Exercise Sheet n°11

Exercise 1:

Show that if Γ is a consistent set of closed formulas which contains ZF, then Γ is not finitely axiomatizable; formally, if $F \subseteq \Gamma$ is finite, we can extend F to a finite set $\widetilde{F}$ such that the formula:

$$\psi : \exists \alpha \left(\bigwedge_{\varphi \in \widetilde{F}} \varphi \right)^{V_\alpha}$$

is such that $\Gamma \vdash \psi$ and $F \nvdash \psi$.

Exercise 2:

Let $\mathbb{P}$ be a notion of forcing and $p \in \mathbb{P}$. Show the following facts:

1. $p \Vdash \varphi \vee \psi$ if and only if $\{q \mid (q \Vdash \varphi) \vee (q \Vdash \psi)\}$ is dense below p.
2. $p \Vdash \varphi \rightarrow \psi$ if and only if $\neg \exists q \leq p \ (q \Vdash \varphi \wedge q \Vdash \neg \psi)$.
3. $p \Vdash \forall v \, \varphi(v)$ if and only if $p \Vdash \varphi(\tau)$ for all $\mathbb{P}$-names $\tau \in V^{\mathbb{P}}$.

Exercise 3:

Let M be a countable transitive model of "ZFC", and $\mathbb{P} \in M$ a non empty partial order.

An element $p \in \mathbb{P}$ is called an *atom of* $\mathbb{P}$ if and only if:

$$\neg \exists q, r \in \mathbb{P} \ (q \leq p \wedge r \leq p \wedge q \perp r).$$

1. Show that if $\mathbb{P}$ has an atom, then there exists a filter $G \in M$ which is $\mathbb{P}$-generic over M.[1]
2. Suppose that $\mathbb{P}$ is atomless. Show that: $\{G : G \text{ is } \mathbb{P}\text{-generic over } M\}$ has cardinality $2^{\aleph_0}$ in V.

   Hint: Modify the proof of the lemma stating the existence of filters which are generic over a countable set and contain a fixed element of the poset.

Exercise 4:

Show that the forcing relation preserves all logical consequences. Namely, let $\varphi(x_1, \ldots, x_n)$, $\psi(x_1, \ldots, x_n)$, be any formulas built on the language of set theory, $\mathbf{M}$ any *c.t.m.* of "**ZFC**", $\mathbb{P}$ any notion of forcing on $\mathbf{M}$, and $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$. Show that if

$$\vdash_c \forall x_1 \ldots \forall x_n \ \big(\varphi(x_1, \ldots, x_n) \longrightarrow \psi(x_1, \ldots, x_n)\big)$$

holds, then for each $p \in \mathbb{P}$, we have

$$\big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}} \text{ implies } \big(p \Vdash_* \psi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

[1] This result is the viceversa of the Lemma, seen in class, which states that if $\mathbb{P}$ is atomless any G which is $\mathbb{P}$-generic over M verifies $G \notin M$.