consider <u>in **M**</u> the following set:

$$Q = \left\{q \in \mathbb{P} \mid q \Vdash_* \psi(\tau_1, \ldots, \tau_n)\right\}.$$

We show that Q is dense below p. Towards a contradiction, let us assume that there exists some $s \leq p$ such that for all $t \leq s$

$$t \nVdash_* \psi(\tau_1, \ldots, \tau_n).$$

This implies

$$s \Vdash_* \neg\psi(\tau_1, \ldots, \tau_n).$$

Since $s \leq p$ and $p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)$, we also have (by Lemma 306 from the lecture notes):

$$s \Vdash_* \varphi(\tau_1, \ldots, \tau_n)$$

we end up with

$$s \Vdash_* \big(\varphi(\tau_1, \ldots, \tau_n) \wedge \neg\psi(\tau_1, \ldots, \tau_n)\big).$$

(By Lemma 286 from the lecture notes) there exists some filter G $\mathbb{P}$-generic over $\mathbf{M}$ such that $s \in G$. By the Truth Lemma, we have

$$\Big(s \in G \wedge \big(s \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}} \wedge \big(s \Vdash_* \neg\psi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}\Big)$$

implies

$$\mathbf{M}[G] \models \varphi(\tau_1, \ldots, \tau_n) \text{ and } \mathbf{M}[G] \models \neg\psi(\tau_1, \ldots, \tau_n)$$

Now, since

$$\vdash_{c} \forall x_1 \ldots \forall x_n \ \big(\varphi(x_1, \ldots, x_n) \longrightarrow \psi(x_1, \ldots, x_n)\big)$$

we have

$$\models \forall x_1 \ldots \forall x_n \ \big(\varphi(x_1, \ldots, x_n) \longrightarrow \psi(x_1, \ldots, x_n)\big)$$

and in particular

$$\mathbf{M}[G] \models \forall x_1 \ldots \forall x_n \ \big(\varphi(x_1, \ldots, x_n) \longrightarrow \psi(x_1, \ldots, x_n)\big)$$

which yields

$$\mathbf{M}[G] \models \big(\varphi(\tau_1, \ldots, \tau_n) \longrightarrow \psi(\tau_1, \ldots, \tau_n)\big)$$

By *modus ponens* this gives

$$\mathbf{M}[G] \models \psi(\tau_1, \ldots, \tau_n)$$

which yields the following contradiction

$$\mathbf{M}[G] \models \big(\psi(\tau_1, \ldots, \tau_n) \wedge \neg\psi(\tau_1, \ldots, \tau_n)\big).$$

So, we have shown that Q is dense below p, and by Lemma 309 we obtain

$$\big(p \Vdash_* \psi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$