Exercise Sheet n°13

Exercise 1: Starting from any transitive model $\mathbf{M}$ of **ZFA**, any countable infinite set of atoms $\mathbb{A}$, $\mathscr{G}$ the group of permutations of $\mathbb{A}$, and $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ be the filter generated[1] by

$$\{fix_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\mathbb{A})\}$$

where[2]

$$fix_{\mathscr{G}}(F) = \{\pi \in \mathscr{G} \mid \forall \mathbb{a} \in F \ \breve{\pi}(\mathbb{a}) = \mathbb{a}\}.$$

The submodel of $\mathbf{M}$ of all its hereditarily symmetric sets is the permutation model known as the basic Fraenkel Model: $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} = \mathbf{M} \cap \mathbf{HS}_{\mathscr{F}}$.

1. Show that $\mathscr{F}$ is a normal filter. Namely,
   (a) $\mathcal{G} \in \mathscr{F}$,
   (b) if $\mathcal{H} \in \mathscr{F}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $\mathcal{K} \in \mathscr{F}$,
   (c) if $\mathcal{H} \in \mathscr{F}$ and $\mathcal{K} \in \mathscr{F}$, then $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$,
   (d) if $\mathcal{H} \in \mathscr{F}$, then $\pi \circ \mathcal{H} \circ \pi^{-1} \in \mathscr{F}$,
   (e) for each atom $\mathbb{a} \in \mathbb{A}$, $\{\pi \in \mathcal{G} \mid \pi(\mathbb{a}) = \mathbb{a}\} \in \mathscr{F}$.

2. We recall that for any $x \in \mathbf{M}$, $sym_{\mathscr{G}}(x) = \{\pi \in \mathcal{G} \mid \breve{\pi}(x) = x\}$.
   With same notation as in the previous exercise, show that for any $F \in \mathcal{P}_{fin}(\mathbb{A})$ and any $S \subseteq \mathbb{A}$, if $fix_{\mathscr{G}}(F) \subseteq sym_{\mathscr{G}}(S)$, then S is either finite or co-finite and
   - if S is finite, then $S \subseteq F$;
   - if S is co-finite, then $(\mathbb{A} \setminus S) \subseteq F$.

   (Hint: distinguish between $S \cap (\mathbb{A} \setminus F) = \emptyset$ and $S \cap (\mathbb{A} \setminus F) \neq \emptyset$.)

3. Show that inside the basic Fraenkel model, the set of atoms is Dedekind-finite. Namely,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \aleph_0 \overset{1-1}{\not\hookrightarrow} \mathbb{A}.$$

   (Hint: assume there exists $f : \aleph_0 \xrightarrow{1-1} \mathbb{A}$ and consider the set $S = \{f(2n) \in \mathbb{A} \mid n \in \omega\}$.)

[1] This means $\mathcal{H} \in \mathscr{F}$ if and only if there exist $F_0, \ldots, F_n \in \mathcal{P}_{fin}(\mathbb{A})$ such that

$$\mathcal{H} \supseteq \bigcap_{i \leq n} fix_{\mathscr{G}}(F_i).$$

[2] We recall that given any permutation $\pi : \mathbb{A} \xleftrightarrow{bij.} \mathbb{A}$, the functional $\breve{\pi} : \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A}) \to \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})$ is defined as: $\breve{\pi}(\emptyset) = \emptyset$; if $\mathbb{a} \in \mathbb{A}$, then $\breve{\pi}(\mathbb{a}) = \pi(\mathbb{a})$; if $x \notin \mathbb{A} \cup \{\emptyset\}$, then $\breve{\pi}(x) = \{\breve{\pi}(y) \mid y \in x\}$.

4. Show that the basic Fraenkel model satisfies

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \aleph_0 \overset{1-1}{\not\lesssim} \mathcal{P}(\mathbb{A}).$$

(Hint: assume there exists $f : \aleph_0 \xrightarrow{1-1} \mathcal{P}(\mathbb{A})$ such $sym_{\mathscr{G}}(f)$ belongs to $\mathscr{F}$ in order to get a contradiction.)

Exercise 2: Starting from any transitive model $\mathbf{M}$ of **ZFA** whose set of atoms consists in a countable set $\mathbb{A}$ equipped with a binary relation $<_{\mathbf{M}} \subseteq \mathbb{A} \times \mathbb{A}$ which is a dense ordering without least nor greatest element. i.e., $(\mathbb{A}, <_{\mathbf{M}})$ is isomorphic to $(\mathbb{Q}, <)$.
We let $\mathscr{G}$ be the group of all order preserving permutations of $\mathbb{A}$. i.e.,

$$\mathscr{G} = \left\{\pi : \mathbb{A} \xrightarrow{bij.} \mathbb{A} \mid \forall \mathbb{a} \in \mathbb{A}\, \forall \mathbb{b} \in \mathbb{A}\, \left(\mathbb{a} <_{\mathbf{M}} \mathbb{b} \longleftrightarrow \pi(\mathbb{a}) <_{\mathbf{M}} \pi(\mathbb{b})\right)\right\}.$$

Let $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ be the filter generated by $\left\{fix_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\mathbb{A})\right\}$, which can be proved to be normal. The *ordered Mostowski model* $\boldsymbol{\mathcal{M}}$ is the corresponding permutation model.

1. Show that the set $<_{\mathbf{M}} = \left\{(\mathbb{a}, \mathbb{b}) \in \mathbb{A} \times \mathbb{A} \mid \mathbb{a} <_{\mathbf{M}} \mathbb{b}\right\}$ belongs to $\boldsymbol{\mathcal{M}}$.

2. For any set y, we call **support** of y any $F_y \in \mathcal{P}_{fin}(\mathbb{A})$ which satisfies $fix_{\mathscr{G}}(F_y) \subseteq sym_{\mathscr{G}}(y)$.

   Notice that if F_y is a support of y and $F_y \subseteq F \in \mathcal{P}_{fin}(\mathbb{A})$ holds, then $fix_{\mathscr{G}}(F) \subseteq fix_{\mathscr{G}}(F_y) \subseteq sym_{\mathscr{G}}(y)$ holds as well, so that F is also a support of y.

   (a) Show that if F and F' are two supports of y, then $F \cap F'$ is also a support of y.

   (Hint: notice that given a permutation $\pi \in fix_{\mathscr{G}}(F \cap F')$ there exists permutations $\rho_1, \ldots, \rho_k \in fix_{\mathscr{G}}(F)$ and $\rho'_1, \ldots, \rho'_k \in fix_{\mathscr{G}}(F')$ — for some k large enough — such that $\rho_1 \circ \rho'_1 \circ \rho_2 \circ \rho'_2 \circ \ldots \rho_k \circ \rho'_k = \pi$.)

   (b) Show that for each set $x \in \boldsymbol{\mathcal{M}}$, there exists some $\subseteq$-least support of x.

   (c) Show that the following class is symmetric:

   $$\left\{(x, E) \in \boldsymbol{\mathcal{M}} \times \mathcal{P}_{fin}(\mathbb{A}) \mid E \text{ is the } \subseteq\text{-least support of } x\right\}.$$

   (Hint: for any $\pi \in \mathscr{G}$, look for a support of $\breve{\pi}\big(x, E\big) = \big(\breve{\pi}(x), \breve{\pi}(E)\big)$.)

3. Show that for all $F \in \mathcal{P}_{fin}(\mathbb{A})$, if F has n elements, then there exist exactly 2^{2n+1} sets of the form $S \subseteq \mathbb{A}$ such that F is a support of S.

   (Hint: Assume $F = \{\mathbb{a}_1, \ldots, \mathbb{a}_n\}$ with $\mathbb{a}_1 <_{\mathbf{M}} \ldots <_{\mathbf{M}} \mathbb{a}_n$ and show first that for every integer $1 \leq i < n$ each interval $]\mathbb{a}_i, \mathbb{a}_{i+1}[$ satisfies either $]\mathbb{a}_i, \mathbb{a}_{i+1}[\subseteq S$ or $]\mathbb{a}_i, \mathbb{a}_{i+1}[\cap S = \emptyset$. Show also that the same property holds also for $]-\infty, \mathbb{a}_1[$ and $]\mathbb{a}_n, +\infty[$. Conclude.)

Exercise 3: The goal of this **difficult exercise** is to show that in the *Mostowski model* $\boldsymbol{\mathcal{M}}$ there exists some mapping $f : \mathcal{P}_{\mathit{fin}}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A})$.

1. For all support $F = \{\mathbb{a}_1, \ldots, \mathbb{a}_n\}$ with $\mathbb{a}_1 <_{\mathbf{M}} \ldots <_{\mathbf{M}} \mathbb{a}_n$, define a mapping

$$\begin{array}{ccc} {}^{2n+1}2 & \longrightarrow & \mathcal{P}(\mathbb{A}) \\ \chi & \mapsto & S_\chi \end{array}$$

   so that $\left\{S_\chi \mid \chi \in {}^{2n+1}2\right\}$ is the set of all subsets of $\mathbb{A}$ which have F as support.

2. Show that in the *Mostowski model* $\boldsymbol{\mathcal{M}}$ there exists some mapping

$$f : \mathcal{P}_{\mathit{fin}}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A}).$$