Solution Sheet n°10

Solution of exercise 1:

1. Since $p_2 \perp p_3$ and G is a filter with $p_2 \in G$, necessarily $p_3 \notin G$. Moreover since $p_1 \geq p_2 \in G$, we have also $p_1 \in G$. Thus $(\sigma)_G = \{\emptyset\} = 1$, $(\tau)_G = \{\emptyset\} = 1$ and $(\nu)_G = \{(\tau)_G, (\sigma)_G, \emptyset\} = 2$.

   Similarly, $(\sigma)_{G'} = \{\emptyset\} = 1$, $(\tau)_{G'} = 2$ and $(\nu)_{G'} = \{0, 2\}$.

2. We have $p_2 \Vdash_* \check{1} = \sigma$ if and only if both (a) and (b) below are satisfied:

   (a) for all $(\pi_1,s_1) \in \check{1}$, the following set is dense below p_2:

   $$D_\alpha(\pi_1,s_1,\sigma) = \Big\{q \in \mathfrak{C} \,|\, q \leq s_1 \longrightarrow \exists(\pi_2,s_2) \in \sigma \quad \big(q \leq s_2 \;\wedge\; q \Vdash_* \pi_1 = \pi_2\big)\Big\}$$

   (b) for all $(\pi_2,s_2) \in \sigma$, the following set is dense below p_2:

   $$D_\beta(\pi_2,s_2,\check{1}) = \Big\{q \in \mathfrak{C} \,|\, q \leq s_2 \longrightarrow \exists(\pi_1,s_1) \in \check{1} \quad \big(q \leq s_1 \;\wedge\; q \Vdash_* \pi_2 = \pi_1\big)\Big\}.$$

   Since $\check{1} = \{(\emptyset, \emptyset)\}$ (where $\emptyset$ also denotes the maximal element of $\mathfrak{C}$), and $\sigma = \{(\emptyset, p_2), (\emptyset, p_3)\}$, we have to show that:

   (a) for all $(\pi_1,s_1) \in \check{1}$, the following set is dense below p_2:

   $$\begin{aligned} D_\alpha(\emptyset,\emptyset,\sigma) &= \Big\{q \in \mathfrak{C} \,|\, q \leq \emptyset \longrightarrow \exists(\pi_2,s_2) \in \sigma \quad \big(q \leq s_2 \;\wedge\; q \Vdash_* \emptyset = \pi_2\big)\Big\} \\ &= \Big\{q \in \mathfrak{C} \,|\, \exists(\pi_2,s_2) \in \sigma \quad \big(q \leq s_2 \;\wedge\; q \Vdash_* \emptyset = \pi_2\big)\Big\} \\ &\supseteq \Big\{q \in \mathfrak{C} \,|\, \big((\emptyset, p_2) = (\pi_2,s_2) \in \sigma \;\wedge\; q \leq s_2 \;\wedge\; q \Vdash_* \emptyset = \pi_2\big)\Big\} \\ &= \Big\{q \in \mathfrak{C} \,|\, q \leq p_2 \;\wedge\; q \Vdash_* \emptyset = \emptyset\Big\} \\ &= \Big\{q \in \mathfrak{C} \,|\, q \leq p_2\Big\} \end{aligned}$$

   (which is obvious since $D_\alpha(\emptyset,\emptyset,\sigma)$ contains the whole cone below p_2);

   (b) for all $(\pi_2,s_2) \in \sigma$, the following set is dense below p_2:

   $$D_\beta(\pi_2,s_2,\check{1}) = \Big\{q \in \mathfrak{C} \,|\, q \leq s_2 \longrightarrow \exists(\pi_1,s_1) \in \check{1} \quad \big(q \leq s_1 \;\wedge\; q \Vdash_* \pi_2 = \pi_1\big)\Big\}$$

   which comes down to checking that the following two sets are dense below p_2:

   i.
   $$\begin{aligned} D_\beta(\emptyset, p_2, \check{1}) &= \{q \in \mathfrak{C} \mid q \leq p_2 \to (q \leq \emptyset \wedge q \Vdash \emptyset = \emptyset)\} \\ &= \mathfrak{C} \end{aligned}$$

   ii.
   $$\begin{aligned} D_\beta(\emptyset, p_3, \check{1}) &= \{q \in \mathfrak{C} \mid q \leq p_3 \to (q \leq \emptyset \wedge q \Vdash \emptyset = \emptyset)\} \\ &= \mathfrak{C}. \end{aligned}$$

3. We have, by point (1), that $(\tau)_G = 1$ and $(\tau)_{G'} = 2$. If it were the case that $p_1 \Vdash \check{2} = \tau$, since $p_1 \in G$, by the Truth Lemma $M[G] \models 2 = (\tau)_G$ but this does not hold. Similarly, if $p_1 \Vdash \neg\check{2} = \tau$, then since $p_1 \in G'$ and $M[G'] \models (\tau)_{G'} = 2$ we obtain a contradiction to the Truth Lemma.

Solution of exercise 2:

1. By Lemma 287 seen during class, it is enough to show that for all $p \in \mathfrak{C}$ there exists $q, r \in \mathfrak{C}$ with $q \leq p$, $r \leq p$ and $q \perp r$. Let $p \in \mathfrak{C}$; since $\mathrm{Dom}(p)$ is finite, there exists $n \in \omega \setminus \mathrm{Dom}(p)$. It is therefore enough to take $q, r \in \mathfrak{C}$ defined by $\mathrm{Dom}(q) = \mathrm{Dom}(r) = \mathrm{Dom}(p) \cup \{n\}$ and:
$$q(k) = r(k) = p(k) \quad \text{if } k \in \mathrm{Dom}(p), \quad q(n) = 0, \text{ and } \quad r(n) = 1.$$
2. For each $n \in \omega$, the set $D_n = \{p \in \mathfrak{C} \mid n \in \mathrm{Dom}(p)\} \in M$ is dense. Indeed, the set D_n is definable by comprehension in M (because $\mathfrak{C} \in M$ and $\omega \in M$) and moreover for all $p \in \mathfrak{C}$ if $p \notin D_n$, i.e. $n \notin \mathrm{Dom}(p)$, then $q \in \mathfrak{C}$ defined by $q = p \cup \{(n, 0)\}$ satisfies $q \leq p$ and $q \in D_n$. Since G is $\mathfrak{C}$-generic over M, G intersects each D_n and therefore $\{n \in \omega \mid \exists p \in G \; n \in \mathrm{Dom}(p)\} = \omega$. Moreover each pair $p, q \in G$ is compatible and therefore, if $n \in \mathrm{Dom}(p) \cap \mathrm{Dom}(q)$, then $p(n) = q(n)$. Thus $f = \bigcup : \omega \to 2$.
3. Let $g : \omega \to 2$ with $g \in M$. The set $D_g = \{p \in \mathfrak{C} \mid p \not\subseteq g\}$ is dense and belongs to M. Therefore, there exists $p \in \mathfrak{C}$ with $p \not\subseteq g$ and $p \in G$ and therefore $f_G \neq g$. So the function $f_G : \omega \to 2$ does not belong to M.
4. For each $n \in \omega \subseteq M$, we have that $\check{n}$ is a $\mathfrak{C}$-name and therefore σ is also a $\mathfrak{C}$-name. Moreover,
$$\begin{aligned}(\sigma)_G =& \{(\check{n})_G \mid (\check{n}, p) \in \sigma \wedge p \in G\} \\ =& \{n \mid p(n) = 1 \wedge p \in G\} \\ =& \{n \in \omega \mid f_G(n) = 1\}.\end{aligned}$$
5. If $X = (\sigma)_G = \{n \in \omega \mid f_G(n) = 1\} \in M$, then the following set must belong to M:
$$\{(n, i) \in \omega \times 2 \mid (n \in X \wedge i = 1) \vee (n \notin X \wedge i = 0)\}.$$
But this set is nothing else than the function f_G. So we can conclude by point 3.
6. All finite subsets of ω belong to M since they are definable by comprehension (by $\{k \in \omega \mid k = n_0 \vee k = n_1 \vee \cdots \vee k = n_l\}$). Therefore their complements also belong to M. Since $(\sigma)_G \notin M$, it has to necessarily be infinite and co-infinite.

Solution of exercise 3:

1. Since $f \in {}^{\omega}\omega \cap M[G]$, by absoluteness of ω, $M[G] \models f : \omega \to \omega$, i.e. $M[G] \models \forall x \in \omega \; \exists! y \in \omega \; f(x) = y$. By the Truth Lemma, there exists $p_0 \in G$ with
$$p_0 \Vdash \forall x \in \check{\omega} \; \exists! y \in \check{\omega} \; \underset{\sim}{f}(x) = y.$$
2. Since $p_0 \Vdash \forall x \in \check{\omega} \; \exists! y \in \check{\omega} \; \underset{\sim}{f}(x) = y$ the same holds for every p_k. Thus, for each k we have $p_k \Vdash \exists! y \; \underset{\sim}{f}(\check{k}) = y$. Notice that the set
$$\{q \leq p_k \mid \exists n \in \omega \quad q \Vdash \underset{\sim}{f}(\check{k}) = \check{n}\}$$
is dense below p_k and in particular is non empty. Therefore, g is well defined and since the enumeration $k \mapsto p_k$ belongs to M, the function g also belongs to M.
3. Notice first of all that for all $q \leq p_0$ the set $\{r \in \mathfrak{C} \mid r \leq q\}$ is infinite and therefore for all n there exists $k \geq n$ with $p_k \leq q$. Now let $n \in \omega$ and $q \leq p_0$; we want to find $k \geq n$ and $r \leq q$ with $r \Vdash \underset{\sim}{f}(\check{k}) = \check{g}(\check{k})$. Let $k \geq n$ with $p_k \leq q$. By definition of g, there exists $r \leq p_k \leq q$ with $r \Vdash \underset{\sim}{f}(\check{k}) = \check{g}(\check{k})$ as we wanted.

4. By the previous point for each $n \in \omega$, the following set is dense below p_0:

$$\Big\{q \in \mathfrak{C} \Vdash \exists y \in \check{\omega} \ \big(y \check{\geq} \check{n} \ \wedge \ \underset{\sim}{f}(y) = \check{g}(y)\big)\Big\}.$$

By Lemma 309, we have

$$p_0 \Vdash \exists y \in \check{\omega} \ \big(y \check{\geq} \check{n} \ \wedge \ \underset{\sim}{f}(y) = \check{g}(y)\big),$$

hence,

$$p_0 \Vdash \forall x \in \check{\omega} \, \exists y \in \check{\omega} \ \ \big(y \check{\geq} x \ \wedge \ \underset{\sim}{f}(y) = \check{g}(y)\big).$$

By the Truth Lemma, since $p_0 \in G$, we have that

$$M[G] \models \forall x \in \omega \, \exists y \in \omega \ \ \big(y \geq x \ \wedge \ f(y) = g(y)\big)$$

which we can abbreviate as

$$M[G] \models \forall n \, \exists k \geq n \ f(k) = g(k).$$

Therefore, f does not dominate g.