Exercise Sheet n°5

Exercise 1:

1. Let $x \in \mathbf{WF}$. Show that $rk(x)$ is the least $\alpha \in \mathbf{ON}$ such that $x \subseteq \mathbf{W}(\alpha)$.
2. Let $x,y \in \mathbf{WF}$, $\alpha \in \mathbf{ON}$, $f \in {}^{x}y$ and R an ordering on $\mathbf{W}(\alpha)$. For each of the following sets z, show that $z \in \mathbf{WF}$ and compute or give an upper bound for $rk(z)$ as a function of $rk(x)$ and $rk(y)$.

   (a) $\{x,y\}$;
   (b) (x,y);
   (c) $\langle x,y\rangle$;
   (d) $\langle x,y,x\rangle$;
   (e) $\bigcup x$;
   (f) $\mathcal{P}(x)$;
   (g) $x \cup y$;
   (h) $x \times y$;
   (i) f;
   (j) ${}^{x}y$;
   (k) $\mathbf{W}(\alpha)$;
   (l) R.

3. Find $x,y \in \mathbf{WF}$ such that $rk\big(\bigcup x\big) = rk(x)$ but $rk\big(\bigcup y\big) < rk(y)$.
4. In the Exercice Sheet 2, we constructed a set-theoretical representation of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.
   (a) Show that all these sets, as well as their elements, are in $\mathbf{WF}$;
   (b) Compute the rank of each of these sets as well as their elements.
5. Can you think of *another* set-theoretical representation of $\mathbb{Q}$ such that $rk\big(\mathbb{Q}\big) = \omega$? What about *another* set-theoretical representation of $\mathbb{R}$ such that $rk\big(\mathbb{R}\big) = \omega + 1$? And *another* set-theoretical representation such that $rk\big(\mathbb{R}\big) = \omega$?

Exercise 2: By a Remark in the Lecture Notes, we can represent any $x \in \mathbf{WF}$ by a well-founded tree $T_x \subseteq A^{<\omega}$, where A is a set.

1. Define the well-founded tree $T_{\bigcup x}$ that represents the set $\bigcup x$.
2. Define the well-founded tree $T_{tc(x)}$ that represents the set $tc(x)$.

Exercise 3:

1. Show that the (class-)function $\aleph : \alpha \mapsto \aleph_\alpha$ admits a fixed point, i.e., there exists $\kappa \in \mathbf{WF}$ such that $\kappa = \aleph_\kappa$.
2. Show that any strong inaccessible cardinal κ is a fixed point of $\aleph : \alpha \mapsto \aleph_\alpha$.

Exercise 4: Let $\mathbf{C}$ be a class characterized by a formula $\varphi_{\mathbf{C}}$. For each of the following formulas, give its corresponding relativization to $\mathbf{C}$.

1. $\exists x\ x = x$;
2. $\forall x\ \exists y\ x \in y$;
3. $\forall x\ \forall y\big(\forall z(z \in x \leftrightarrow z \in y) \to x = y\big)$;
4. $\forall x\ \exists y\ \forall z\big(\forall u(u \in z \to u \in x) \to z \in y\big)$.

Exercise 5:

Show that the axiom of choice is equivalent to the following statement:

Axiom of Choice (The product of non empty sets is non empty)**.** Let $(A_i)_{i\in I}$ be a collection of sets indexed by a set I, such that $A_i \neq \emptyset$ for all $i \in I$. Then the cartesian product $\prod_{i\in I} A_i$ is non empty.

We recall the following result in topology.

Theorem (Tychonoff's Theorem)**.** *Let $(X_i)_{i\in I}$ be a collection of topological spaces indexed by a set I such that X_i is compact for all $i \in I$. Then the product space $\prod_{i\in I} X_i$ is compact.*

The proof of Tychonoff's theorem uses the axiom of choice. The goal of this exercise is to show that (AC) is necessary. Let $(A_i)_{i\in I}$ be a collection of sets indexed by a set I such that $A_i \neq \emptyset$ for all $i \in I$. Let α be a set which does not belong to $\bigcup A_i$ (why is it possible?). We define, for each $i \in I$, the set $X_i = A_i \cup \{\alpha\}$ which we endow with the topology $\tau_i = \{\emptyset, \{\alpha\}, X_i\}$.

1. Show that for each $i \in I$, (X_i, τ_i) is a compact topological space.
2. Show that for each $i \in I$ the set
$$C_i = \left\{ (x_j)_{j\in I} \in \prod_{j\in I} X_j \;\middle|\; x_i \in A_i \right\}$$
is a non empty closed subset of the product space $\prod_{j\in I} X_j$.
3. Use Tychonoff's theorem to show that $\bigcap_{i\in I} C_i$ is non empty.
4. Conclude that Tychonoff's theorem implies the axiom of choice.