Exercise Sheet n°3

Exercise 1: For each of the following ordinals α, describe a well-ordering on the set $\mathbb{N}$ of natural numbers of order-type α:

1. ω;
2. $\omega + 1$;
3. $\omega + \omega$;
4. $\omega + \omega + 17$;
5. $\omega \cdot 3$;
6. ω^2;
7. $\omega^2 + \omega \cdot 2 + 17$.

Exercise 2:

Compute the following expressions:

1. $3 + \omega$
2. $\omega + 3$
3. $\omega + 15 + \omega + 9 + 3 + \omega$
4. $\omega \cdot 3$
5. $3 \cdot \omega$
6. $(\omega \cdot 3) \cdot (\omega \cdot 5)$
7. $\omega^2 \cdot \omega$
8. $\omega \cdot \omega^2$
9. $(\omega + 3) \cdot 4$
10. $4 \cdot (\omega + 3)$
11. $(\omega + 3) \cdot \omega$
12. $\omega \cdot (\omega + 3)$
13. $10 \cdot \omega \cdot 7 \cdot 3 \cdot \omega$
14. $\omega^3 \cdot \omega^2 \cdot 9 \cdot \omega + 7 \cdot \omega^4 + 3 \cdot (\omega + 2)$
15. $2 \cdot \omega^3 \cdot 3 + \omega^6 + (\omega + 3) \cdot 12$

Exercise 3:

Show that the definition of the ordinal addition by transfinite recursion is equivalent to the following definition in terms of order-type: we write .

Definition (Ordinal addition)**.** *Let α, β be ordinals, then $\alpha + \beta$ is the unique ordinal γ which is isomorphic to the well order $(\alpha \times \{0\} \cup \beta \times \{1\}, <)$ where $(\gamma, i) < (\eta, j)$ iff $i < j$ or $i = j$ and $\gamma < \eta$.*

Exercise 4:

Let $\alpha, \beta, \gamma, \xi$ be ordinals, show the following properties by induction:

1. If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.
2. If $\alpha < \beta$, then there exists a unique ordinal δ such that $\alpha + \delta = \beta$.
3. If $\alpha \neq 0$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.
4. Euclidian division: if α is an ordinal and $\xi > 0$, then there exist two unique ordinals θ and $\rho < \xi$ such that $\alpha = \xi \cdot \theta + \rho$.
5. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
6. If $\alpha > 1$ and $\beta < \gamma$, then $\alpha^\beta < \alpha^\gamma$.
7. $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.
8. $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.

Hints: the points (1), (3), (5), (6), (7) and (8) can be proved by transfinite induction on γ. For (2), consider the smallest ordinal δ' such that $\alpha + \delta' > \beta$, show that δ' is a successor, and consider $\delta = pred(\delta')$. For (4), consider the smallest ordinal θ' such that $\xi \cdot \theta' > \alpha$, show that θ' is a successor, take $\theta = pred(\theta')$ and use (2). Moreover, if B is a non empty set of ordinals, one can use the following properties:

$$\alpha + \sup\{\beta : \beta \in B\} = \sup\{\alpha + \beta : \beta \in B\}$$

$$\alpha \cdot \sup\{\beta : \beta \in B\} = \sup\{\alpha \cdot \beta : \beta \in B\}$$

$$\alpha^{\sup\{\beta : \beta \in B\}} = \sup\{\alpha^\beta : \beta \in B\}$$

Exercise 5:

Prove the following:

Proposition (Cantor's Normal Form of base ω)**.** *Each ordinal $\alpha > 0$ can be written in a unique way in the form:*

$$\alpha = \omega^{\beta_1} \cdot n_1 + \ldots + \omega^{\beta_k} \cdot n_k,$$

with $k > 0$, $\alpha > \beta_1 > \beta_2 > \ldots > \beta_k \geq 0$, and $n_1, \ldots, n_k$ strictly positive natural numbers. This is called Cantor's normal form of base ω *of the ordinal α.*

Hint: Procede by induction on α and use the euclidian division for ordinals.

We can remark that this proposition is still true if we replace ω by whichever other ordinal ξ; each ordinal $\alpha > 0$ can be written in a unique way in the form:

$$\alpha = \xi^{\beta_1} \cdot \gamma_1 + \ldots + \xi^{\beta_k} \cdot \gamma_k,$$

with $k > 0$, $\alpha \geq \beta_1 > \beta_2 > \ldots > \beta_k \geq 0$, and $\gamma_1, \ldots, \gamma_k$ strictly between 0 and ξ. This is called *Cantor's normal form of base ξ* of the ordinal α.