Exercise Sheet n°10

We denote by $\mathfrak{C}$ the set:

$$\{p : F \to 2 \mid F \text{ is a finite subset of } \omega\},$$

partially ordered by the relation $p \leq q$ if and only if p extends q, i.e.

$$p \leq q \iff \text{dom}(p) \supseteq \text{dom}(q) \text{ and } p_{\restriction \text{dom}(q)} = p.$$

This notion of forcing is called Cohen's Forcing.

Exercise 1: Let us consider the $\mathfrak{C}$-conditions $p_1 = \{(0,0)\}$, $p_2 = \{(0,0),(1,0)\}$, $p_3 = \{(0,0)(1,1)\}$ and the $\mathfrak{C}$-names:

$$\begin{aligned}\sigma =& \{(\emptyset, p_2), (\emptyset, p_3)\},\\ \tau =& \{(\sigma, p_3), (\emptyset, p_1)\},\\ \nu =& \{(\tau, p_1), (\sigma, p_2), (\emptyset, p_2), (\emptyset, p_3), (\tau, p_3)\}.\end{aligned}$$

Let M be a countable transitive model of "ZFC" with $\mathfrak{C} \in M$, G a filter which is $\mathfrak{C}$-generic over of M such that $p_2 \in G$ and F be $\mathfrak{C}$-generic over of M such that $p_3 \in F$.

1. Compute $(\sigma)_G$, $(\tau)_G$ and $(\nu)_G$, as well as $(\sigma)_F$, $(\tau)_F$ and $(\nu)_F$.
2. Show that $p_2 \Vdash \check{1} = \sigma$.
3. Show that $p_1 \nVdash \check{2} = \tau$ and that $p_1 \nVdash \neg\check{2} = \tau$.

Exercise 2: Let M be a countable transitive model of "ZFC" with $\mathfrak{C} \in M$ and G a filter which is $\mathfrak{C}$-generic over of M.

1. Show that $G \notin M$.
2. Show that $f_G = \bigcup G$ is a function : $\omega \to 2$.
3. Show that $f_G \notin M$.
4. Show that:
$$\sigma = \{(\check{n}, p) \mid p \in \mathfrak{C} \wedge p(n) = 1\} \in M^{\mathfrak{C}},$$
where $M^{\mathfrak{C}}$ is the set of all $\mathfrak{C}$-names that belong to M. Also show that
$$(\sigma)_G = \{n \in \omega \mid f_G(n) = 1\}.$$
5. Show that $(\sigma)_G \notin M$.
6. Show that $\{n \in \omega \mid f_G(n) = 1\}$ is infinite and co-infinite.

Exercise 3: We say that a function $f : \omega \to \omega$ dominates a function $g : \omega \to \omega$ if there exists some integer n such that for all integers $k \geq n$ $f(k) > g(k)$ holds.

We will show the following:

Theorem. *Cohen's Forcing does not add a dominant real.*
i.e. for any $\mathfrak{C}$*-generic filter* G *over of* M *there does not exist a real* $f \in {}^{\omega}\omega \cap M[G]$ *which dominates all functions* $g \in {}^{\omega}\omega \cap M$.

Let G be a filter which is $\mathfrak{C}$-generic over of a countable transitive model of "ZFC" M, $f \in {}^{\omega}\omega \cap M[G]$ and $\underset{\sim}{f}$ be a $\mathfrak{C}$-name for f.

1. Show that there exists $p_0 \in G$ such that

$$p_0 \Vdash \forall x \in \check{\omega}\ \exists! y \in \check{\omega}\ \underset{\sim}{f}(x) = y.$$

We fix a $p_0 \in \mathfrak{C}$ which satisfies the previous point. Since $\mathfrak{C}$ is countable in M, we can fix, in M, an enumeration $\{p_k \mid k \in \omega\}$ of the set $\{q \in \mathfrak{C} \mid q \leq p_0\}$.

2. Show that we can define a function $g \in {}^{\omega}\omega$ in M by:

$$g(k) = \min \big\{ n \mid \exists p \leq p_k\ p \Vdash \underset{\sim}{f}(\check{k}) = \check{n} \big\}.$$

3. Show that for all $n \in \omega$ the set $\big\{ q \in \mathfrak{C} \mid \exists k \in \omega\ \ q \Vdash \big(\check{k} \check{\geq} \check{n} \wedge \underset{\sim}{f}(\check{k}) = \check{g}(\check{k}) \big) \big\}$ is dense below p_0.

4. Show that $M[G] \models \forall x \in \omega\, \exists y \in \omega\ \ \big(y \geq x\ \wedge\ f(y) = g(y) \big)$ and conclude.