Exercise Sheet n°12

During the lecture we will establish that if **ZF** is consistent, then we can have a model of **ZF** in which the set of reals ($\mathbb{R}$) is a countable union of countable sets. The exercises of this week aim at showing some surprising consequences of having $\mathbb{R}$ as a countable union of countable sets (Exercise 5).

For any sets A,B we use the notations: $A \simeq B$ for "there exists $f : A \xrightarrow{bij.} B$"; $A \overset{1-1}{\lesssim} B$ if there exists $f : A \xrightarrow{1-1} B$; $A \overset{\text{onto}}{\lesssim} B$ if there exists $f : B \xrightarrow{onto} A$.

Exercise 1: Give a proof of the following statements:

1. $\mathbf{ZFC} \vdash_c \forall A \forall B \left(A \overset{\text{onto}}{\lesssim} B \longrightarrow A \overset{1-1}{\lesssim} B\right)$;
2. $\mathbf{ZFC} \vdash_c \forall A \forall B \Big(\big(A \overset{1-1}{\lesssim} B \wedge B \overset{\text{onto}}{\lesssim} A\big) \longrightarrow A \simeq B\Big)$;
3. $\mathbf{ZF} \vdash_c \big(\mathbf{AC} \longleftrightarrow \forall A \forall B$ "for all $g : B \xrightarrow{onto} A$, there exists $f : A \xrightarrow{1-1} B$ s.t. $g \circ f = id$"$\big)$;
4. $\mathbf{ZF} \vdash_c \forall A \forall B$ "if there exists $f : A \xrightarrow{1-1} B$, there exists $g : B \xrightarrow{onto} A$";
5. $\mathbf{ZF} \vdash_c \forall A \forall B$ "if there exists $f : A \xrightarrow{1-1} B$, then there exists $g : \mathscr{P}(A) \xrightarrow{1-1} \mathscr{P}(B)$";

Exercise 2:
Show that Hartog's Lemma is provable in **ZF** (without **AC**). Namely,

Given any set A, there exists some ordinal α such that $\alpha \overset{1-1}{\not\lesssim} A$.

(Hint: Consider $\mathcal{W} = \{(B, <_B) \subseteq A \times (A \times A) \mid (B, <_B)$ *is a well-ordering*$\}$.*)*

Exercise 3:
Briefly show that **ZF** proves that the following sets[1] are all equipotent:

- $\mathbb{R}$
- ${}^{\omega}\omega$
- ${}^{\omega}2$
- ${}^{\omega}\mathbb{R}$
- ${}^{\omega}({}^{\omega}\omega)$
- ${}^{\omega}({}^{\omega}2)$.

(Hint: you may use Cantor-Schröder-Bernstein Theorem which asserts that $A \simeq B$ holds if $A \overset{1-1}{\lesssim} B$ and $B \overset{1-1}{\lesssim} A$ both hold, and was proved without **AC***).*

Exercise 4: Give a proof of the following statements:

1. $\mathbf{ZF} \vdash_c \omega_1 \overset{\text{onto}}{\lesssim} {}^{\omega}2$
   (Hint: view some of the infinite sequences of 0's and 1's that contain infinitely many 1's as "coding" the well-orderings of the integers).

[1]We recall that ${}^{\omega}\omega = \{f : \mathbb{N} \to \mathbb{N}\}$ and ${}^{\omega}2 = \{f : \mathbb{N} \to \{0,1\}\}$.

2. $\mathbf{ZF} \vdash_c {}^\omega 2 \,\dot\cup\, \omega_1 \overset{\text{onto}}{\lesssim} {}^\omega 2$.
   $({}^\omega 2 \,\dot\cup\, \omega_1 := ({}^\omega 2 \times \{0\}) \cup (\omega_1 \times \{1\})$ is the disjoint union of ${}^\omega 2$ and ω_1.)

Exercise 5: Give a proof of the following statements:

1. $\mathbf{ZF} \vdash_c$ "if $\mathbb{R}$ is a countable union of countable sets, then $\omega_1 \overset{1-1}{\not\lesssim} {}^\omega 2$".

   (Hints:

   (a) notice that since $\mathbf{ZF}$ *proves* $\mathbb{R} \simeq {}^\omega({}^\omega 2)$*, the assumption "if* $\mathbb{R}$ *is a countable union of countable sets... " is equivalent to "if* ${}^\omega({}^\omega 2)$ *is a countable union of countable sets... "*

   (b) Assume towards a contradiction that ${}^\omega({}^\omega 2) = \bigcup_{n<\omega} G_n$*, where* $(G_n)_{n\in\omega}$ *is some family of countable sets, and there also exists some mapping* $f : \omega_1 \xrightarrow{1-1} {}^\omega 2$*.*

   (c) Show that $H_n = \{s \in {}^\omega 2 \mid \exists S \in G_n\ \exists k < \omega\ S(k) = s\}$ *satisfies* $H_n \overset{1-1}{\lesssim} \omega$*.*

   (d) Define $h : \omega \to {}^\omega 2$ *by* $h(n) = f(\alpha_n)$ *where*

   $$\alpha_n = \min\{\alpha \in \omega_1 \mid f(\alpha) \notin H_n\}.$$

   (e) Show that $h \in {}^\omega({}^\omega 2) = \bigcup_{n<\omega} G_n$ *leads to a contradiction.)*

2. $\mathbf{ZF} \vdash_c$ "if $\mathbb{R}$ is a countable union of countable sets, then there exists some partition $\mathcal{R}$ of $\mathbb{R}$ such that $\mathbb{R} \overset{1-1}{\lesssim} \mathcal{R}$ but $\mathcal{R} \overset{1-1}{\not\lesssim} \mathbb{R}$".

   (Hints:

   (a) notice that the statement $\mathcal{R} \overset{1-1}{\not\lesssim} \mathbb{R}$ *and* $\mathbb{R} \overset{1-1}{\lesssim} \mathcal{R}$ *is equivalent to the existence of some partition* $\mathcal{C}$ *of* ${}^\omega 2$ *such that* ${}^\omega 2 \overset{1-1}{\lesssim} \mathcal{C}$ *and* $\mathcal{C} \overset{1-1}{\not\lesssim} {}^\omega 2$*.*

   (b) make use of ${}^\omega 2 \,\dot\cup\, \omega_1 \overset{\text{onto}}{\lesssim} {}^\omega 2$ *and take any* $f : {}^\omega 2 \xrightarrow{\text{onto}} {}^\omega 2 \,\dot\cup\, \omega_1$ *to form the partition*

   $$\mathcal{C} = \Big\{\{s \in {}^\omega 2 \mid f(s) = x\} \mid x \in {}^\omega 2 \dot\cup \omega_1\Big\} = \Big\{f^{-1}[\{x\}] \mid x \in {}^\omega 2 \dot\cup \omega_1\Big\}.$$

   i. Show directly that ${}^\omega 2 \overset{1-1}{\lesssim} \mathcal{C}$*.*

   ii. Show that $\mathcal{C} \overset{1-1}{\lesssim} {}^\omega 2$ *leads to a contradiction.)*