

Partial Mock Exam

(about 1/2 size)

An A4 two side sheet of personal notes is allowed.
Points are only given as an indication of the length
and/or the difficulty of each exercise.

Last name:

First name:

/73 points

Section:

Problem 1: (38 points)

2 pt

Question 1.1: One the following ordinal equalities is true. Which one?

- $3 \cdot \omega = \omega \cdot 3$
- $\omega \cdot 2 + \omega^2 = \omega \cdot 3 + 2 + \omega^2$
- $\omega^2 + \omega^3 = \omega^2 + \omega^2$
- $4 \cdot (3 + \omega) = (3 + \omega) \cdot 4$

2 pt

Question 1.2: Let \oplus and \otimes denote the cardinal operations. One of the following cardinal equalities is provably false in ZFC. Which one?

- $\aleph_0 \otimes \aleph_0 = \aleph_0$
- $\aleph_\omega \oplus \aleph_1 = \aleph_{\omega+1}$
- $\aleph_2 \otimes 2^{(2^{\aleph_0})} = 2^{(2^{\aleph_0})}$
- $8 \oplus \aleph_1 = \aleph_1 \otimes 8$

2 pt

Question 1.3: Assuming ZFC is consistent. Let $\text{cof}(\alpha)$ denote the cofinality of α . One of the following ordinal equality is provably true in ZFC. Which one?

- $\text{cof}(\omega^\omega) = \omega^\omega$
- $\text{cof}(\omega_2) = \omega$
- $\text{cof}(\omega_1 + \omega^\omega) = \omega_1$
- $\text{cof}(\aleph_\omega) = \omega$

4 pt

Question 1.4: Working in ZFC and assuming ZFC is consistent, we consider the structure (V_ω, \in) . Among the following axioms, check those that **do hold** in this structure.

- Foundation axiom
- Pairing axiom
- Axiom of infinity
- Power Set axiom

3 pt

Question 1.5: Check the correct equalities among the following.

- $\text{rk}(\omega) = \omega$
- $\text{rk}(\omega^\omega) = \omega + 1$
- $\text{rk}(\omega^\omega) = \omega$
- $\text{rk}(\{0, 2\}) = 3$
- $\text{rk}(\{0, 2\}) = 2$
- $\text{rk}(\{\omega_1\}) = \omega_1 + 1$

3 pt

Question 1.6: Assuming ZFC is consistent. Which of the following statements is provable in ZFC?

- there exists a surjection $f : \mathbb{R} \rightarrow \mathcal{P}(\aleph_0)$
- there exists a injection $f : \omega_2 \rightarrow \mathbb{R}$
- there exists an injection $f : \omega_1 \rightarrow \mathcal{P}(\mathbb{R})$
- there exists a bijection $f : \mathbb{R} \rightarrow \mathcal{P}(\aleph_0)$
- there exists a surjection $f : \omega_2 \rightarrow \mathbb{R}$
- there exists a surjection $f : \omega_2 \rightarrow \omega_1$

3 pt

Question 1.7: Let \mathbf{L} be the class of constructible sets. Assuming ZF is consistent, which of the following statements is provable in ZF?

- (Power Set Axiom) $^\mathbf{L}$
- (“*there exists some well-ordering of \mathbb{R} of order type ω_2* ”) $^\mathbf{L}$
- (AC) $^\mathbf{L}$
- (Axiom of Extensionality) $^\mathbf{L}$
- $(2^{(\aleph_0)} = \aleph_2)^\mathbf{L}$
- (“ *\mathbb{R} is a countable union of countable sets*”) $^\mathbf{L}$

3 pt

Question 1.8: Let \mathbf{M} be a transitive class model of ZFC. Which of the following formulas are absolute for \mathbf{M} ?

- “ α is an ordinal”

- “ α is an infinite ordinal”
- “ α is a limit ordinal”
- “ α is some uncountable ordinal”
- “ α is some countable ordinal”
- “ $f : x \rightarrow y$ is some injection”

4 pt

Question 1.9: Assuming ZFC is consistent. Among the following statements, check those which are consistent with ZFC:

- There exists some regular cardinal κ such that $\aleph_0 \leq \kappa < 2^{\aleph_0}$.
- There exists no singular cardinal κ such that $\aleph_0 < \kappa < 2^{(\aleph_0)}$.
- There exists some singular cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$.
- There exists some singular cardinal κ and some regular cardinal λ such that $\aleph_0 < \kappa \leq \lambda < 2^{\aleph_0}$.
- There exists some singular regular κ and some singular cardinal λ such that $\aleph_0 < \kappa \leq \lambda < 2^{\aleph_0}$.
- There exists exactly one singular cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$.
- There exists no regular cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$.
- There exist exactly two singular cardinals κ, λ such that $\aleph_0 < \kappa < \lambda < 2^{\aleph_0}$.

5 pt

Question 1.10: Assuming **ZF** is consistent. Among the following statements, check those which hold in all models of **ZF**:

(We recall that $A \overset{1-1}{\precsim} B$ stands for “*there exists an injection from A to B* ”.)

<input type="checkbox"/> $\mathbb{R} \overset{1-1}{\precsim} \omega \mathbb{R}$.	<input type="checkbox"/> $\omega 2 \overset{1-1}{\precsim} \mathbb{R}$.
<input type="checkbox"/> $\omega \mathbb{R} \overset{1-1}{\precsim} \mathbb{R}$.	<input type="checkbox"/> $\mathbb{R} \overset{1-1}{\precsim} {}^2(\omega 2)$.
<input type="checkbox"/> $\mathbb{R} \overset{1-1}{\precsim} \mathbb{R}_2$.	<input type="checkbox"/> ${}^2(\omega 2) \overset{1-1}{\precsim} \mathbb{R}$.
<input type="checkbox"/> $\mathbb{R}_2 \overset{1-1}{\precsim} \mathbb{R}$.	<input type="checkbox"/> $\mathbb{R} \overset{1-1}{\precsim} {}^{(\omega 2)}2$.
<input type="checkbox"/> $\mathbb{R} \overset{1-1}{\precsim} \omega 2$.	<input type="checkbox"/> $(\omega 2) \overset{1-1}{\precsim} \mathbb{R}$.

2 pt

Question 1.11: Assuming **ZF** is consistent. Among the following statements, check those which hold in all models of **ZF** + “ \mathbb{R} is a countable union of countable sets”:

- there is a partition \mathcal{R} of \mathbb{R} such that $\mathbb{R} \overset{1-1}{\precsim} \mathcal{R}$
- there is a partition \mathcal{R} of \mathbb{R} such that $\mathbb{R} \overset{1-1}{\not\precsim} \mathcal{R}$
- there is a partition \mathcal{R} of \mathbb{R} such that $\mathcal{R} \overset{1-1}{\not\precsim} \mathbb{R}$
- there is a partition \mathcal{R} of \mathbb{R} such that both $\mathbb{R} \overset{1-1}{\precsim} \mathcal{R}$ and $\mathcal{R} \overset{1-1}{\not\precsim} \mathbb{R}$ hold.

2 pt

Question 1.12: Assuming **ZF** is consistent. Let M be any countable transitive model of “**ZFC**”, $(\mathbb{P}, \leq, \mathbb{1})$ any partial order over M , \mathcal{G} any subgroup of the group of automorphisms of \mathbb{P} , and \mathcal{F} a normal filter on \mathcal{G}

For all subgroups \mathcal{H}, \mathcal{K} of \mathcal{G} and all $\pi \in \mathcal{G}$ which of the following assertion holds?

- $\mathcal{G} \in \mathcal{F}$
- if $\mathcal{H} \in \mathcal{F}$ and $\mathcal{K} \subseteq \mathcal{H}$, then $\mathcal{K} \in \mathcal{F}$
- if $\mathcal{H} \in \mathcal{F}$ and $\mathcal{K} \in \mathcal{F}$, then $\mathcal{H} \cup \mathcal{K} \in \mathcal{F}$
- if $\mathcal{H} \in \mathcal{F}$, then $\pi^{-1} \circ \mathcal{H} \circ \pi \in \mathcal{F}$

3 pt

Question 1.13: Assuming **ZFA** is consistent. Let \mathbf{M} be any transitive model of **ZFA** with \mathbb{A} as set of atoms, \mathcal{G} any subgroup of the group of permutations of \mathbb{A} , \mathcal{F} any normal filter on \mathcal{G} , $\mathbf{HS}_{\mathcal{F}}$ the class of all hereditarily symmetric sets, and $\mathbf{M}^{\mathbf{HS}_{\mathcal{F}}} = \mathbf{M} \cap \mathbf{HS}_{\mathcal{F}}$ the induced permutation model. Among the following statements, check those which hold:

- $\mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$ is transitive.
- $\mathcal{P}^{\infty}(\emptyset) \subseteq \mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$;
- $\mathbb{A} \subseteq \mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$.

- $\mathbb{A} \in \mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$.
- $\mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$ satisfies **ZFA**.
- $\mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$ satisfies **AC**.

/38 points

Problem 2: (6 points)

2 pt

Question 2.1: Write a first order formula $\varphi(x,A,B)$ in the language of set theory such that $\varphi(x,A,B)$ holds if and only if x belongs to $\bigcup(A \cap B)$.

4 pt

Question 2.2: In the language of set theory, write a first order formula with one free variable which describes the class of transitive \in -well-founded sets.

/6 points

Problem 3: (29 points)

We let \mathbb{P} denote the following forcing notion:

$$\{p : F \rightarrow \omega \mid F \text{ is any finite subset of } \omega\}$$

partially ordered by $p \leq q$ if and only if p extends q , i.e. $\text{dom}(p) \supseteq \text{dom}(q)$ and the restriction of p to $\text{dom}(q)$ equals q .

We assume that M is a countable transitive model of “ZFC” that contains \mathbb{P} .

4 pt

Question 3.1: Among the following sets, check the ones which are \mathbb{P} -conditions.

<input type="checkbox"/> $p_0 = 0$	<input type="checkbox"/> $p_5 = \{(0, 0), (3, 1)\}$
<input type="checkbox"/> $p_1 = \{(0, 0)\}$	<input type="checkbox"/> $p_6 = \{(0, p_0), (3, p_3)\}$
<input type="checkbox"/> $p_2 = \{\{0\}\}$	<input type="checkbox"/> $p_7 = \{p_0, p_3\}$
<input type="checkbox"/> $p_3 = \omega$	<input type="checkbox"/> $p_8 = \{(p_0, 0), (p_3, 3)\}$
<input type="checkbox"/> $p_4 = \{(0, 0), (0, 3)\}$	

3 pt

Question 3.2: Assume that σ and τ are \mathbb{P} -names and p and q are \mathbb{P} -conditions. Among the following sets, check the ones which are \mathbb{P} -names.

<input type="checkbox"/> $\tau_0 = 0$	<input type="checkbox"/> $\tau_3 = \{(\sigma, p), (\tau, q)\}$
<input type="checkbox"/> $\tau_1 = \{(0, 0)\}$	<input type="checkbox"/> $\tau_4 = \{(p, \sigma), (q, \tau)\}$
<input type="checkbox"/> $\tau_2 = \{0, \tau_1\}$	<input type="checkbox"/> $\tau_5 = \{(p, p), (q, q)\}$

4 pt

Question 3.3: Let G be \mathbb{P} -generic over M with $p = \{(0, 0), (1, 2), (5, 7)\} \in G$.

1. Does $q_0 = \{(0, 0), (5, 7)\} \in G$ hold?

yes no it depends

2. Does $q_1 = \{(1, 0)\} \in G$ hold?

yes no it depends

3. Does $q_2 = \{(2, 1)\} \in G$ hold?

yes no it depends

4. Does $q_3 = \{(0, 0), (1, 2), (2, 3), (3, 4), (5, 7)\} \in G$ hold?

yes no it depends

6 pt

Question 3.4: We now consider the following \mathbb{P} -conditions:

$$p_0 = \{(0, 0)\}, \quad p_1 = \{(0, 0), (1, 1)\}, \quad \text{and} \quad p_2 = \{(0, 0), (1, 2)\}.$$

We consider the following \mathbb{P} -names

$$\tau_0 = \{(\emptyset, p_0)\}, \quad \tau_1 = \{(\emptyset, p_1)\}, \quad \tau_2 = \{(\tau_0, p_0), (\emptyset, p_1)\}, \quad \tau_3 = \{(\tau_2, p_2)\}.$$

Let G be any \mathbb{P} -generic filter over M with $p_1 \in G$.

Compute $(\tau_0)_G$, $(\tau_1)_G$, $(\tau_2)_G$, and $(\tau_3)_G$.

Does $p_1 \Vdash \tau_0 \in \tau_1$ hold?

yes no it depends

Does $p_1 \Vdash (\tau_1 \in \tau_2 \wedge \tau_3 \in \tau_2)$ hold?

yes no it depends

8 pt

Question 3.5: Let G be any \mathbb{P} -generic filter over M and

$$\sigma = \{(\check{n}, p) \mid p \in \mathbb{P} \wedge n \in \omega \wedge p(n) = 1\}.$$

Does $(\sigma)_G$ belong to M ?

yes no it depends

4 pt

Question 3.6: Show that for all $n \in \omega$ the set

$$D_n = \{p \in \mathbb{P} \mid \exists k \in \omega (k > n \wedge p(k) > k)\}$$

is dense in \mathbb{P} and that D_n belongs to M .

/29 points