# Exercise Sheet n°6

## Exercise 1:

The goal of this exercise is to show the independence of the axiom of foundation $AF$ in the system $ZFC_{\text{fin}}^{-} = ZFC_{\text{fin}} \setminus \{AF\}$. More precisely, we show that there exists a model of $ZFC_{\text{fin}}^{-} \cup \{AF\}$ and another model of $ZFC_{\text{fin}}^{-} \cup \{\neg AF\}$.

Let $\varphi$ be a bijection between $\mathbb{N}$ and the set of finite parts of $\mathbb{N}$, and let $\in_\varphi \subseteq \mathbb{N} \times \mathbb{N}$ be the binary relation defined by $x \in_\varphi y$ iff $x \in \varphi(y)$.

1. Show (briefly) that the structure $U_\varphi = (\mathbb{N}, \in_\varphi)$ is a model of $ZFC_{\text{fin}}^{-}$.
2. Show that, if for all $p, q \in \mathbb{N}$, we have that $p \in_\varphi q$ implies $p < q$ (the usual order on $\mathbb{N}$), then $U_\varphi$ is a model of $ZFC_{\text{fin}}^{-} \cup \{AF\}$.
3. Exhibit a bijection $\varphi$ such that $U_\varphi$ is a model of $ZFC_{\text{fin}}^{-} \cup \{\neg AF\}$.

## Exercise 2:

1. Find a non transitive class $N$ such that "$N \not\models$ axiom of extensionality".
2. Show the following proposition:

**Proposition.** *Let $M$ be a transitive class, then:*

*(1) "$M \models$ axiom of extensionality";*

*(2) If $M \subseteq$ **WF**, then "$M \models$ axiom of foundation";*

*(3) "$M \models$ pairing axiom" iff $\forall a, b \in M\ \exists c \in M \quad \{a, b\} \subseteq c$;*

*(4) "$M \models$ union axiom" iff $\forall a \in M\ \exists b \in M \quad \bigcup a \subseteq b$;*

*(5) "$M \models$ powerset axiom" iff $\forall a \in M\ \exists c \in M \quad \mathcal{P}(a) \cap M \subseteq c$;*

*(6) "$M \models$ comprehension axioms" iff for every formula $\varphi := \varphi(x, z, \overline{w})$ with free variables among $x, z, w_1, ..., w_n$, we have:*

$$\forall z \in M\ \forall w_1 \in M\ ...\forall w_n \in M \quad \{x \in z : \varphi(x, z, \overline{w})^M\} \in M$$

*;*

*(7) "$M \models$ replacement axioms" iff for every formula $\varphi(x, y, w_1, \dots, w_n)$ with free variables among $x, y, w_1, \dots, w_n$, we have:*

$$\forall c_1 \in M\ \dots \forall c_n \in M\ \forall A \in M \left[\forall x \in A\ \exists! y \in M \quad \varphi^M(x, y, c_1, \dots, c_n) \rightarrow \exists B \in M(\{y : \exists x \in A\ \varphi^M(x, y, c_1, \dots, c_n)\} \subseteq B)\right].$$

## Exercise 3:

Denote by $ZF^-$ the axiomatic system $ZF$ without the axiom of foundation.

1. Working in $ZF^-$, show that "$V_\omega \models ZF - Inf + \neg Inf$".

   *Hint: Use exercise 2 to show that $V_\omega$ satisfies $Z - Inf$. For the axiom of infinity, suppose towards contradiction that $V_\omega$ satisfies it, and show that this implies that $\omega \in V_\omega$, from which a contradiction follows.*

2. Working in $ZF^-$, show that "$\mathbf{WF} \models ZF$".

   *Hint: Notice that $\omega$ contains $0$, is closed under successor and belongs to $\mathbf{WF}$.*

3. Working in $ZFC$, show that "$V_\lambda \models ZC$", for all $\lambda > \omega$ limite.

   *Hint: Use the fact that the formula «$R$ is a well order on $a$» is absolute for the transitive models of $ZF$-$P$.*

Points 1 and 2 above lead to the following consistency results:

**Theorem.**

*(1)* $ZF^- \vdash Con(ZF^-) \rightarrow Con(ZF - Inf + \neg Inf)$;

*(2)* $ZF^- \vdash Con(ZF^-) \leftrightarrow Con(ZF)$.