Exercise Sheet n°9

Exercise 1: Show that if F is a set of formulae of the set theory which is closed under sub-formulae and $\mathbf{M}$ is a class, then the following are equivalent:

1. each $\varphi \in F$ is absolute for $\mathbf{M}$,
2. for each $\varphi \in F$ of the form[1] $\exists x\, \psi(x, \vec{y})$.:

$$\forall \vec{y} \in \mathbf{M}\big(\exists x\, \psi(x, \vec{y}) \leftrightarrow \exists x \in \mathbf{M}\ \psi(x, \vec{y})\big).$$

Exercise 2:

Theorem (Reflexion Theorem)**.** *Let F be a finite set of formulae of the set theory, then:*

1. $ZF \vdash \forall\alpha\, \exists\beta > \alpha$ *"the formulae of F are absolute for V_β".*
2. $ZF \vdash \forall M_0\, \exists M \supseteq M_0$ *"the formulae of F are absolute for M".*

In particular, if $ZF \vdash \varphi$ for all $\varphi \in F$, then:

1. $ZF \vdash \forall\alpha\, \exists\beta > \alpha$ "$V_\beta \models F$".
2. $ZF \vdash \forall M_0\, \exists M \supseteq M_0$ "$M \models F$".

Hint: Without loss of generality, suppose that F is closed under sub-formulae. For each $\varphi(\vec{y}) \in F$ of the form $\exists x\, \psi(x, \vec{y})$ with $\vec{y} = (y_1, \ldots, y_n)$ define $G_\varphi : V^n \to \mathbf{ON}$ by:

$$G_\varphi(y_1, \ldots, y_n) = \begin{cases} 0 & \text{if } \neg\varphi(y_1, \ldots, y_n) \\ \min\{\eta \in \mathbf{ON} \mid \exists x \in V_\eta\ \psi(x, y_1, \ldots, y_n)\} & \text{if } \varphi(y_1, \ldots, y_n). \end{cases}$$

Then define $H_\varphi : \mathbf{ON} \to \mathbf{ON}$ by:

$$H_\varphi(\xi) = \sup\{G_\varphi(y_1, \ldots, y_n) \mid y_1, \ldots, y_n \in V_\xi\}.$$

Finally if $\widetilde{F}$ is the set of formulae in F which are of the form $\exists x\psi$, define β_k by induction:

$$\begin{array}{lcl} \beta_0 & = & \alpha \\ \beta_{n+1} & = & \sup\big(\{\beta_n + 1\} \cup \{H_\varphi(\beta_n) \mid \varphi \in \widetilde{F}\}\big) \end{array}$$

Show that the limit ordinal $\beta = \sup\{\beta_k \mid k \in \omega\}$ is as wished.

[1] more precisely, logically equivalent to a formula of the form $\exists x\, \psi(x, \vec{y})$

Exercise 3: Prove the following theorem:

Theorem. *Let $\varphi_1, \dots, \varphi_n$ be formulae of set theory, then:*

$$ZFC \ \vdash \ \forall M_0 \Big[M_0 \ \textit{transitive} \ \to \exists M \Big(M \supseteq M_0 \ \wedge \ M \ \textit{transitive} \\ \wedge \ |M| \leq \max(\omega, |M_0|) \ \wedge \ \varphi_1, \dots, \varphi_n \ \textit{are absolute for } M \Big) \Big].$$

In particular, for any finite set $\varphi_1, \dots, \varphi_n$ of axioms of ZFC:

$$ZFC \ \vdash \ \forall M_0 \Big[M_0 \ \textit{transitive} \ \to \exists M \big(M \supseteq M_0 \ \wedge \ M \ \textit{transitive} \\ \wedge \ |M| \leq \max(\omega, |M_0|) \ \wedge \ \bigwedge_{i=1}^{n} \varphi_i^M \big) \Big].$$

Hint: Enlarge the list $\varphi_1, \dots, \varphi_n$ in such a way that it becomes closed under sub-formulae and it contains the axiom of extensionality. By the reflexion theorem, there exists an ordinal β such that $V_\beta \supset M_0$ and $\varphi_1, \dots, \varphi_n$ are absolute for V_β. Using the axiom of choice, for each existential formula $\varphi_i(y_1, \dots, y_{l_i}) = \exists x \varphi_j(x, y_1, \dots, y_{l_i})$, define a function $H_i : V_\beta^{l_i} \to V_\beta$ such that: $\varphi_j(H_i(y_1, \dots, y_{l_i}), y_1, \dots, y_{l_i})$ if $\exists x \in V_\beta \ \varphi_j(x, y_1, \dots, y_{l_i})$ and $H_i(y_1, \dots, y_{l_i}) = 0$ if not. Let $\overline{M}$ be the smallest set containing M_0 and closed under all the functions H_i (i.e. $\forall \vec{y} \in \overline{M}^{l_i}$, $H_i(\vec{y}) \in \overline{M}$). Notice that $\varphi_1, \dots, \varphi_n$ are absolute for $\overline{M}$ and that $|\overline{M}| \leq \max(\omega, |M_0|)$. On the other hand, $\overline{M}$ could be not transitive. To overcome this, notice that $(\overline{M}, \in)$ is extensional and consider the Mostowski collapse M of $\overline{M}$. Finally, show that M satisfies all the required properties.

Deduce the following results:

Corollary. *Let T be a theory which extends ZFC and let $\varphi_1, \dots, \varphi_n$ be a finite set of axioms of T, then:*

$$T \ \vdash \ \exists M \Big(M \ \textit{transitive} \ \wedge \ |M| = \omega \ \wedge \ \bigwedge_{i=1}^{n} \varphi_i^M \Big).$$

In particular, for any finite set of axioms $\varphi_1, \dots, \varphi_n$ of ZFC:

$$ZFC \ \vdash \ \exists M \Big(M \ \textit{transitive} \ \wedge \ |M| = \omega \ \wedge \ \bigwedge_{i=1}^{n} \varphi_i^M \Big).$$

Exercise 4: Let $\varphi_1, \dots, \varphi_n$ be a finite set of axioms of ZFC from which it is possible to define the sets ω and ω_1, to prove that $\mathcal{P}(\omega)$ and ω_1 are not countable, and to prove all the "classic" results of absoluteness for the transitive class models of (a great enough finite number of axioms of) $ZF - P$. Show that there exists a countable transitive set M such that $ZFC \vdash$ "$(\mathcal{P}(\omega))^M$ and $(\omega_1)^M$ are countable". Explain this apparent paradox (usually referred to as "Skolem's paradox").