# Exercise Sheet n°7

## Exercise 1:

Prove *some* of the following points:

Let $\mathbf{M}$ be a transitive class model of $ZF - \text{Pow} - \text{Inf}$. Then the following formulae and operations are absolute for $\mathbf{M}$.

*(a)* $\{x, y\}$
*(b)* $\bigcup x$
*(c)* $x \cup y$
*(d)* $\langle x, y \rangle$
*(e)* $\emptyset$
*(f)* $S(x)$
*(g)* $x$ is transitive
*(h)* $\alpha$ is an ordinal
*(i)* $\alpha$ is a limit ordinal
*(j)* $\alpha$ is a successor ordinal
*(k)* $\alpha$ is a finite ordinal
*(l)* $R$ is a binary relation on $A$
*(m)* $f$ is a function from $A$ to $B$
*(n)* $f$ is a bijection between $A$ and $B$

Prove that the notion of cardinality, however, is not absolute for the transitive class models of $ZFC$. More precisely, prove that:

**Proposition.** *Let* $\mathbf{M}$ *be a transitive class model of* $ZFC$. *Then for all* $a \in \mathbf{M}$, $\text{Card}(a) \leq \text{Card}^{\mathbf{M}}(a)$.

## Exercise 2:

Show the following proposition:

**Proposition.** *(i)* $(ZF^-)$ *"*$V_{\omega+\omega} \models ZF - Repl + \neg Repl$*".*

*(ii)* $(ZFC^-)$ *"*$V_{\omega+\omega} \models ZFC - Repl + \neg Repl$*".*

*Hint: in* $V_{\omega+\omega}$, *consider the functional* $\varphi(n, \alpha) = \exists f\ (n \in \omega \wedge Ord(\alpha) \wedge$ *"*$f$ *is an isomorphism between* $\alpha$ *and* $\omega + n$*"). Notice that the image of this functional cannot be a set belonging to* $V_{\omega+\omega}$.

These results directly imply the relative consistency of the replacement axioms relative to the other axioms, that is:

**Theorem.** *(i)* $Con(ZF^-) \rightarrow Con(ZF - Repl + \neg Repl)$,
*i.e.* $Con(ZF^-) \rightarrow$ *"*$ZF - Repl \nvdash Repl$*".*

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## Exercise 3:

Working inside the theory $ZFC + \exists\kappa(\kappa \text{ strongly inaccessible})$, show that if $\kappa$ is a strongly inaccessible cardinal, then "$V_\kappa \models ZFC$".

It follows that, in $ZFC + \exists\kappa(\kappa \text{ strongly inaccessible})$, it is possible to exhibit a set model of $ZFC$, that is:

$$ZFC + \exists\kappa(\kappa \text{ strongly inaccessible}) \vdash Con(ZFC).$$

*Hint: For replacement, show first of all that if $\kappa$ is a strongly inaccessible cardinal, then for all $\alpha < \kappa$, we have $|V_\alpha| < \kappa$. Consider a set $A \in V_\kappa$ and a formula $\varphi(x, y, \vec{z})$ which is functional on $A$ in $V_\kappa$, then prove that the image of $A$ by $f$ belongs to $V_\kappa$, using the previous point.*

Deduce the following theorem:

**Theorem.**

$$Con(ZFC) \text{ implies } Con(ZFC + \neg\exists\kappa \text{ “}\kappa \text{ strongly inaccessible”}).$$

*In particular, if $ZFC$ is consistent, then $ZFC$ cannot prove the existence of a strongly inaccessible cardinal.*

*Hint: use the fact that for all limit ordinals $\lambda$, the formula “$\kappa$ is a strongly inaccessible cardinal” is absolute for $V_\lambda$.*