Exercise Sheet n°8

Recall that for any set x, the transitive closure of x is defined by $cl(x) = \bigcup\{\bigcup^n x : n \in \omega\}$, where $\bigcup^0 x = x$ and $\bigcup^{n+1} x = \bigcup(\bigcup^n x)$. The transitive closure of x is the smallest transitive set contenaing x. For every infinite cardinal κ, we let $H_\kappa = \{x : |cl(x)| < \kappa\}$, that is: H_κ is the class of sets which are hereditarily of cardinality $< \kappa$.

1. Show that for every infinite cardinal κ, $H_\kappa \subseteq V_\kappa$, so H_κ is actually a set.

   Hint: For $t = cl(x)$, show that the set of ordinals $S = \{rk(y) : y \in t\}$ is an ordinal. To do so, considerer the smallest ordinal α which does not belong to S, suppose towards contradiction that $S \neq \alpha$ and find a contradiction by considering $y \in t$ with $rk(y) = \min(S \setminus \alpha)$. Show that $\alpha < \kappa$. Conclude by showing that $rk(x) \leq \alpha < \kappa$.

2. Using the axiom of choice, show that if κ is an infinite regular cardinal, then $\forall x(x \in H_\kappa \leftrightarrow x \subseteq H_\kappa \wedge |x| < \kappa)$.

   Hint: Use the property $cl(x) = x \ \cup \ \bigcup\{cl(y) : y \in x\}$.

3. Working in ZFC, show that if κ is an uncountable regular cardinal, then H_κ is a model of $ZFC - P$.

   Hint: Show that H_κ is transitive and use Exercise 2 of Sheet 7. For the axioms of replacement and choice, use point 2. For the axiom of infinity, show that $H_\kappa \cap \mathbf{ON} = \kappa$, and use this property to show that $\omega \in H_\kappa$.

4. Working in ZFC, show that if κ is a regular uncountable cardinal which is not strongly inaccessible, then H_κ is a model of $ZFC - P + \neg P$.

   Hint: Show that "$H_\kappa \models P$" iff $\forall x \in H_\kappa(\mathcal{P}(x) \in H_\kappa)$, and use this property to deduce that "$H_\kappa \models \neg P$" for κ not strongly inaccessible.

   We then have the following relative consistency result:

   Theorem. *$Con(ZFC) \rightarrow Con(ZFC - P + \neg P)$, i.e.*
   $ZFC \vdash$ "$ZFC - P \nvdash P$".

5. Working in ZFC, show that if κ is a strongly inaccessible cardinal then H_κ is a set model of ZFC.

   Hint: Show that for a strongly inaccessible κ, we have $\forall \alpha < \kappa \ (|V_\alpha| < \kappa)$. Deduce that $H_\kappa = V_\kappa$. Conclude that "$H_\kappa \models P$".

   We have thus reproved the following:

   Theorem. *$ZFC + \exists\kappa(\kappa$ strongly inaccessible$) \ \vdash \ Con(ZFC)$.*

6. Deduce the following result that we have also already proved:

   Theorem. *$Con(ZFC) \rightarrow Con(ZFC + \neg\exists\kappa(\kappa$ strongly inaccessible$))$,*
   i.e. $Con(ZFC) \rightarrow$ "$ZFC \nvdash \exists\kappa(\kappa$ strongly inaccessible$)$".