Exercise Sheet n°2

Exercise 1:

The goal of this exercise is to construct, in the theory of Zermelo-Fraenkel (ZF), a set theoretical representation of the structures $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.

1. We call *natural number* an ordinal α such that for all ordinals $\beta \leq \alpha$ either $\beta = 0$ or β is a successor. Show that the successor of a natural number is itself a natural number and that the elements of a natural number are natural numbers.

   Show that the set $\mathbb{N}$ of natural numbers exists. Which axioms of ZF are being used?

   Hint: Show that the set whose existence is assured by the axiom of infinity contains all natural numbers.

2. Check that $(\mathbb{N}, 0, s)$ so defined verifies the principe of recursion, then define the addition and the multiplication between natural number by recursion.

3. To construct the set of integers $\mathbb{Z}$, the idea is to represent the difference $n - m$ by a couple of integers (n, m). We then define the following relation on $\mathbb{N} \times \mathbb{N}$:

   $$(n, m) \sim (n', m') \text{ iff } n + m' = n' + m.$$

   Show that it is an equivalence relation, then show the existence of the quotient set, that is, $\mathbb{Z}$. Show that the addition component by component $\mathbb{N}^2$ is compatible with the relation $\sim$ and then that $\mathbb{Z}$ endowed with this operation is a group.

4. Draw the inspiration from the previous construction to construct the field $\mathbb{Q}$ of the rational numbers.

5. Knowing that each real number can be defined as the set of rational numbers which are strictly less then itself (construction by Dedekind cuts), show the existence of the set of real numbers $\mathbb{R}$.

Exercise 2:

Given a set A, we recall that $\bigcup A$ denotes the set of elements of the elements of A, that is to say $x \in \bigcup A$ iff there exists $z \in A$ such that $x \in z$. We then call *choice function on* A a map $f : A \setminus \{\emptyset\} \longrightarrow \bigcup A$ such that $f(x) \in x$ for all $x \in A \setminus \{\emptyset\}$. We can state different types of choice axioms:

Axiom of choice (AC): each set E admits a choice function.

Axiom of dependant choice (DC): if R is a binary relation on E such that $\forall x \in E\ \exists y \in E\ \langle x, y \rangle \in R$, then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of E that satisfy $\langle x_n, x_{n+1} \rangle \in R$ for all $n \in \mathbb{N}$.

Axiom of countable choice (CC): each countable set E admits a choice function.

1. Show (by induction) that all finite sets admit a choice function.
2. Show that **AC** implies **DC**, and that **DC** implies **CC**.

Hint: For the first implication, consider a choice function on the power set $\mathcal{P}(E)$. For the second, consider an enumeration $\{e_0, e_1, e_2, \ldots\}$ of E, and the binary relation R on the set of choice functions on all the finite subsets of the form $\{e_0, e_1, \ldots, e_n\}$ defined by: $\langle f_i, f_j \rangle \in R$ iff $dom(f_j) \supsetneq dom(f_i)$ and $f_j\restriction_{dom(f_i)} = f_i$.