Exercise Sheet n°4

Exercise 1: Let X be a set, κ an infinite cardinal and $\mathcal{O}$ a set of operations on X of arity κ, i.e. a set of functions $f : X^\kappa \to X$. A subset C of X is called *closed* under the operations in $\mathcal{O}$ if for all $f \in \mathcal{O}$ and all $(x_\alpha)_{\alpha<\kappa}$ in C^κ we have $f((x_\alpha)_{\alpha<\kappa}) \in C$.

For $E \subseteq X$, the *closure of* E *relative to* $\mathcal{O}$, in symbols $\mathrm{cl}_{\mathcal{O}}(E)$, is the smallest subset of X which is closed under the operations in $\mathcal{O}$ and contains E.

For $E \subseteq X$, we define the following sets by transfinite recursion:

$$\begin{aligned} E_0 &= E; \\ E_{\alpha+1} &= E_\alpha \cup \{f((x_\xi)_{\xi<\kappa}) \mid f \in \mathcal{O} \text{ and } (x_\xi)_{\xi<\kappa} \in (E_\alpha)^\kappa\}; \\ E_\lambda &= \bigcup_{\zeta<\lambda} E_\zeta \quad \text{if } \lambda \text{ is limit.} \end{aligned}$$

1. Show by induction that $E_\alpha \subseteq \mathrm{cl}_{\mathcal{O}}(E)$ for all ordinals α.
2. Show that E_{κ^+} is closed under the operations in $\mathcal{O}$.
3. Conclude that
$$\mathrm{cl}_{\mathcal{O}}(E) = \bigcup_{\alpha<\kappa^+} E_\alpha.$$

Here are some applications.

4. Show that the set of open subsets of $\mathbb{R}$ has cardinality $2^{\aleph_0}$.
5. Show that the set of Borel subsets of $\mathbb{R}$ has cardinality $2^{\aleph_0}$.

   Hint: The Borel subsets are the smallest family of sets of reals containing the open sets and closed under countable intersection and complementation.

Let us shift our attention towards functions. We recall that $f : \mathbb{R} \to \mathbb{R}$ is the pointwise limit of a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ if for all $x \in \mathbb{R}$ we have $f(x) = \lim_{n\to\infty} f_n(x)$.

6. Compute the cardinality of the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is Borel if for all open subsets U of $\mathbb{R}$, $f^{-1}(U)$ is a Borel set. We denote by $\mathfrak{B}$ the closure of the set of continuous functions under pointwise limit.

7. Show that the set of Borel functions is closed under pointwise limit.

It is actually also possible to prove the converse: the set of Borel functions is exactly $\mathfrak{B}$. We will not prove it here, but we will use it for the following point.

8. Compute the cardinality of the set of Borel functions.

Exercise 2:

The measure problem was presented for the first time by Henri Lebesgue in 1902. He posed the following question:

> Does there exist a function m which associates to each bounded set of real numbers X a non-negative real number $m(X)$ such that the following conditions hold?
>
> **i)** m is not always 0;
>
> **ii)** m is invariant under translation, i.e. for every bounded set of real numbers X and every real number r:
>
> $$m(X) = m(\{x + r \mid x \in X\});$$
>
> **iii)** m is σ-additive, i.e. for every countable family $\langle X_n \mid n \in \omega \rangle$, if each X_n is a bounded set of real numbers, the X_n are pairwise disjoints, and $\bigcup_{n\in\omega} X_n$ is a bounded set of real numbers, then
>
> $$m(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} m(X_n).$$

Show the following points:

1.1 If m is a solution to Lebesgue's measure problem, then for all bounded set of real numbers X and all $Y \subseteq X$, $m(Y) \leq m(X)$.

1.2 Every solution to Lebesgue's measure problem is determined by its values on the subsets of the unit interval $[0, 1]$.

We now show the following result which constitutes the first historic use of the axiom of choice to construct a set of real numbers, after its formulation by Ernst Zermelo[1].

Theorem (Giuseppe Vitali, 1907)**.** *Assuming the axiom of choice, there does not exists a function solving the measure problem of Lebsegue.*

Suppose towards contradiction that there exists a function m which is a solution to Lebesgue's measure problem. We define the following equivalence relation on the real numbers. For x and y real numbers:

$$x \sim y \quad \text{iff} \quad x - y \text{ is rational.}$$

2.1 Notice that, for every real number x, the equivalence class of x is the orbit of x under the action of the rational numbers on the real numbers by addition, i.e. the class $[x]$ of x is $\{x + r \mid r \text{ is rational number}\}$.

2.2 Using the axiom of choice, define a set $\mathcal{V}$ containing, for each real number x, exactly an element of the intersection of the class of x with the interval $[0, 1]$. We call this Vitali's set.

2.3 Show that for all distinct rational numbers r and s, the translates $\mathcal{V}+r = \{x+r \mid x \in \mathcal{V}\}$ and $\mathcal{V} + s$ of Vitali's set are disjoint.

2.4 Show that

$$m([0, 1]) \leq m\left(\bigcup_{r\in\mathbb{Q}\cap[-1,1]} \mathcal{V} + r \right) \leq m([-1, 2]).$$

2.5 Show that $m(\mathcal{V})$ cannot be strictly positive.

2.6 Show that $m(\mathcal{V})$ cannot be zero.

[1] Akihiro KANAMORI: The higher infinite. Large cardinals in set theory from their beginnings. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1994, p. 22.

«*For Lebesgue, Vitali's construction raised doubts not so much about the possibilities for a measure but about AC.*»[2]

Notice that the three conditions imposed by Lebesgue's problem are necessary to prove the previous theorem.

In 1930, Stefan Banach proposed a generalisation of the measure problem posed by Lebesgue which got rid of all the geometrical considerations:

> Does there exist a non empty set S and a function $m : \mathcal{P}(S) \to [0, 1]$ such that the following properties hold?
>
> **a)** $m(S) = 1$;
>
> **b)** $m(\{x\}) = 0$ for all $x \in S$;
>
> **c)** m is σ-additive, i.e. for every countable family $\langle X_n \mid n \in \omega\rangle$ of pairwise disjoint subsets of S:
> $$m(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} m(X_n).$$

We call a solution to the measure problem of Banach a *measure* on S. The condition **b)** takes the place of the condition **ii)** in Lebesgue's problem. It takes care of excluding trivial solutions:

3.1 Notice that for all $x \in S$ the function m_x defined, for all $E \subseteq S$, by $m_x(E) = 1$ if $x \in E$ and $m_x(E) = 0$ if not, satisfies the points **a)** and **c)** but does not satisfy **b)**.

The following is a characteristic property of measures.

4.1 Let m be a measure on a set S. Show that every family $T \subseteq \mathcal{P}(S)$ of pairwise disjoint sets contains at most a countable number of sets of positve measure.

Hint: Argue by contradiction and consider the fact that $T = \bigcup_{n\in\omega} T_n$*, where* $T_n = \{X \in T \mid m(X) > \frac{1}{n}\}$*. Use the axiom of countable choice.*

Banach noticed that only the cardinality of the set S is relevant with regards to its measure problem and that it is therefore reasonable to generalise the property **iii)** in the following way.

For a cardinal λ, a measure m on a set S is λ-*additive* if for every ordinal $\gamma < \lambda$ and every family $\langle X_\alpha \mid \alpha < \gamma\rangle$ of pairwise disjoint subsets of S:

$$m(\bigcup_{\alpha<\gamma} X_\alpha) = \sum_{\alpha<\gamma} m(X_\alpha),$$

where the member on the right is by definition the supremum of the sums of finite sub-collections, i.e.

$$\sum_{\alpha<\gamma} m(X_\alpha) = \sup\left\{ \sum_{k<n} m(X_{f(k)}) \;\middle|\; n \in \omega \text{ and } f \in {}^{n}\gamma \right\}.$$

Notice that, in these terms, σ-additivity is $\aleph_1$-additivity.

Moreover we have the following fact (the proof of which is facultative):

Lemma. *If κ is the smallest cardinal such that there exists a measure on κ, then every measure on κ is κ-additive.*

This naturally leads us to the following definition.

Definition. *A cardinal $\kappa > \omega$ is* real-valued measurable *if there exists a κ-additive measure on κ.*

[2] KANAMORI, op. cit., p. 22.

Let κ be a real-valued measurable cardinal and m a κ-additive measure on κ.

5.1 Show that for all $X \subseteq \kappa$ with $|X| < \kappa$, we have $m(X) = 0$;

5.2 Show that κ is a regular cardinal.

In 1930, Stanisław M. Ulam noticed the importance of a specific kind of measures on cardinals, those which have an atom. For a measure m on a cardinal κ, a subset $A \subseteq \kappa$ is called an *atom* if $m(A) > 0$ and for all $B \subseteq A$ either $m(B) = m(A)$ or $m(B) = 0$.

We have the following theoremes (which we do not prove here[3]).

Theorem (Ulam, 1930)**.** *If κ is real-valued measurable, then κ is weakly inaccessible.*

Theorem (Ulam, 1930)**.** *If there exists a κ-additive measure without atoms on κ, then $\kappa \leq 2^{\aleph_0}$.*

The existence of a real-valued measurable cardinal with an atomless measure thus strongly contradicts the continuum hypothesis. Why?

Let κ be a cardinal and m be a κ-additive measure with an atom $A \subseteq \kappa$.

6.1 Show that the function μ defined by

$$\mu(X) = \frac{m(X \cap A)}{m(A)}, \text{ for all } X \subseteq \kappa,$$

is a κ-additive measure on κ with values in $\{0, 1\}$.

6.2 Show that the collection $U_\mu = \{X \subseteq \kappa \mid \mu(X) = 1\}$ is a non-principal ultrafilter on κ, i.e.

- $\kappa \in U_\mu$ and $\emptyset \notin U_\mu$;
- if $X \in U_\mu$ and $Y \subseteq \kappa$ with $X \subseteq Y$, then $Y \in U_\mu$;
- if $X \in U_\mu$ and $Y \in U_\mu$, then $X \cap Y \in U_\mu$;
- for all $X \subseteq \kappa$, $X \in U_\mu$ or $\kappa \setminus X \in U_\mu$;
- for all $\alpha \in \kappa$, $\{\alpha\} \notin U_\mu$.

Show that moreover, U_μ is *κ-complete*, i.e. that for $\gamma < \kappa$ and every family $\langle X_\alpha \mid \alpha < \gamma \rangle$, if for all $\alpha \in \gamma$ $X_\alpha \in U_\mu$, then $\bigcap_{\alpha<\gamma} X_\alpha \in U_\mu$.

This in turn leads us to one of the most important definitions of the *theory of large cardinals*.

Definition. *A cardinal $\kappa > \omega$ is* measurable *if there exists a non-principal κ-complete ultrafilter on κ.*

7.1 Show the following theorem:

Theorem (Ulam-Tarski, 1930)**.** *Any measurable cardinal κ is strongly inaccessible.*

Hint: Suppose towards contradiction that there exists $\lambda < \kappa$ and an injective function $f : \kappa \to {}^{\lambda}2$. Then consider the family composed, for $\alpha < \lambda$, of the sets $X_\alpha = \{\xi \in \kappa \mid f(\xi)(\alpha) = i_\alpha\}$, where $i_\alpha \in \{0, 1\}$ is such that $X_\alpha \in U$.

[3]for the proofs, see: Akihiro KANAMORI, op. cit., p. 24.