# Part V

# Forcing

# Chapter 13

# Forcing Conditions and Generic Filters

## 13.1 Introduction

Assuming that **ZFC** is consistent, the aim of this chapter is to prove that there exists a model of **ZFC** that does not satisfies the Continuum Hypothesis. In other words, we are going to prove that **ZFC** + ¬**CH** is not inconsistent assuming **ZFC** is consistent. To do so, we proceed by contraposition and prove:

If **ZFC** + ¬**CH** is inconsistent, then **ZFC** is already inconsistent.

i.e.,

$$\mathbf{ZFC} + \neg\mathbf{CH} \vdash_c \bot \implies \mathbf{ZFC} \vdash_c \bot.$$

Now, if **ZFC** + ¬**CH** $\vdash_c \bot$ is satisfied, then such a proof of its inconsistency involves only finitely many formulas. Therefore, there exist $\varphi_1, \ldots, \varphi_n$ in **ZFC** + ¬**CH** such that for any closed formula $\varphi$, we have

$$\varphi_1, \ldots, \varphi_n \vdash_c (\varphi \wedge \neg\varphi).$$

i.e.,

$$\vdash_c \Big((\varphi_1 \wedge \ldots \wedge \varphi_n) \longrightarrow (\varphi \wedge \neg\varphi)\Big).$$

From Lemma 167, the following implication holds for any non-empty class **M**:

$$\vdash_c \Big((\varphi_1 \wedge \ldots \wedge \varphi_n) \longrightarrow (\varphi \wedge \neg\varphi)\Big) \text{ implies } \vdash_c \Big((\varphi_1 \wedge \ldots \wedge \varphi_n) \longrightarrow (\varphi \wedge \neg\varphi)\Big)^{\mathbf{M}}.$$

Notice that

$$\Big((\varphi_1 \wedge \ldots \wedge \varphi_n) \longrightarrow (\varphi \wedge \neg\varphi)\Big)^{\mathbf{M}} = \Big((\varphi_1 \wedge \ldots \wedge \varphi_n)\Big)^{\mathbf{M}} \longrightarrow \Big((\varphi \wedge \neg\varphi)\Big)^{\mathbf{M}}$$
$$= \Big((\varphi_1)^{\mathbf{M}} \wedge \ldots \wedge (\varphi_n)^{\mathbf{M}}\Big) \longrightarrow \Big((\varphi)^{\mathbf{M}} \wedge (\neg\varphi)^{\mathbf{M}}\Big)$$
$$= \Big((\varphi_1)^{\mathbf{M}} \wedge \ldots \wedge (\varphi_n)^{\mathbf{M}}\Big) \longrightarrow \Big((\varphi)^{\mathbf{M}} \wedge \neg(\varphi)^{\mathbf{M}}\Big)$$

By using *forcing* methods, one can prove that there exists some $\mathbf{N}$ such that

$$\mathbf{ZFC} \vdash_c \Big((\varphi_1)^{\mathbf{N}} \wedge \ldots \wedge (\varphi_n)^{\mathbf{N}}\Big).$$

Since we also have

$$\vdash_c \Big((\varphi_1)^{\mathbf{N}} \wedge \ldots \wedge (\varphi_n)^{\mathbf{N}}\Big) \longrightarrow \Big((\varphi)^{\mathbf{N}} \wedge \neg(\varphi)^{\mathbf{N}}\Big)$$

we obtain

$$\mathbf{ZFC} \vdash_c \Big((\varphi)^{\mathbf{N}} \wedge \neg(\varphi)^{\mathbf{N}}\Big)$$

thus

$$\mathbf{ZFC} \vdash_c \bot.$$

As for the proof of

$$\mathbf{ZFC} \vdash_c \exists \mathbf{N}\Big((\varphi_1)^{\mathbf{N}} \wedge \ldots \wedge (\varphi_n)^{\mathbf{N}}\Big).$$

there exist only a finite number of formulas $\psi_1, \ldots, \psi_k$ from **ZFC** that are really needed to conduct the proof. So, it really is

$$\psi_1, \ldots, \psi_k \vdash_c \exists \mathbf{N}\Big((\varphi_1)^{\mathbf{N}} \wedge \ldots \wedge (\varphi_n)^{\mathbf{N}}\Big).$$

So, what we will do in practice is consider any transitive countable model[1] $\mathbf{M}$ (given by the Montague's Reflection Principle (see page 217) such that

$$\mathbf{M} \models (\psi_1 \wedge \ldots \wedge \psi_k).$$

By forcing, we will obtain a transitive model

$$\mathbf{N} = \mathbf{M}[G] \models (\varphi_1 \wedge \ldots \wedge \varphi_n).$$

[1] Notice that both $\mathbf{M}$ and $\mathbf{N} = \mathbf{M}[G]$ will be sets.

## 13.2 Montague's Reflection Principle

**Montague's Reflection Principle** (Lévy & Montague)**.** *Let $\varphi_0, \ldots, \varphi_n$ be any $\mathcal{L}_{\text{ST}}$-formulas.*

$$\mathbf{ZF} \vdash_c \forall\alpha \in \mathbf{On}\ \exists\beta > \alpha \quad \text{“}\varphi_0, \ldots, \varphi_n \quad \textit{are absolute for } \mathbf{V}(\beta),\ \mathbf{V}.\text{”}$$

*Proof of Montague's Reflection Principle*: The proof is similar to the proof of Theorem 273. First, without loss of generality we may assume that the set of formulas $\{\varphi_0, \ldots, \varphi_n\}$ is closed under sub-formulas and only contains formulas using $\neg, \wedge$ as connectors and $\exists$ as quantifiers.

For each integer $i \leqslant n$ such that $\varphi_i$ is of the form $\exists x\ \varphi_j(x, y_1, \ldots, y_{k_i})$, we define a class-function $\mathbf{G}_i : \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{k_i} \to \mathbf{On}$ by

$$\begin{aligned} \mathbf{G}_i(y_1, \ldots, y_{k_i}) &= 0 \text{ if } \Big(\neg\exists x\ \varphi_j(x, y_1, \ldots, y_{k_i})\Big)^{\mathbf{V}} \\ &= \text{least } \theta \text{ s.t. } \exists x \in \mathbf{V}(\theta)\ \ \big(\varphi_j(x, y_1, \ldots, y_{k_i})\big)^{\mathbf{V}} \end{aligned}$$

Then, for each integer $i \leqslant n$ we define a class-function $\mathbf{F}_i : \mathbf{On} \to \mathbf{On}$ by

$$\begin{aligned} &\mathbf{F}_i(\xi) = \sup\{\mathbf{G}_i(y_1, \ldots, y_{k_i}) \mid y_1, \ldots, y_{k_i} \in \mathbf{V}(\xi)\} \text{ if } \mathbf{G}_i \text{ is defined} \\ &\mathbf{F}_i(\xi) = 0 \text{ otherwise.} \end{aligned}$$

Given any ordinal $\alpha$, one defines the strictly increasing sequence $(\beta_k)_{n\in\omega}$ and a limit ordinal $\beta$ by:

- $\beta_0 = \alpha$
- $\beta_{k+1} = \sup\big\{\beta_k + 1, \mathbf{F}_1(\beta_k), \ldots, \mathbf{F}_n(\beta_k)\big\}$
- $\beta = \sup_{k\in\omega} \beta_k$

We show — by induction on the height of the formula — that for each integer $i \leqslant n$, one has

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \Big(\varphi_i(y_1, \ldots, y_{k_i})^{\mathbf{V}(\beta)} \longleftrightarrow \varphi_i(y_1, \ldots, y_{k_i})^{\mathbf{V}}\Big) \tag{13.1}$$

**If $\varphi_{\mathbf{i}}$ is an atomic formula:**

- If $\varphi_{\mathbf{i}}$ is $y_1 = y_2$, then one has $(y_1 = y_2)^{\mathbf{V}(\beta)} = (y_1 = y_2)^{\mathbf{L}} = (y_1 = y_2)$, hence

$$\forall y_1 \in \mathbf{V}(\beta)\ \forall y_2 \in \mathbf{V}(\beta)\ \ \Big((y_1 = y_2)^{\mathbf{V}(\beta)} \longleftrightarrow (y_1 = y_2)^{\mathbf{L}}\Big)$$

comes down to

$$\forall y_1 \in \mathbf{V}(\beta)\ \forall y_2 \in \mathbf{V}(\beta)\ \left(y_1 = y_2 \longleftrightarrow y_1 = y_2\right)$$

which trivially holds.

- If $\varphi_{\mathbf{i}}$ is $y_1 \in y_2$, then one has $(y_1 \in y_2)^{\mathbf{V}(\beta)} = (y_1 \in y_2)^{\mathbf{L}} = (y_1 = y_2)$, hence

$$\forall y_1 \in \mathbf{V}(\beta)\ \forall y_2 \in \mathbf{V}(\beta)\ \left((y_1 \in y_2)^{\mathbf{V}(\beta)} \longleftrightarrow (y_1 \in y_2)^{\mathbf{L}}\right)$$

comes down to

$$\forall y_1 \in \mathbf{V}(\beta)\ \forall y_2 \in \mathbf{V}(\beta)\ \left(y_1 \in y_2 \longleftrightarrow y_1 \in y_2\right)$$

which trivially holds as well.

- If $\varphi_{\mathbf{i}}$ is either $y_1 = y_1$ or $y_1 \in y_1$, theses cases are taken care of by the previous cases by taking $y_2 = y_1$.

So, in any case, when $\varphi_i$ is an atomic formula, the formula 11.1 is satisfied.

**If $\varphi_{\mathbf{i}} := \neg\varphi_{\mathbf{j}}(\mathbf{y_1}, \ldots, \mathbf{y_{k_i}})$:** by induction hypothesis, one has

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta)\ \left(\varphi_j(y_1, \ldots, y_{k_i})^{\mathbf{V}(\beta)} \longleftrightarrow \varphi_j(y_1, \ldots, y_{k_i})^{\mathbf{V}}\right)$$

which yields

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta)\ \left(\neg\left(\varphi_j(y_1, \ldots, y_{k_i})\right)^{\mathbf{V}(\beta)} \longleftrightarrow \neg\left(\varphi_j(y_1, \ldots, y_{k_i})\right)^{\mathbf{V}}\right)$$

and finally gives

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta)\ \left(\left(\neg\varphi_j(y_1, \ldots, y_{k_i})\right)^{\mathbf{V}(\beta)} \longleftrightarrow \left(\neg\varphi_j(y_1, \ldots, y_{k_i})\right)^{\mathbf{V}}\right)$$

which shows that formula 13.1 is satisfied.

**If $\varphi_{\mathbf{i}} := (\varphi_{\mathbf{j}}(\mathbf{y_1}, \ldots, \mathbf{y_{k_i}}) \wedge \varphi_{\mathbf{k}}(\mathbf{y_1}, \ldots, \mathbf{y_{k_i}}))$:** by induction hypothesis, one has both

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta)\ \left(\varphi_j(y_1, \ldots, y_{k_i})^{\mathbf{V}(\beta)} \longleftrightarrow \varphi_j(y_1, \ldots, y_{k_i})^{\mathbf{V}}\right)$$

and

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta)\ \left(\varphi_k(y_1, \ldots, y_{k_i})^{\mathbf{V}(\beta)} \longleftrightarrow \varphi_k(y_1, \ldots, y_{k_i})^{\mathbf{V}}\right).$$

Now, given any $y_1, \ldots y_{k_i} \in \mathbf{V}(\beta)$, one has that both formulas $\varphi_j(y_1, \ldots, y_{k_i})$ and $\varphi_k(y_1, \ldots, y_{k_i})$ hold in $\mathbf{V}(\beta)$ if and only if they both hold in $\mathbf{V}$. Therefore, $\left(\varphi_j(y_1, \ldots, y_{k_i}) \wedge \varphi_k(y_1, \ldots, y_{k_i})\right)$ holds in $\mathbf{V}(\beta)$ if and only if it holds in $\mathbf{V}$. This shows that formula 13.1 is satisfied.

**If $\varphi_{\mathbf{i}} := \exists \mathbf{x}\ \varphi_{\mathbf{j}}(\mathbf{x}, \mathbf{y_1}, \ldots, \mathbf{y_{k_i}})$:** we have to check that

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \left( \left( \exists x\ \varphi_j(x, y_1, \ldots, y_{k_i}) \right)^{\mathbf{V}(\beta)} \longleftrightarrow \left( \exists x\ \varphi_j(x, y_1, \ldots, y_{k_i}) \right)^{\mathbf{V}} \right)$$

i.e.,

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \left( \exists x \in \mathbf{V}(\beta)\ \ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{V}(\beta)} \longleftrightarrow \exists x \in \mathbf{V}\ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{V}} \right)$$

Clearly, the direction

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \left( \exists x \in \mathbf{V}(\beta)\ \ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{V}(\beta)} \longrightarrow \exists x \in \mathbf{V}\ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{V}} \right)$$

is taken care of by the induction hypothesis. So, we show

$$\forall y_1 \in \mathbf{V}(\beta) \ldots \forall y_{k_i} \in \mathbf{V}(\beta) \quad \left( \exists x \in \mathbf{V}\ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{V}} \longrightarrow \exists x \in \mathbf{V}(\beta)\ \ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{V}(\beta)} \right)$$

We fix $y_1 \in \mathbf{V}(\beta), \ldots, y_{k_i} \in \mathbf{V}(\beta)$. For some large enough integer $p$, one has

$$\{y_1, \ldots, y_{k_i}\} \subseteq \mathbf{V}(\beta_p).$$

By construction, there exists $x \in \mathbf{V}(\mathbf{G}_i(y_1, \ldots, y_{k_i}))$ such that $\left(\varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{V}}$. Since $\mathbf{G}_i(y_1, \ldots, y_{k_i}) \leqslant \mathbf{F}_i(\beta_p) \leqslant \beta_{p+1}$, it follows that there exists $x \in \mathbf{V}(\beta_{p+1}) \subseteq \mathbf{V}(\beta)$ such that $\left(\varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{V}}$.

Finally, by induction hypothesis, there exists $x \in \mathbf{V}(\beta)$ such that $\left(\varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{V}(\beta)}$.

□ Montague's Reflection Principle

**Corollary 285.** *For every finite set of formulas $\{\varphi_0, \ldots, \varphi_n\} \subseteq$ **ZFC**,*

$$\mathbf{ZFC} \vdash_c \exists \mathbf{M} \quad \left( |\mathbf{M}| = \aleph_0 \ \wedge\ \text{“}\mathbf{M}\ \textit{is transitive}\text{”} \ \wedge\ \Big( \bigwedge_{0 \leqslant i \leqslant n} \varphi_i \Big)^{\mathbf{M}} \right).$$

Notice that, although we use the notation $\mathbf{M}$ and not $M$ for the countable transitive model that satisfies all formulas in $\{\varphi_0, \ldots, \varphi_n\}$, $\mathbf{M}$ is not a proper class: it is really some set!

*Proof of Corollary* 285: Either **ZFC** is inconsistent, in which case it proves anything. Or, **ZFC** is consistent, and by Montague's Reflection Principle, since $\bigwedge_{0 \leqslant i \leqslant n} \varphi_i$ holds in $\mathbf{V}$, there exists some ordinal $\beta$ such that $\bigwedge_{0 \leqslant i \leqslant n} \varphi_i$ holds in $\mathbf{V}(\beta)$. Then, since the language of set theory is finite, and $\mathbf{V}(\beta)$ is infinite[2] by Löwenheim-Skolem Theorem (see [2, 3, 4, 5, 6, 33]), there exists

[2]The fact $\mathbf{V}(\beta)$ is infinite relies for instance on the construction of $\beta$ in the proof of Montague's Reflection

some countable model $\mathbf{N}$ such that

$$\mathbf{N} \models \bigwedge_{0 \leqslant i \leqslant n} \varphi_i.$$

Notice that, although $\mathbf{V}(\beta)$ is transitive, this may not be the case with $\mathbf{N}$. However, the Mostowski Collapsing Theorem (page 113) grants the existence of both some transitive class $\mathbf{M}$, and an isomorphism $\iota : (\mathbf{N}, \in) \xleftrightarrow{isom.} (\mathbf{M}, \in)$. Finally, being isomorphic, $\mathbf{N}$ and $\mathbf{M}$ are elementary equivalent, i.e., they satisfy the same closed formulas, which yields $\mathbf{M}$ is a transitive countable set that satisfies

$$\mathbf{M} \models \bigwedge_{0 \leqslant i \leqslant n} \varphi_i.$$

□ 285

## 13.3 Posets and Generic Filters

**Notation 286.** *we write "* **ZF** *"(respectively "* **ZFC** *") for "finitely many axioms from* **ZF** *" (respectively "finitely many axioms from* **ZFC** *").*

A proof is something that only makes use of finitely many axioms or instances of axiom schemas. For instance, we showed that the empty set exists using the axiom of **Extensionality** and one instance of the **Comprehension Schema**.

So, later on, it could happen that we write something like " **ZF**" $\vdash_c$ "∅ exists" to indicate both that **ZF** $\vdash_c$ "∅ exists" and " **ZF**"refers to the axioms that were necessary to conduct the proof. An other example, would be the proof of the existence of a class-function as in Theorem 53 :

> Given any $\mathbf{F} : \mathbf{V} \to \mathbf{V}$, there exists a unique $\mathbf{G} : \mathbf{On} \to \mathbf{V}$ such that for each ordinal $\alpha$
>
> $$\forall \alpha \;\; \mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha).$$

Strictly speaking, this theorem is a theorem schema: there are infinitely many theorems, one for every class-function $\mathbf{F}$. Indeed a class-function $\mathbf{F}$ refers to some formula $\varphi_{\mathbf{F}}$, and the result consists in constructing another formula $\varphi_{\mathbf{G}}$ which satisfies the required property and showing that the class-function $\mathbf{G}$ it represents is unique.

Although the whole construction only requires finitely many axioms or instances of axiom schemas, but we do not bother precisely indicating which one we used, reason why we use the notation " **ZF**" for "these finitely many axioms that a hard work could precisely point out, but we don't really care as long as there are only finitely many of them".

A *countable transitive model* (*c.t.m.*) of " **ZFC** " is a countable transitive model of a " *sufficiently large number of axioms of* **ZFC** ". A nice way of thinking of " **ZFC** " is to imagine that it contains all the following axioms:

Principle on page 217, and also in that one wants the **Infinity** Axiom to be part of the set of formulas $\{\varphi_0, \ldots, \varphi_n\}$; or also again, by simply setting $\alpha$ to be infinite in the application of the Montague's Reflection Principle on page 217 which yields the ordinal $\beta$.

- **Set Existence**
- **Extensionality**
- **Pairing**
- **Union**
- **Infinity**
- **Power Set**
- **Foundation**
- **Choice**

and in addition, finitely many instances[3] from the following two axiom schemas:

- **Comprehension Schema**
- **Replacement Schema**.

**Definition 287** (Notion of Forcing).

- *A **notion of forcing** is a partial order $(\mathbb{P}, \leqslant)$. It is often abbreviated as $\mathbb{P}$, and refereed to as a* poset.

  *We also use the notation $(\mathbb{P}, \leqslant, \mathbb{1})$ when the poset admits a maximum element $\mathbb{1}$.*

- *The elements of $\mathbb{P}$ are called **conditions**.*

- *Given two conditions $p, q \in \mathbb{P}$, we say that $p$ is* **stronger** than *$q$ if $p \leqslant q$.*

**Definition 288** (Poset). *Let $(\mathbb{P}, \leqslant, \mathbb{1})$ be a poset with maximal element $\mathbb{1}$, and let $p, q \in \mathbb{P}$. We say that*

- *$p$ and $q$ are* comparable *if either $p \leqslant q$ or $q \leqslant p$ holds;*
- *$p$ and $q$ are **compatible** if there exists $r \in \mathbb{P}$ such that $r \leqslant p$ and $r \leqslant q$;*
- *we write $p \perp q$ when $p$ and $q$ **incompatible**. i.e., when they are not compatible;*
- *a subset $A \subseteq \mathbb{P}$ is an (strong) **antichain** if for all $p, q \in A$, $p \perp q$ holds;*
- *a subset $D \subseteq \mathbb{P}$ is **dense** in $\mathbb{P}$ if for all $p \in \mathbb{P}$ there exists $q \in D$ such that $q \leqslant p$.*

**Example 289.** *Let $\mathbb{P} = \mathscr{P}(X) \smallsetminus \{\varnothing\}$, with $p \leqslant q$ if and only if $p \subseteq q$. In this case, one has*

- $\mathbb{1} = X$

[3] These are typically the instances that were necessary to conduct the proofs of the results that we now need to use in a particular proof.

- *if* $p \cap q \neq \varnothing$, *then* $p \cap q \leqslant p, q$
- $p \perp q$ *if and only if* $p \cap q = \varnothing$
- $\left\{ \{p\} \mid p \in X \right\}$ *is both an antichain and dense in* $\mathbb{P}$.

**Example 290.** *We let* $\mathbb{P}$ *be the following notion of forcing:*

$$\mathbb{P} = \left\{ f \;:\; \aleph_2 \times \omega \to \{0,1\} \mid f \text{ a partial function whose domain is finite} \right\}$$
$$= \left\{ f \subseteq \aleph_2 \times \omega \times \{0,1\} \;\middle|\; \left( \begin{array}{c} |f| < \omega \\ \wedge \\ \forall \alpha < \aleph_2 \, \forall n < \omega \, \forall i < 2 \, \big( (\alpha, n, i) \in f \longrightarrow (\alpha, n, 1-i) \notin f \big) \end{array} \right) \right\}$$

*with*

$$f \leqslant g \iff f \supseteq g$$

*and*

$$\mathbb{1} = \varnothing.$$

*Notice that*

(1) $f \leqslant g$ *holds iff both* $dom(g) \subseteq dom(f)$ *and* $f \upharpoonright dom(g) = g$;

(2) $f$ *and* $g$ *are compatible iff* $f \upharpoonright (dom(f) \cap dom(g)) = g \upharpoonright (dom(f) \cap dom(g))$;

(3) $f$ *and* $g$ *are incompatible iff there exists* $(\alpha, k) \in dom(f) \cap dom(g)$ *such that* $f(\alpha, k) \neq g(\alpha, k)$.

**Definition 291** (Filter)**.** *Let* $(\mathbb{P}, \leqslant, \mathbb{1})$ *be a notion of forcing.*

$$G \subseteq \mathbb{P} \text{ is a } \textbf{filter} \text{ on } \mathbb{P} \iff \begin{cases} \forall p \in G \; \forall q \in G \;\; \exists r \in G \;\; \big( r \leqslant p \wedge r \leqslant q \big) \\ \text{and} \\ \forall p \in G \; \forall q \in \mathbb{P} \;\; \big( p \leqslant q \longrightarrow q \in G \big). \end{cases}$$

As shown in the figure below, if $p$ and $q$ are inside the filter, then not only all forcing conditions inside the *cone* above $q$ or the cone above $q$ belong to the filter, but there exists some $r$ below both $p$ and $q$ which belong to the filter, and therefore the whole cone above $r$ is included inside the filter.

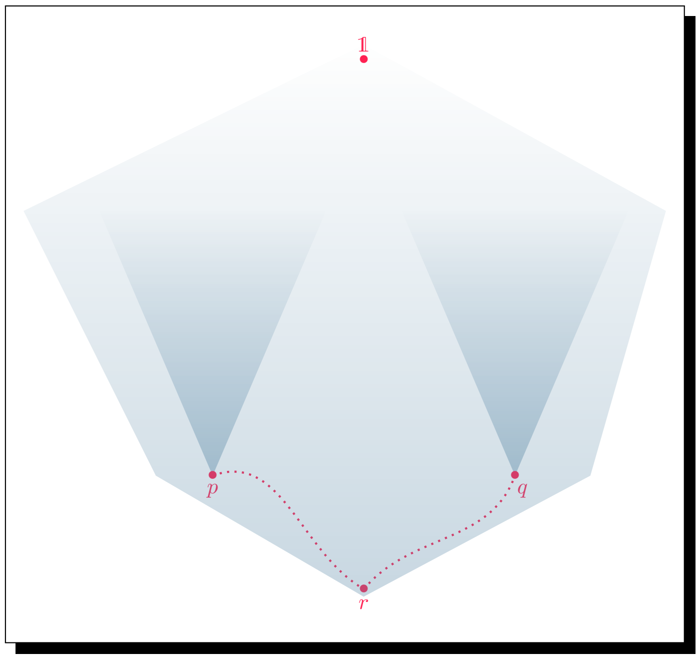



Figure 13.1: $r$ wittnesses that $p$ and $q$ are compatible.

All the filters we will consider will be non-constructive. We will essentially claim that " there exists some filter $G$..." by mean of a proof by contradiction. i.e., the proofs will be of the form: assuming that such a filter does not exist, leads to some contradiction; therefore, such a filter exists...

So, asking for samples of such filters is useless for the reason that the ones that could be constructed would be of no interest for our purpose.

**Definition 292** (Genericity). *Let* $(\mathbb{P}, \leqslant, \mathbb{1})$ *be a notion of forcing and* $\mathbf{M}$ *be any set (or class).* $G \subseteq \mathbb{P}$ *is* $\mathbb{P}$-generic over $\mathbf{M}$ *if the following two conditions are satisfied:*

*(1)* $G$ *is a filter on* $\mathbb{P}$

*(2)* $G$ *intersects every dense subset of* $\mathbb{P}$ *which belongs to* $\mathbf{M}$.

Notice that the property of being $\mathbb{P}$-*generic over* $\mathbf{M}$ is expressible by some $\mathcal{L}_{\text{ST}}$-formula "$G \subseteq$

$\mathbb{P}$ is $\mathbb{P}$-*generic over* $\mathbf{M}$". Namely,

$$\Big(\text{“}G \text{ is a filter on } \mathbb{P}\text{”} \;\wedge\; \forall D \subseteq \mathbb{P}\ \big((\text{“}D \text{ dense in } \mathbb{P}\text{”} \;\wedge\; D \in \mathbf{M}) \longrightarrow D \cap G \neq \varnothing)\Big).$$

The goal of the following example is to show that filters $G \subseteq \mathbb{P}$ which are $\mathbb{P}$-*generic over* $\mathbf{M}$ do not necessarily exist. In particular, it emphasize the reason why we work with a set $\mathbf{M}$ which is a model of finitely many axioms from **ZFC**.

**Example 293.** *Let* $(\mathbb{P}, \leqslant, \mathbb{1})$ *be the following notion of forcing:*

*(1)* $\mathbb{P}$ *be the set of functions such that the domain is finite and included in* $\omega$ *and the image is included in* $\omega_1$.

$$\mathbb{P} = \Big\{p \subseteq (\omega \times \omega_1) \mid |p| < \omega \text{ and } \forall a, b, c \in \omega \ \ \Big(\big((a,b) \in p \wedge (a,c) \in p\big) \longrightarrow b = c\Big); \Big\}$$

*(2)* $p \leqslant q$ *if and only if* $p \supseteq q$ *(*$p$ *extends* $q$*, for* $p, q \in \mathbb{P}$*);*

*(3)* $\mathbb{1} = \varnothing$.

*We want to show that there is no filter* $G$ *which is* $\mathbb{P}$*-generic over* $\mathbf{V}$.

*So, towards a contradiction, assume* $G$ *is* $\mathbb{P}$*-generic over* $\mathbf{V}$*. Set* $f = \bigcup G$ *and notice that* $f$ *is a binary relation since it is a set of couples of the form* $(n, \alpha)$ *with* $n$ *an integer and* $\alpha$ *some countable ordinal. We now show that* $f \subseteq \omega \times \omega_1$ *is much better than any subset of* $\omega \times \omega_1$ *since it satisfies* $f : \ \omega \xrightarrow{onto} \omega_1$*. i.e.,*

- $f$ *is a function*
- $dom(f) = \omega$
- $ran(f) = \omega_1$.

*(1) To show that* $f$ *is a function, simply consider any integer* $n$ *and countable ordinals* $\alpha$ *and* $\beta$ *such that both couples* $(n, \alpha)$ *and* $(n, \beta)$ *belong to* $f = \bigcup G$*. Then consider any* $p, q \in G$ *such that* $(n, \alpha) \in p$ *and* $(n, \beta) \in q$*. Since* $G$ *is a filter, there exists* $r \in G$ *such that* $r \leqslant p, q$ *(i.e.,* $r$ *extends both* $p$ *and* $q$*). So, in particular both* $dom(p) \subseteq dom(r)$ *and* $dom(q) \subseteq dom(r)$ *hold which shows that* $n \in dom(r)$ *and since* $r$ *(as a function) agrees with both* $p$ *and* $q$ *on their respective domains, we have* $r(n) = p(n) = q(n)$*, which shows that* $\alpha = \beta$.

*(2)* $dom(f) = \omega$*, since for all* $n \in \omega$ *the set*

$$D_n = \{p \in \mathbb{P} \mid n \in dom(p)\}$$

*is a set which is dense in* $\mathbb{P}$ *— so both statements "*$D_n$ *is dense in* $\mathbb{P}$*" and "*$D_n$ *belongs to* $\mathbf{V}$*" are satisfied.*

*Since $D_n \in \mathbf{V}$ and $G$ is $\mathbb{P}$-generic over $\mathbf{V}$, it follows that the intersection $D_n \cap G$ is nonempty which yields the existence of some $p \in G$ with $n \in dom(p)$. Therefore, we have $n \in dom(\bigcup G) = dom(f)$ that holds for every integer $n$. Thus $dom(f) = \omega$.*

*(3) $ran(f) = \omega_1$, since for all ordinals $\alpha < \omega_1$, the set:*

$$E_\alpha = \{p \in \mathbb{P} \mid \alpha \in ran(p)\}$$

*is dense in $\mathbb{P}$ and belongs to $\mathbf{V}$, hence there exists some $p \in E_\alpha \cap G$, showing that $\alpha$ belongs to the domain of $f = \bigcup G$.*

*So, we have obtained $f = \bigcup G : \omega \xrightarrow{onto} \omega_1$, which contradicts several results[4] that we obtained working within* **ZFC**. *This shows that our assumption fails. i.e., there is no $\mathbb{P}$-generic filter over $\mathbf{V}$.*

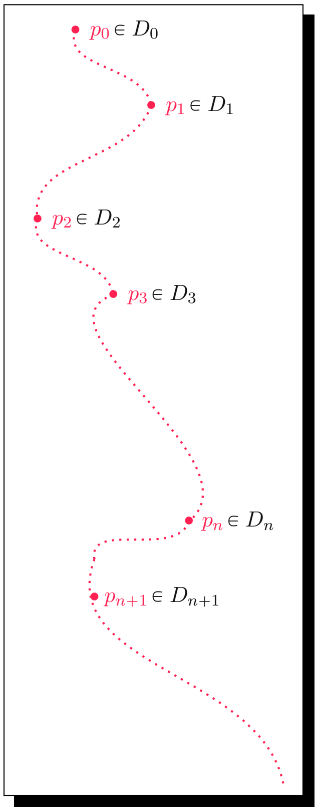
That may seem a problem at first glance, but since our aim it to consider countable transitive models of "**ZFC**", the latter result is not in our scope. To, the contrary, when $\mathbf{M}$ is a countable set as opposed to the whole universe $\mathbf{V}$, we have a positive result.

**Lemma 294** (**ZFC**). *Let $\mathbf{M}$ be any countable set, $\mathbb{P}$ any poset in $\mathbf{M}$, and $p \in \mathbb{P}$. There exists some filter $G$ which is $\mathbb{P}$-generic over $\mathbf{M}$ and such that $p \in G$.*

Notice that we do not claim that $G$ belongs to $\mathbf{M}$. In fact, most of the time we will have $G \notin \mathbf{M}$, simply because when $G \in \mathbf{M}$ holds, the generic extension obtained by forcing is no different than the ground model $\mathbf{M}$ one starts with, and therefore it is useless.

*Proof of Lemma* 294:

[4] For instance, that $\omega_1$ is a regular cardinal; or that any surjection $s : A \xrightarrow{onto} B$ yields an injection $i : B \xrightarrow{1-1} A$ such that $s \circ i = id$. .




Let us consider (in $\mathbf{V}$) an enumeration $(D_n)_{n\in\omega}$ of the sets of the form $D$ that satisfy both

(1) $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$. (2) $D \in \mathbf{M}$.

Notice that such an enumeration exists since we are working within **ZFC** and that it is countable since $\mathbf{M}$ is a countable set.

Let $p_0 \in D_0$ be such that $p_0 \leqslant p$; we define by induction on $\omega$ a sequence $(p_n)_{n\in\omega}$ such that:

$$p_{n+1} \leqslant p_n \qquad \text{and} \qquad p_{n+1} \in D_{n+1}.$$

Let us consider $G$, the filter generated by $(p_n)_{n\in\omega}$:

$$G = \{q \in \mathbb{P} \mid \exists n \in \omega \;\; p_n \leqslant q\}.$$

Since the formula " $D$ is dense in $\mathbb{P}$ " is absolute — i.e., " $D$ is dense in $\mathbb{P}$ "$\longleftrightarrow$ $\big($" $D$ is dense in $\mathbb{P}$ "$\big)^{\mathbf{M}}$ — $G$ is a filter whose intersection with the dense sets of $\mathbf{M}$ is nonempty. $G$ is thus $\mathbb{P}$-generic over $\mathbf{M}$ and $p \in G$.

□ 294

We said that in most cases, the generic filter does not belong to the ground model $\mathbf{M}$. Indeed, the cases that matter the least are those where filter exists inside the ground model. The next lemma gives an easy condition that the poset $\mathbb{P}$ needs to satisfy in order for any filter $G$ to not belong to the ground model $\mathbf{M}$.

**Lemma 295** (**ZFC**)**.** *Let* $\mathbf{M}$ *be any transitive model of "***ZFC***", and* $\mathbb{P} \in \mathbf{M}$ *a notion of forcing that satisfies* $\forall p \in \mathbb{P}\ \exists r, q \in \mathbb{P}\ \ \big(q \leqslant p \ \wedge\ r \leqslant p \ \wedge\ q \perp r\big)$.

*If* $G$ *is* $\mathbb{P}$*-generic over* $\mathbf{M}$*, then* $G \notin \mathbf{M}$.

Notice that in this lemma, we do not simply consider any transitive set $\mathbf{M}$, but rather a transitive model of "**ZFC**". The reason for this, is that we need $\mathbf{M}$ to satisfy some very basic properties. For instance, we need that given $\mathbb{P}$ and $G$ that belong to $\mathbf{M}$, the set $\mathbb{P} \setminus G$ also belongs to $\mathbf{M}$.

*Proof of Lemma* 295: . Suppose, for the sake of contradiction, that $G \in \mathbf{M}$.

- We let $D = \mathbb{P} \setminus G$ and $\mathbf{M}$ satisfy enough axioms from **ZFC** such that $D \in \mathbf{M}$ holds.
- We then show that $D$ is dense in $\mathbb{P}$: take any $p \in \mathbb{P}$, there exist $r, q \in \mathbb{P}$ such that $q \leqslant p$, $r \leqslant p$ and $q \perp r$. But it cannot be the case that both $r$ and $q$ belong to $G$, for otherwise,

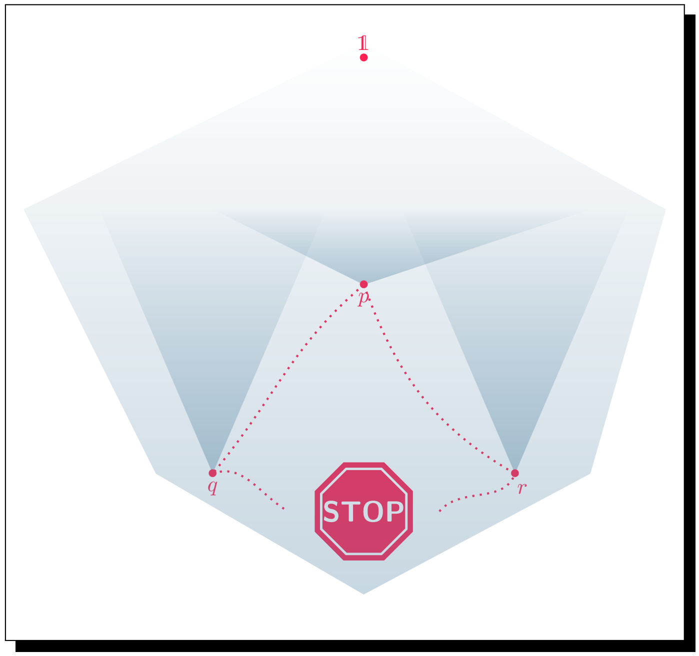



Figure 13.2: $p \geqslant q, r$ with $q \perp r$.

since $G$ is a filter on $\mathbb{P}$, there would exist $s \in \mathbb{P}$ such that $s \leqslant r$ and $s \leqslant q$ both hold, which would contradict the fact that $p$ and $r$ are incompatible. It follows that either $r$ or $q$ is a member of $D$, therefore $D$ is dense in $\mathbb{P}$.

Finally, we have the following contradiction:

(1) $G$ is $\mathbb{P}$-generic over $\mathbf{M}$ (2) $D \in \mathbf{M}$ and $D$ is dense (3) $D \cap G = (\mathbb{P} \setminus G) \cap G = \varnothing.$

□ 295

We need a last result which seems technical at first glance but will prove extremely useful later on.

**Definition 296.** *Let $\mathbb{P}$ be any poset, $E \subseteq \mathbb{P}$, and $p \in \mathbb{P}$.*

$$E \text{ is } \textbf{\textit{dense below}}\ p \iff \forall q \leqslant p\ \exists r \in E \quad r \leqslant q.$$

So, being dense below $p$ is really what it says it is: being dense but only with regards to the sub-poset formed of all forcing conditions that lies below $p$.

**Lemma 297.** *Let* $\mathbf{M}$ *be a transitive model of "***ZFC***",* $\mathbb{P}$ *a notion of forcing such that* $\mathbb{P} \in \mathbf{M}$*, and* $G$ *be* $\mathbb{P}$*-generic over* $\mathbf{M}$*. Let also* $p \in \mathbb{P}$ *and* $E \subseteq \mathbb{P}$ *be such that* $E \in \mathbf{M}$*. Then,*

- *either* $G \cap E \neq \varnothing$*, or*
- *there exists* $q \in G$ *such that for all* $r \in E$*,* $r \perp q$*.*

*Furthermore, if* $E$ *is dense below* $p \in G$*, then* $G \cap E \neq \varnothing$*.*

This last statement: "every set which is dense below some element which belongs to the generic filter $G$ also intersects this filter $G$" will be used time and again.

*Proof of Lemma 297:* To prove the first part of the lemma, let

$$D = \underbrace{\{s \in \mathbb{P} \mid \exists r \in E \ s \leqslant r\}}_{D_{\leqslant E}} \cup \underbrace{\{s \in \mathbb{P} \mid \forall r \in E \ s \perp r\}}_{D_{\perp E}}.$$

First, we notice that $D$ is dense. Indeed, take any $s \in \mathbb{P}$. Then,

(1) either there exists $r \in E$ such that $r$ and $s$ are compatible, and so there exists $q \in \mathbb{P}$ with $q \leqslant s$ and $q \leqslant r$, which implies that $q \in D_{\leqslant E} \subseteq D$;

(2) or, for all $r \in E$, we have $r \perp s$ and thus $s \in D_{\perp E} \subseteq D$. Since $s \leqslant s$ holds, this shows that $D$ is dense in $\mathbb{P}$.

Moreover, $D \in \mathbf{M}$ holds because $E \in \mathbf{M}$ and $\mathbf{M}$ is a model of "**ZFC**" which contains enough axioms to show that $D$ exists. As a result of $G$ being $\mathbb{P}$-generic over $\mathbf{M}$, its intersection with $D$ is non-empty. Take any $q \in D \cap G$. Since $q \in D$,

(1) either $q \in D_{\leqslant E}$, i.e., there exists $r \in E$ such that $q \leqslant r$. In that case, since $G$ is a filter, $r \in G$ and $G \cap E \neq \varnothing$;

(2) or, $q \in D_{\perp E}$, i.e., for all $r \in E$, $q \perp r$. In that case, there exists $q \in G$ such that for all $r \in E$, $r \perp q$.

For the second part of the lemma, we assume $p \in G$ and $E$ is dense below $p$. Towards a contradiction we also assume $G \cap E = \varnothing$. Then, the previous result provides some $q \in G$ such that for all $r \in E$, $r \perp q$.
Since $G$ is a filter, there exists $s \in G$ such that $s \leqslant p$ and $s \leqslant q$. But $E$ is dense below $p$, so there exists $r \in E$ such that $r \leqslant s$. We have obtained $r \in E$ such that $r \leqslant q$. This contradicts the property that $q$ satisfies: $\forall r' \in E \ \ r' \perp q$.

□ 297

Our main goal will now be as follows: start from

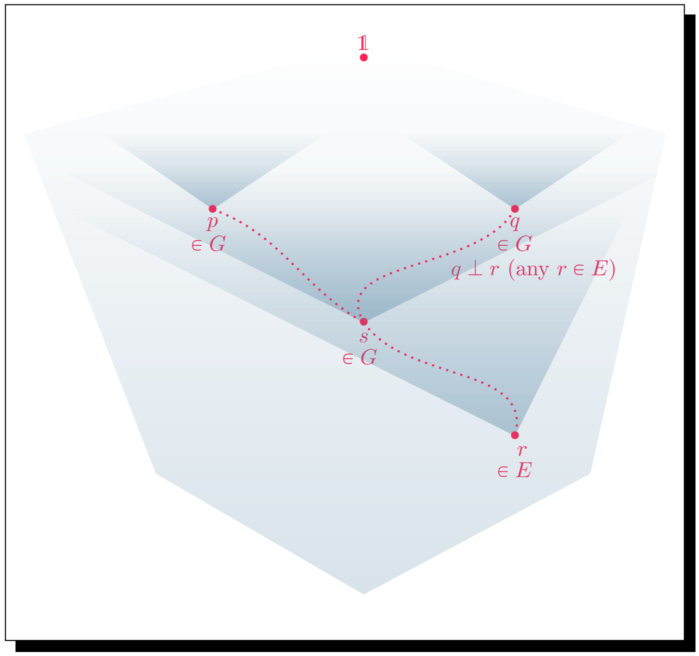



Figure 13.3: $p \geqslant s$, $q \geqslant s$ with $p, q, s \in G$ and $s \geqslant r \in E$ since $E$ is dense below $p$.

(1) any $\mathbf{M}$ which is some *c.t.m.* of "**ZFC**" and

(2) any filter $G$ which is $\mathbb{P}$-generic over $\mathbf{M}$,

and construct a *c.t.m.* of "**ZFC**" $\mathbf{M}[G]$ — called a generic extension of $\mathbf{M}$ — which satisfies the following:

(1) $\mathbf{M} \subseteq \mathbf{M}[G]$; (2) $(\mathbf{On})^{\mathbf{M}} = (\mathbf{On})^{\mathbf{M}[G]}$; (3) $G \in \mathbf{M}[G]$.

# Chapter 14

# $\mathbb{P}$-names and Generic Extensions

## 14.1 $\mathbb{P}$-names

We will see that some of the elements that belong to the generic extension $\mathbf{M}[G]$ will be brand new sets. In the sense that they do not exist inside $\mathbf{M}$, but they are created when going from $\mathbf{M}$ to the the generic extension $\mathbf{M}[G]$. This seems an obvious remark, since any time one considers the strict inclusion of a set into some other one ($A \subsetneq B$) there are elements that belong to the bigger one but not the smaller one.

However, the main difference here is that each and every one of these new elements will happen to already have a *name* in $\mathbf{M}$. They do not exist in $\mathbf{M}$ but in $\mathbf{M}$, they could be called by their names, ALTHOUGH THEY DO NOT EXIST! It is as if in $\mathbf{M}$ one can call many names without knowing what one talks about. Only with the help of a key that allows to decode the names and give rise to the sets they denominate that one can see the relation between the name and the object it depicts.

To view things the other way round, every set that belongs to $\mathbf{M}[G]$ already pre-exists in $\mathbf{M}$ in that it already has a name, even though a key that is required to decode and identify it is missing in $\mathbf{M}$ (this key is the filter $G$).

**Definition 298** ($\mathbb{P}$-name)**.** *$\tau$ is a $\mathbb{P}$-name if and only if $\tau$ is a binary relation and for all $(\sigma, p) \in \tau$, $\sigma$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.*

Notice that $\varnothing$ satisfies this definition, hence $\varnothing$ is a $\mathbb{P}$-name.

Formally, $\mathbb{P}$-names are defined recursively. First consider the following binary relation $E$ on $\mathbb{P}$-names:

$$\sigma E \tau \Longleftrightarrow \exists p \in \mathbb{P}\ (\sigma, p) \in \tau.$$

$E$ is well-founded since:

$$\sigma E \tau \Longrightarrow rk(\sigma) < rk(\tau).$$

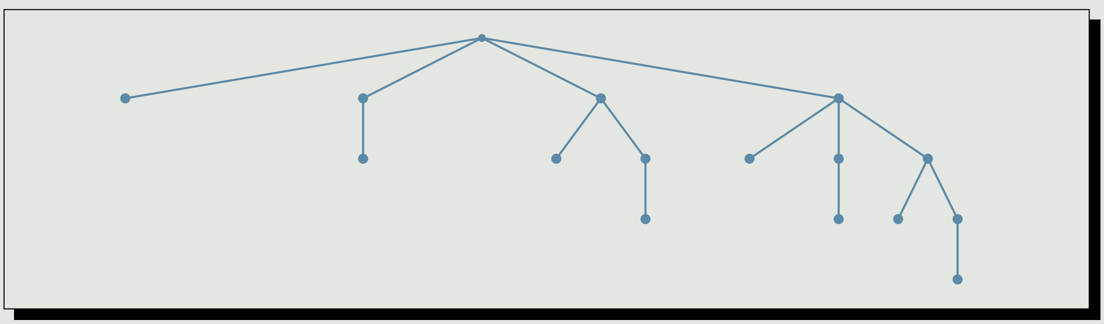
We set:

$$\mathbf{F}(\tau) = \begin{cases} 1, & \text{if } \forall x \in \tau\big(\text{“}x \text{ is a couple } (x_1, x_2)\text{”} \wedge x_2 \in \mathbb{P} \wedge \mathbf{F}(x_1) = 1\big); \\ 0, & \text{otherwise.} \end{cases}$$

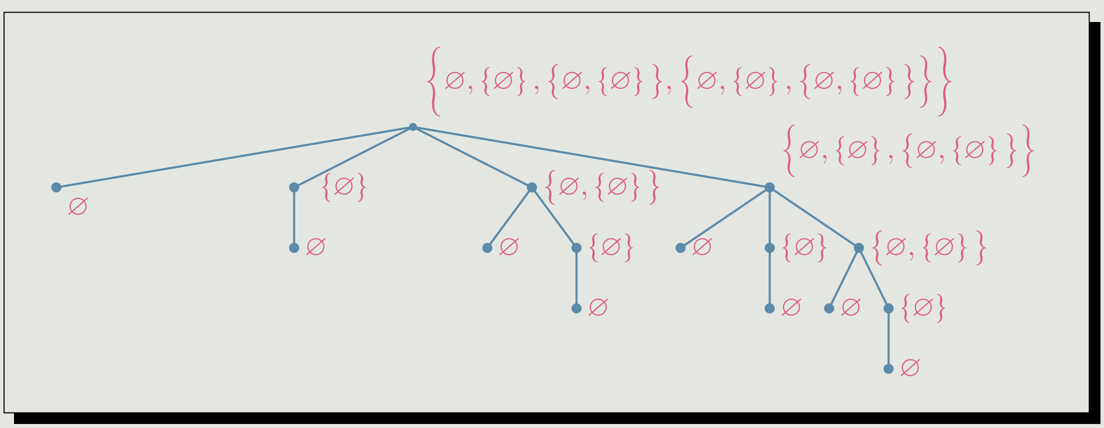
We may define $\mathbf{F}$ as $\mathbf{F}(\tau) = \mathbf{H}(\mathbf{F} \restriction_{\text{pred}_E(\tau)}, \tau, \mathbb{P})$ where all notions used to define $\mathbf{H}$ are $\Delta_0^{0_{rud}}$, hence $\mathbf{H}$ is absolute for transitive models of “**ZFC**”. Then, the class of all $\mathbb{P}$-names is the set

$$\{\tau \mid \mathbf{F}(\tau) = 1\}.$$

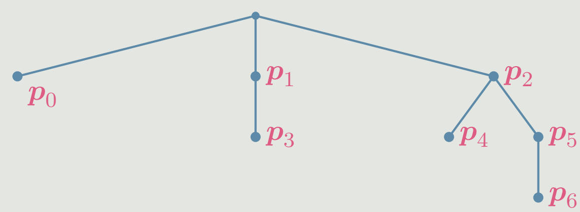
**Example 299.** *In order to get the right intuition about $\mathbb{P}$-names, it is fruitful to go back to the way we represented well-founded sets by well-founded trees. For instance, in Example 152 where we presented a tree*



*that represents the ordinal 4 when we associate to each node $\mathfrak{n}$ the set $\widehat{\mathfrak{n}} = \{\widehat{\mathfrak{c}} \mid \mathfrak{c}$ is a child of $\mathfrak{n}\}$:*



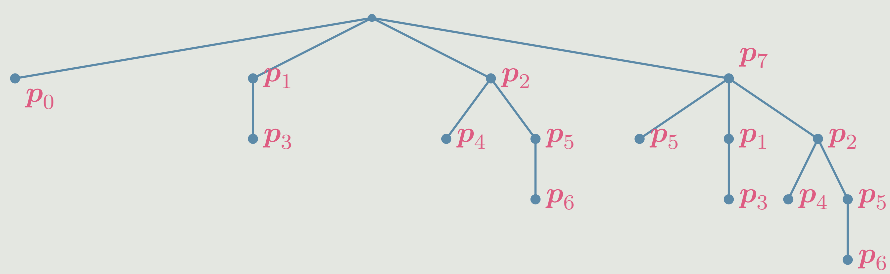

*When represented by trees, $\mathbb{P}$-names are not just well-founded trees, but rather some particular **colored** well-founded trees: those whose nodes* — except for the root — *are* “colored” by forcing conditions. *For instance:*




*This tree represents the* $\mathbb{P}$*-name:*

$$\Big\{(\varnothing, p_0)\ ,\ \ \Big(\big\{(\varnothing, p_3)\big\}\ ,\ p_1\Big)\ ,\ \ \Big(\Big\{(\varnothing, p_4)\ ,\ \Big(\big\{(\varnothing, p_6)\big\}\ ,\ p_5\Big)\Big\}\ ,\ p_2\Big)\Big\}$$

*Some* $\mathbb{P}$*-name which is a coloring of the set* 4*:*




*This tree represents the* $\mathbb{P}$*-name:*

$$\Bigg\{(\varnothing, p_0)\ ,\ \Big(\big\{(\varnothing, p_3)\big\}\ ,\ p_1\Big)\ ,\ \Big(\Big\{(\varnothing, p_4)\ ,\ \Big(\big\{(\varnothing, p_6)\big\}\ ,\ p_5\Big)\Big\}\ ,\ p_2\Big),$$
$$\Bigg(\Big\{(\varnothing, p_5)\ ,\ \Big(\big\{(\varnothing, p_3)\big\}\ ,\ p_1\Big)\ ,\ \Big(\Big\{(\varnothing, p_4)\ ,\ \Big(\big\{(\varnothing, p_6)\big\}\ ,\ p_5\Big)\Big\}\ ,\ p_2\Big)\Big\}, p_7\Bigg)\ \Bigg\}$$

**Definition 300.** *Let* $\mathbf{V}^{\mathbb{P}} = \{\mathbb{P}\text{-names}\}$. *If* $\mathbf{M}$ *is a transitive model of "***ZFC***", then*

$$\mathbf{M}^{\mathbb{P}} = \mathbf{M} \cap \mathbf{V}^{\mathbb{P}}.$$

*By absoluteness,*

$$\mathbf{M}^{\mathbb{P}} = \left\{\tau \in \mathbf{M} \mid (\tau \text{ is a } \mathbb{P}\text{-name})^{\mathbf{M}}\right\}.$$

## 14.2 Generic Extensions

Starting from $\mathbf{M}$ any transitive model of "**ZFC**", we create another model — known as a generic extension — by considering all the $\mathbb{P}$-names that belong to $\mathbf{M}$ ($\mathbf{M}^{\mathbb{P}} = \mathbf{M} \cap \mathbf{V}^{\mathbb{P}}$) and "unscrambling" them with the use a filter $G$ — that is generic over $\mathbf{M}$ — which plays the role of a decryption key.

**Definition 301** (Generic Extension)**.** *Let* $(\mathbb{P}, \leqslant, \mathbb{1})$ *be a notion of forcing*

*(1) Given any* $\tau \in \mathbf{V}^{\mathbb{P}}$, *and* $G \subseteq \mathbb{P}$ *a filter, we recursively define*

$$(\tau)_G = \{(\sigma)_G \mid \exists p \in G \ (\sigma, p) \in \tau\}.$$

*(2) Given* $\mathbf{M}$ *any transitive model of "***ZFC***",* $\mathbb{P} \in \mathbf{M}$ *and* $G \subseteq \mathbb{P}$ *a filter, we define*

$$\mathbf{M}[G] = \left\{(\tau)_G \mid \tau \in \mathbf{M}^{\mathbb{P}}\right\}.$$

**Notation 302.** *Given* $\mathbf{M}$ *any transitive model of "***ZFC***",* $\mathbb{P} \in \mathbf{M}$, $G \subseteq \mathbb{P}$ *any filter* $\mathbb{P}$*-generic over* $\mathbf{M}$, *and* $x \in \mathbf{M}[G]$, *we write* $\underset{\sim}{x}$ *for any* $\mathbb{P}$*-name for* $x$. *i.e.,*

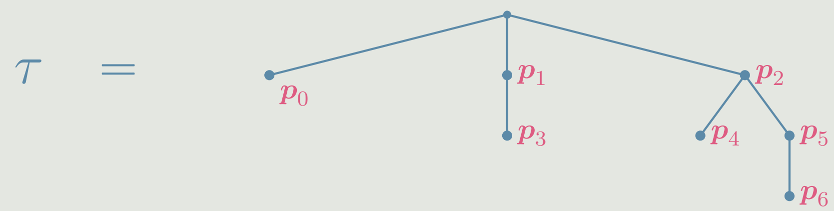
$$\underset{\sim}{x} \in \mathbf{M}^{\mathbb{P}} \text{ and } (\underset{\sim}{x})_G = x.$$

There are two different ways of looking at $\mathbb{P}$-names:

- either we start from the ground model $\mathbf{M}$, pick a $\mathbb{P}$-name $\tau$, and move forward to the generic extension $\mathbf{M}[G]$ to deal with $(\tau)_G$;

- or we start from the generic extension $\mathbf{M}[G]$, pick an element $x$, and move backward to the ground model $\mathbf{M}$ to deal with a $\mathbb{P}$-name $\underset{\sim}{x}$ that has produced $x$.

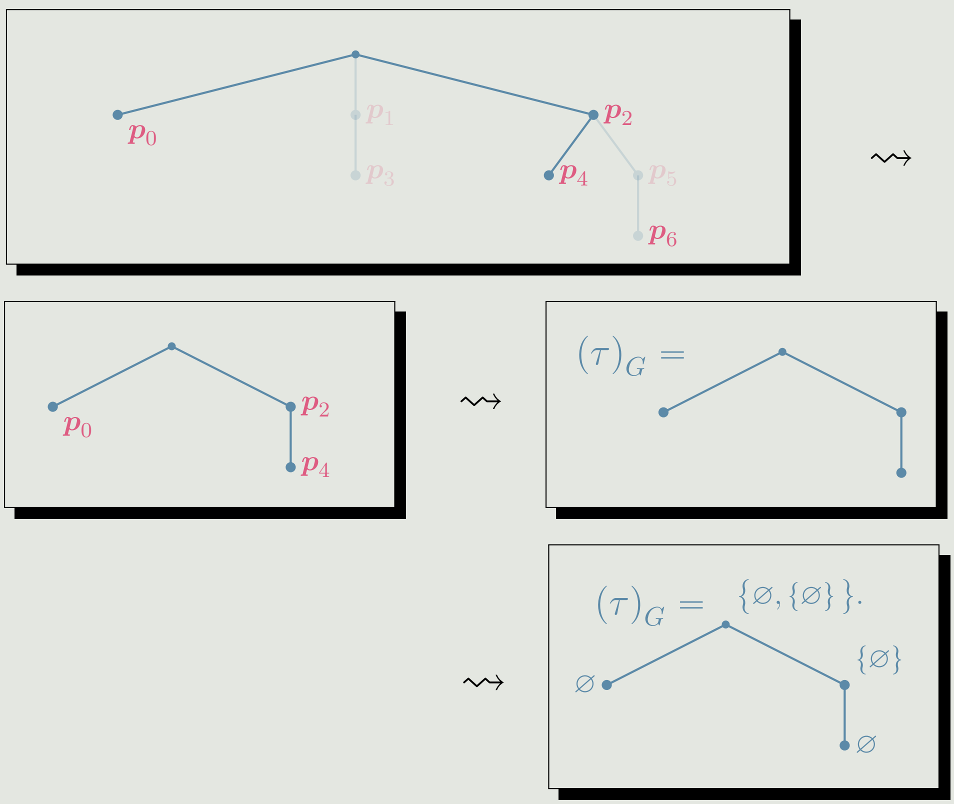
**Example 303.** *Consider the following* $\mathbb{P}$*-name* $\tau$ *that was introduced in Example 299:*




*This tree represents the* $\mathbb{P}$*-name:*

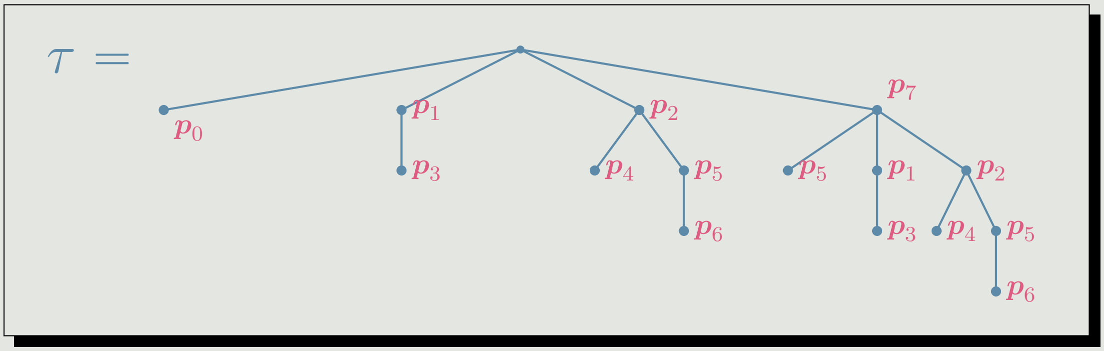
$$\Big\{(\varnothing, p_0)\ ,\ \Big(\big\{(\varnothing, p_3)\big\}\ ,\ p_1\Big)\ ,\ \Big(\Big\{(\varnothing, p_4)\ ,\ \Big(\big\{(\varnothing, p_6)\big\}\ ,\ p_5\Big)\Big\}\ ,\ p_2\Big)\Big\}$$

*If for each integer* $n$*,* $p_n \in G \iff n$ *is even. Then* $(\tau)_G$ *is obtained by removing the nodes colored by forcing conditions not in the filter, then getting rid of the coloring:*




*So, we obtain* $(\tau)_G = \{\varnothing, \{\varnothing\}\}$*.*

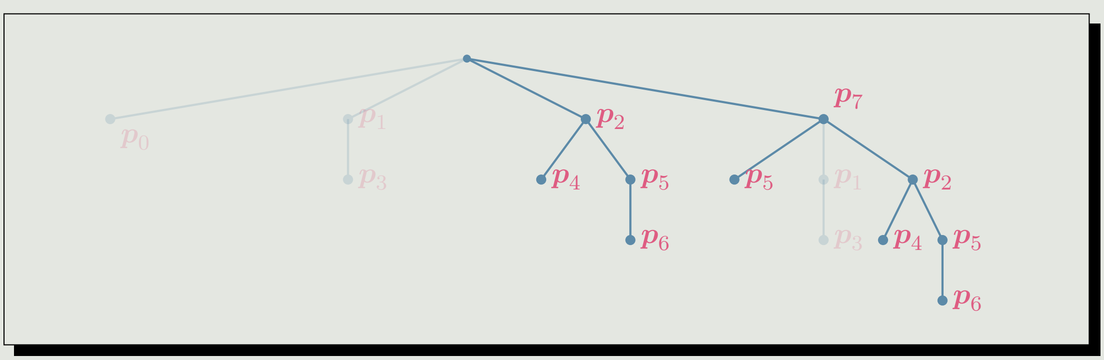
**Example 304.** *Consider the following* $\mathbb{P}$*-name* $\tau$ *that was introduced in Example 299:*



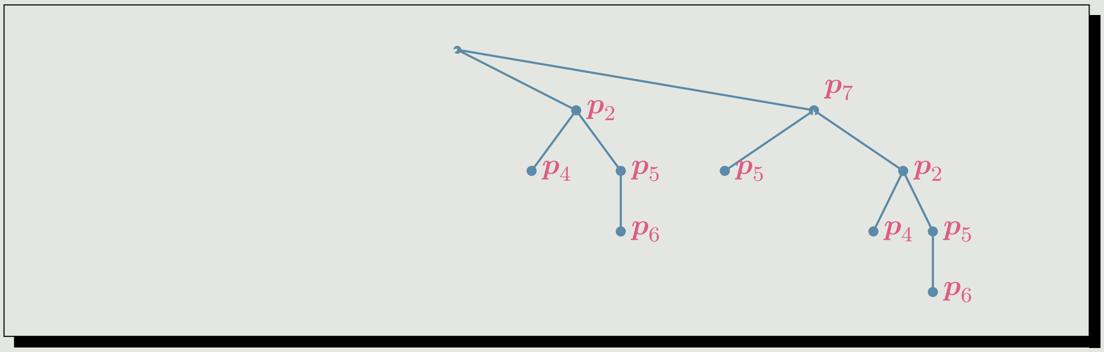

*This tree represents the* $\mathbb{P}$*-name:*

$$\tau = \left\{ \begin{array}{l} (\varnothing, p_0) \ , \\ \quad \Big( \{(\varnothing, p_3)\} \ , \ p_1 \Big) \ , \quad \left( \left\{ (\varnothing, p_4) \ , \ \Big( \{(\varnothing, p_6)\} \ , \ p_5 \Big) \right\} \ , \ p_2 \right), \\ \left( \left\{ (\varnothing, p_5) \ , \ \Big( \{(\varnothing, p_3)\} \ , \ p_1 \Big) \ , \ \left( \left\{ (\varnothing, p_4) \ , \ \Big( \{(\varnothing, p_6)\} \ , \ p_5 \Big) \right\} \ , \ p_2 \right) \right\}, p_7 \right) \end{array} \right\}$$

*If for each integer* $n$*,* $p_n \in G \iff n \notin \{0, 1, 3\}$*. Then* $(\tau)_G$ *is obtained by removing the nodes colored by forcing conditions not in the filter, then getting rid of the coloring:*




$\rightsquigarrow$

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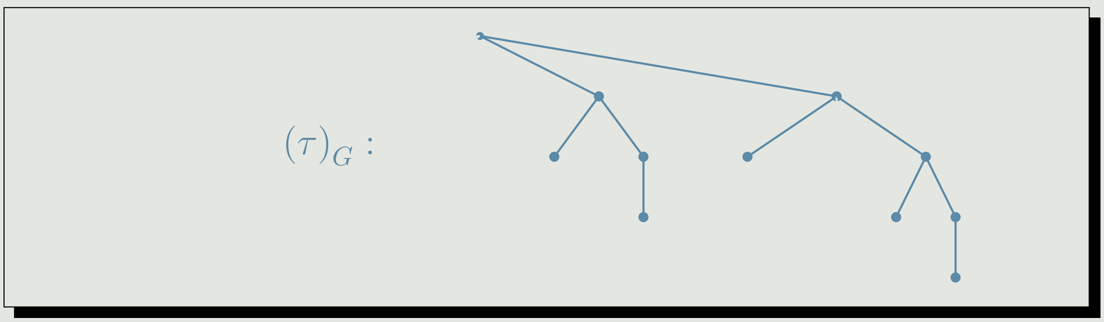


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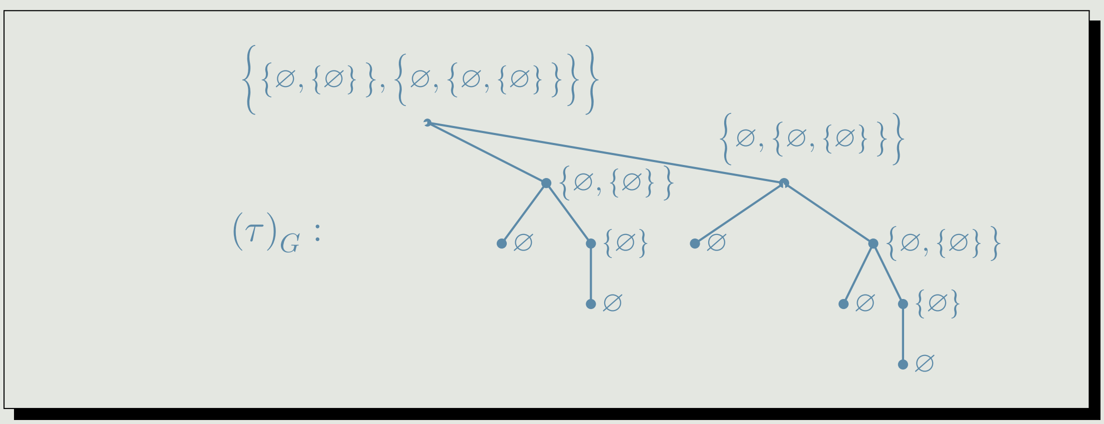
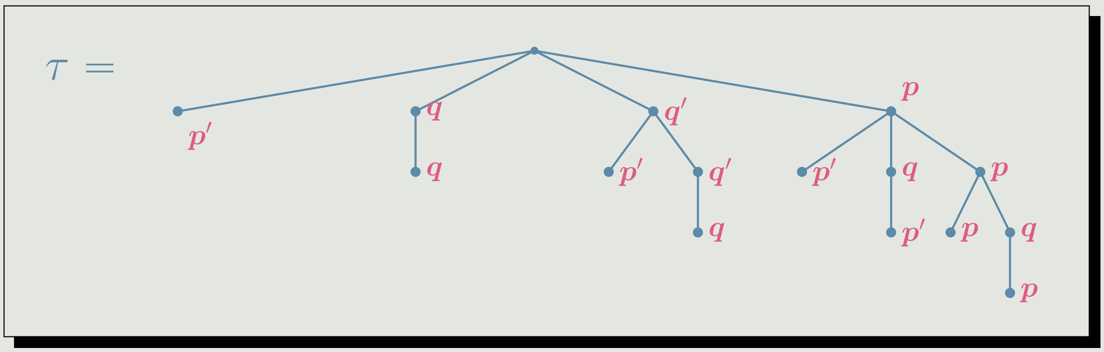



$\rightsquigarrow$

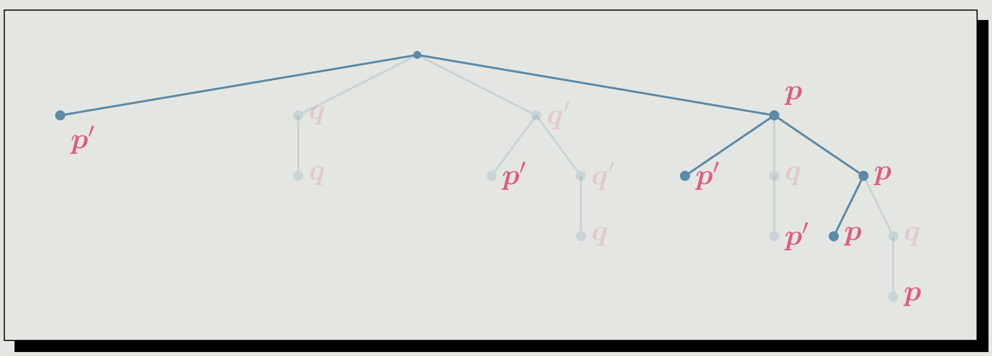
*So, we obtain*

$$(\tau)_G = \Big\{\{\varnothing,\{\varnothing\}\},\Big\{\varnothing,\{\varnothing,\{\varnothing\}\}\Big\}\Big\}.$$

**Example 305.** *Consider*




*with $p, p' \in G$, but $q, q' \notin G$. This yields the following tree:*



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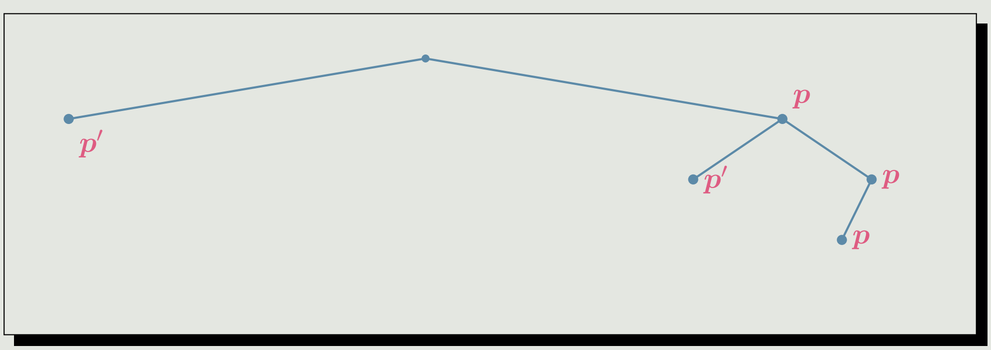

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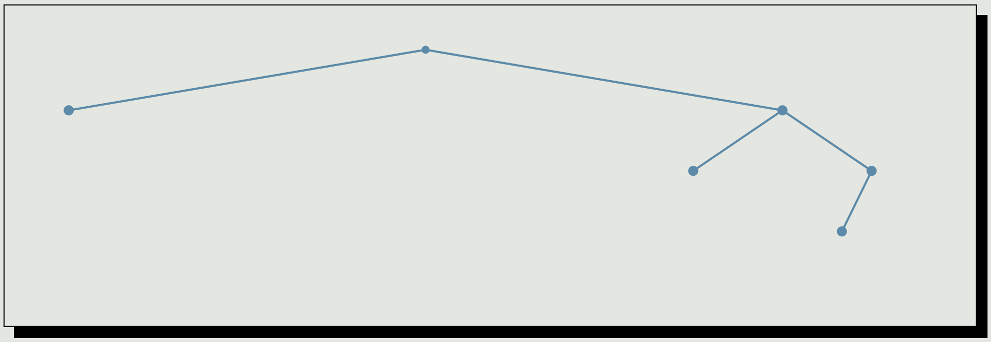
*and by dropping the forcing conditions:*



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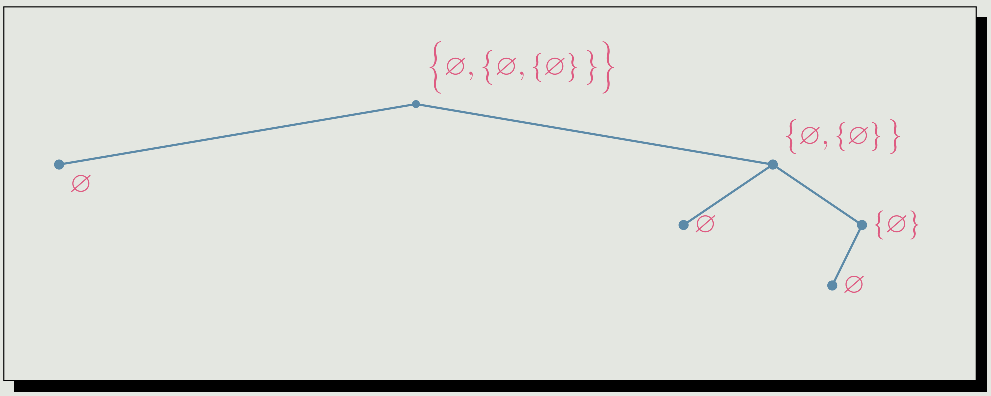



$\rightsquigarrow$

*which yields*

$$(\tau)_G = \Big\{\varnothing, \{\varnothing, \{\varnothing\}\}\Big\}.$$

**Definition 306** (Canonical ℙ-names)**.** *By* $\in$*-induction, we define for any* $x \in \mathbf{M}$*,*

$$\check{x} = \{(\check{y}, \mathbb{1}) \mid y \in x\}.$$

*We will also consider*

$$\Gamma = \{(\check{p}, p) \mid p \in \mathbb{P}\}.$$

The ℙ-names $\check{x}$ are called *canonical names* for sets that belong to $\mathbf{M}$, and the ℙ-names $\Gamma$ is called the *canonical name* for the filter $G$.

**Lemma 307.** *Let* $(\mathbb{P}, \leqslant, \mathbb{1})$ *be a notion of forcing, and* $G \subseteq \mathbb{P}$ *a filter.*

*(1)* $(\check{x})_G = x$

*(2)* $(\Gamma)_G = G$.

An immediate consequence of this lemma is that as long as $\mathbf{M}$ is a model which is closed under the "check" operation[1] — which comes to asking that $\mathbf{M}$ be a model of "**ZFC**", where "**ZFC**" contains the axioms that are necessary to prove that $\mathbf{V}$ is closed under the "check" operation — then both $\mathbf{M} \subseteq \mathbf{M}[G]$ and $G \in \mathbf{M}[G]$ hold.

*Proof of Lemma* 307:

(1) By $\in$-induction, since $\check{\varnothing} = \varnothing$ and $(\varnothing)_G = \varnothing$:

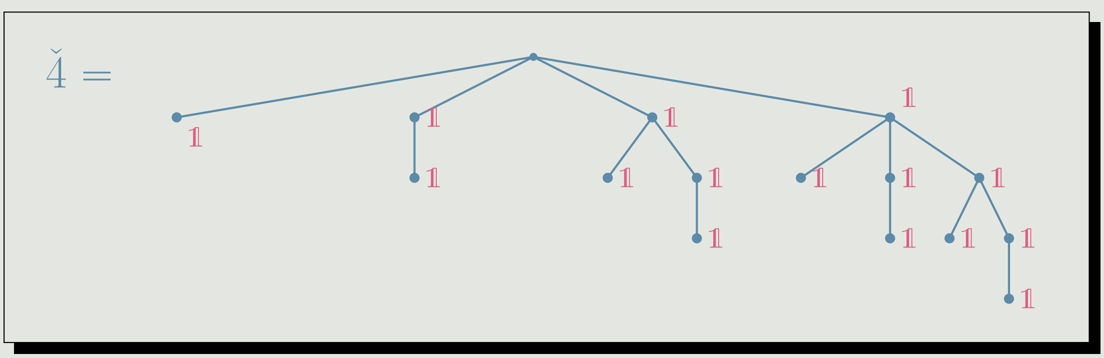
$$\begin{aligned}(\check{x})_G &= \{(\check{y})_G \mid \exists p \in G\ (\check{y}, p) \in \check{x}\}\\ &= \{(\check{y})_G \mid (\check{y}, \mathbb{1}) \in \check{x}\}\\ &= \{y \mid y \in x\}\\ &= x.\end{aligned}$$

(2)

$$\begin{aligned}(\Gamma)_G &= \{(\check{p})_G \mid \exists p \in G\ (\check{p}, p) \in \Gamma\}\\ &= \{p \mid p \in G\}\\ &= G.\end{aligned}$$

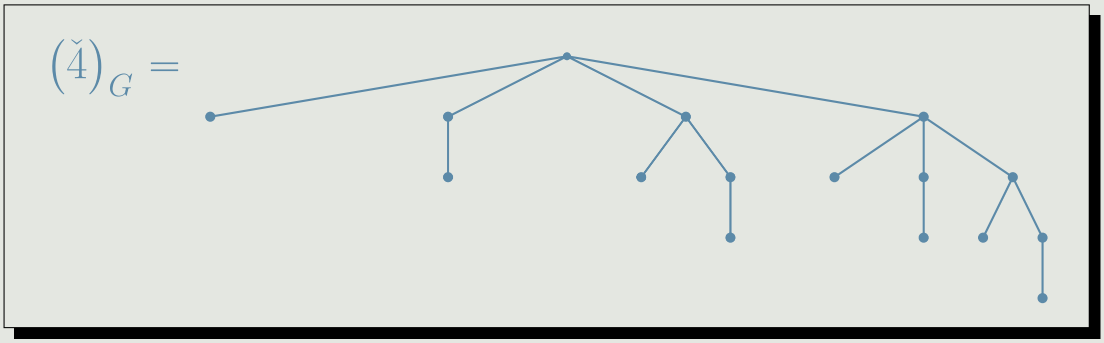
□ 307

**Example 308.** *For instance,* $\check{4}$ *corresponds to:*

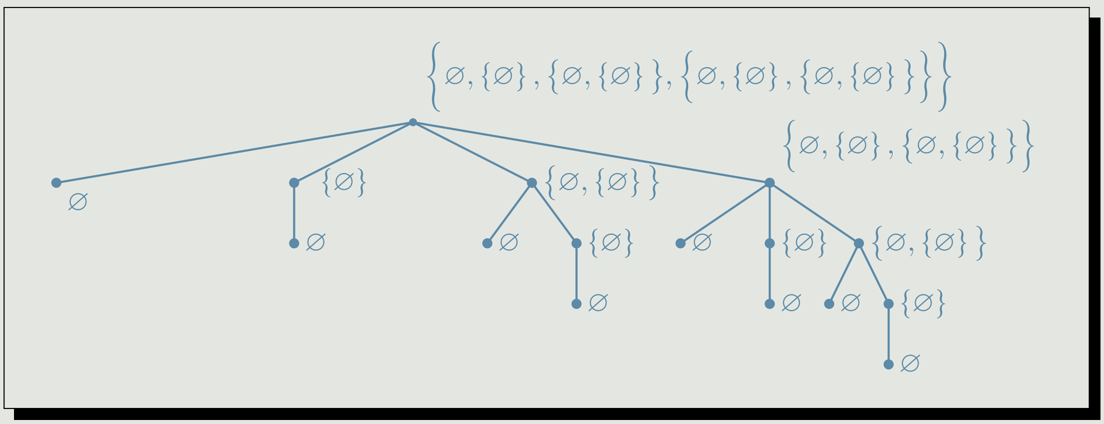



*which yields*

[1]This means that $\mathbf{M}$ satisfies $\check{x} \in \mathbf{M}$ holds for every $x \in \mathbf{M}$



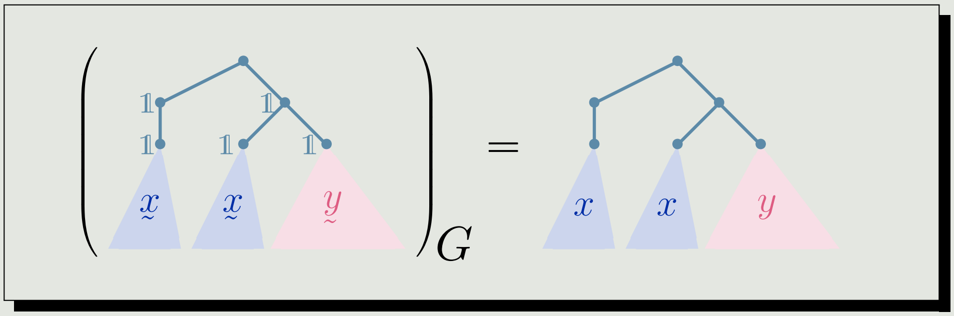

*which is nothing but the ordinal* 4*:*




**Example 309.** *We define couple* : $\mathbf{M}^{\mathbb{P}} \times \mathbf{M}^{\mathbb{P}} \to \mathbf{M}^{\mathbb{P}}$ *so that given any* $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbf{M}^{\mathbb{P}}$*, and any* $G$ $\mathbb{P}$*-generic over* $\mathbf{M}$*, couple*$(\underset{\sim}{x}, \underset{\sim}{y}) = \tau$ *with* $(\tau)_G = \big((\underset{\sim}{x})_G, (\underset{\sim}{y})_G\big)$*. This is the canonical name*

$$\tau = \widecheck{\{\{\underset{\sim}{x}\}, \{\underset{\sim}{x}, \underset{\sim}{y}\}\}} = \Big\{\Big(\{(\underset{\sim}{x}, \mathbb{1})\}, \mathbb{1}\Big), \Big(\{(\underset{\sim}{x}, \mathbb{1}), (\underset{\sim}{y}, \mathbb{1})\}\, \mathbb{1}\Big)\Big\},$$

*as shown in the picture below.*



**Lemma 310.** *If* $\mathbf{M}$ *is a transitive model of "***ZFC***",* $\mathbb{P} \in \mathbf{M}$ *is a notion of forcing, and* $G$ *is* $\mathbb{P}$*-generic over* $\mathbf{M}$*, then*

*(1)* $\mathbf{M} \subseteq \mathbf{M}[G]$

*(2)* $G \in \mathbf{M}[G]$.

*Proof of Lemma* 310: Both statements are consequence of previous Lemma 307. Indeed, since for all $x \in \mathbf{M}$, one has $\check{x} \in \mathbf{M}^{\mathbb{P}}$ and $x = (\check{x})_G \in \mathbf{M}[G]$, it follows that $\mathbf{M} \subseteq \mathbf{M}[G]$. Moreover, $\Gamma \in \mathbf{M}^{\mathbb{P}}$, so $G = (\Gamma)_G \in \mathbf{M}[G]$.

□ 310

**Lemma 311.** *Let* $\mathbf{M}$ *be a transitive model of "***ZFC***",* $\mathbb{P}$ *a notion of forcing, and* $G$ *be* $\mathbb{P}$*-generic over* $\mathbf{M}$*. Then*

*(1)* $\mathbf{M}[G]$ *is transitive,*

*(2) if* $\mathbf{N}$ *is a transitive model of "***ZFC***" with* $\mathbf{M} \subseteq \mathbf{N}$ *such that* $G \in \mathbf{N}$*, then* $\mathbf{M}[G] \subseteq \mathbf{N}$*.*

*Proof of Lemma* 311:

(1) Given any $x \in (\tau)_G \in \mathbf{M}[G]$, by transitivity of $\mathbf{M}$, there exists $\sigma \in \mathbf{M}^{\mathbb{P}}$ and $p \in \mathbb{P}$ (in fact $p \in G$) such that $(\sigma, p) \in \tau$ and $x = (\sigma)_G$. Thus, $x = (\sigma)_G \in \mathbf{M}[G]$.

(2) Recall that

$$(\tau)_G = \{(\sigma)_G \mid \exists p \in G \ (\sigma, p) \in \tau\},$$

and that we defined the class-function $F$ as:

$$\begin{aligned}\mathbf{F} : \mathbf{M}^{\mathbb{P}} &\longrightarrow \mathbf{M}[G]\\ \tau &\longmapsto (\tau)_G\end{aligned}$$

Therefore:

$$\mathbf{F}(\tau) = \mathbf{H}\left(\mathbf{F}\restriction_{\mathrm{pred}_E(\tau)}, G, \tau\right).$$

Since $\mathbf{H}$ is absolute, we have $\big((\tau)_G\big)^{\mathbf{N}} = (\tau)_G$ and therefore $\mathbf{M}[G] \subseteq \mathbf{N}$.

□ 311

If $\mathbf{M}[G]$ is a transitive model of "**ZFC**", the second part of this Lemma states that $\mathbf{M}[G]$ is the smallest transitive model of "**ZFC**" such that both $\mathbf{M} \subseteq \mathbf{M}[G]$ and $G \in \mathbf{M}[G]$ hold.

**Lemma 312.** *Let $\mathbf{M}$ be a transitive model of "**ZFC**", $\mathbb{P}$ a notion of forcing with $\mathbb{P} \in \mathbf{M}$, and $G$ be $\mathbb{P}$-generic over $\mathbf{M}$.*

$$(\mathbf{On})^{\mathbf{M}} = (\mathbf{On})^{\mathbf{M}[G]}$$

*Proof of Lemma 312:* By induction on the rank, we prove that for all $\mathbb{P}$-name $\tau \in \mathbf{M}^{\mathbb{P}}$, one has $rk\big((\tau)_G\big) \leqslant rk(\tau)$. Indeed

$$(\tau)_G = \{(\sigma)_G \mid \exists p \in G \ \ (\sigma, p) \in \tau\},$$

it follows that

$$rk\big((\tau)_G\big) = \sup\left\{rk\big((\sigma)_G\big) + 1 \mid \exists p \in G \ \ (\sigma, p) \in \tau\right\}.$$

By inductive hypothesis, one has

$$\begin{aligned} rk\big((\tau)_G\big) &\leqslant \sup\{rk(\sigma) + 1 \mid \exists p \in G \ \ (\sigma, p) \in \tau\}\\ &\leqslant \sup\{rk(\sigma) + 1 \mid \exists p \in \mathbb{P} \ \ (\sigma, p) \in \tau\}\\ &\leqslant \sup\{rk(z) + 1 \mid z \in \tau\}\\ &\leqslant rk(\tau).\end{aligned}$$

In particular, for any ordinal $\alpha \in \mathbf{M}[G]$, and any $\mathbb{P}$-name $\underset{\sim}{\alpha} \in \mathbf{M}$, one has $\alpha \leqslant rk(\underset{\sim}{\alpha}) \in \mathbf{M}$, and since $\mathbf{M}$ is transitive, it follows that $\alpha \in \mathbf{M}$.

This gives $(\mathbf{On})^{\mathbf{M}[G]} \subseteq (\mathbf{On})^{\mathbf{M}}$ which combined with $\mathbf{M} \subseteq \mathbf{M}[G]$ yields $(\mathbf{On})^{\mathbf{M}} = (\mathbf{On})^{\mathbf{M}[G]}$.

□ 312

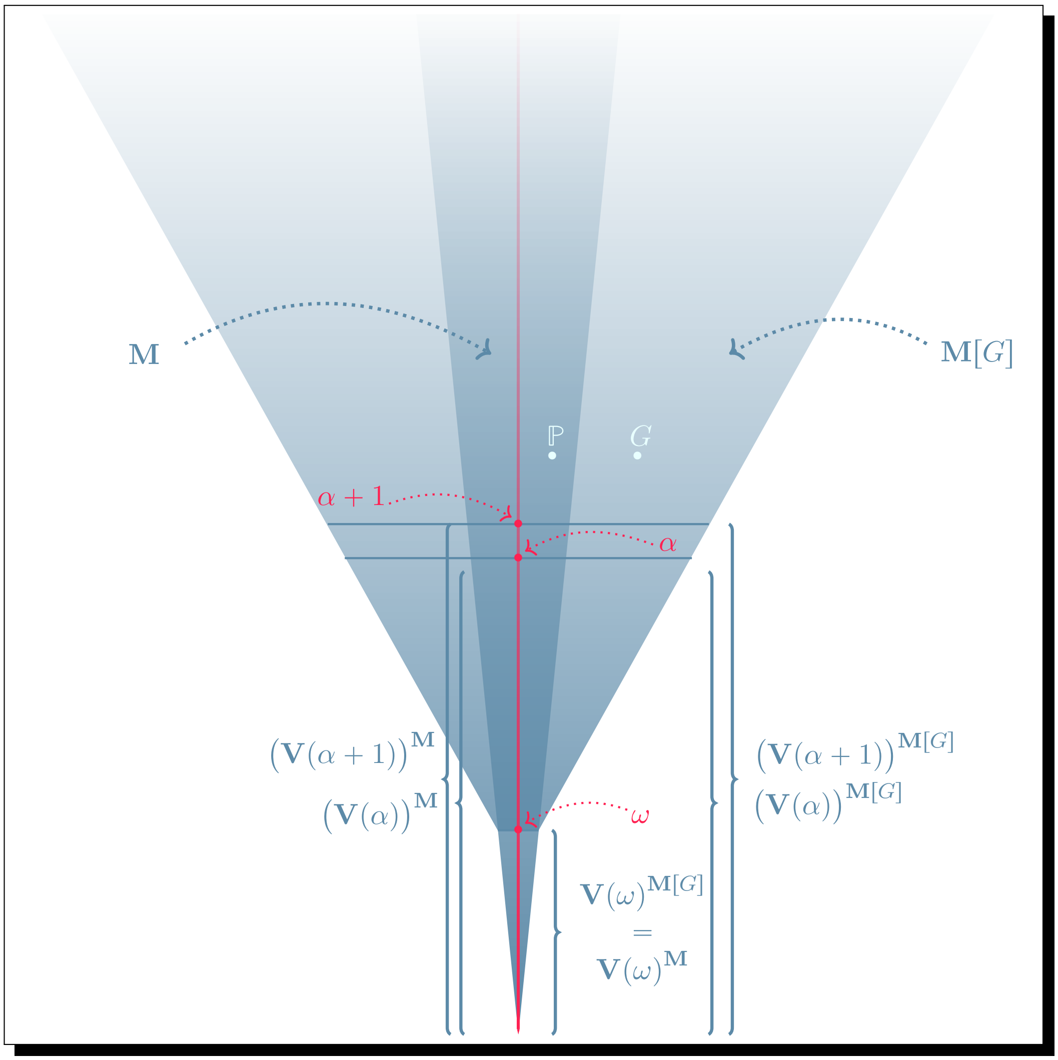



Figure 14.1: The ground model **M** and its generic extension **M**[$G$].

# Chapter 15

# The Truth Lemma

The Truth Lemma is about connecting the truth inside the generic extension to the truth inside **V** and the truth inside **M**. Ultimately we will prove that the relations represented in the following Figure hold (see page 264).

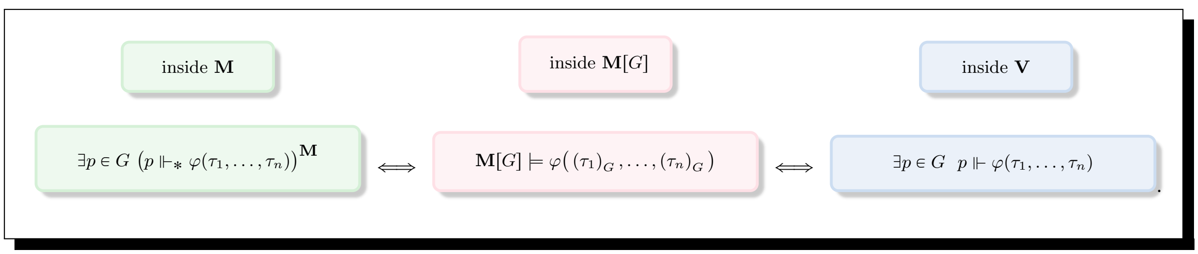



Figure 15.1: Connecting the truth inside $\mathbf{M}[G]$ to the truth inside **V** and the truth inside **M**.

## 15.1 Forcing from inside V

**Definition 313.** *Let* **M** *be a c.t.m. of "***ZFC***",* $\mathbb{P}$ *a notion of forcing with* $\mathbb{P} \in \mathbf{M}$*. Let also* $\varphi(x_1, \ldots, x_n)$ *be any* $\mathcal{L}_{\mathrm{ST}}$*-formula whose free variables are among* $x_1, \ldots, x_n$*,* $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$ *and* $p \in \mathbb{P}$*. We say that* $p$ forces $\varphi(\tau_1, \ldots, \tau_n)$ *and write*

$$p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$$

$$\iff$$

*for all* $G$ $\mathbb{P}$*-generic over* **M** *such that* $p \in G$*, one has*

$$\mathbf{M}[G] \models \varphi\big((\tau_1)_G, \ldots, (\tau_n)_G\big).$$

Notice that, since $G$ may not exist in $\mathbf{M}$, this definition is not made in $\mathbf{M}$, but rather in $\mathbf{V}$.

**Lemma 314.** *Let* $\mathbf{M}$ *be a c.t.m. of "***ZFC***",* $\mathbb{P} \in \mathbf{M}$ *a notion of forcing,* $\varphi(x_1, \ldots, x_n), \psi(x_1, \ldots, x_n)$ *be any* $\mathcal{L}_{\text{ST}}$*-formulas whose free variables are among* $x_1, \ldots, x_n$*, and* $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$*, and* $p \in \mathbb{P}$*.*

*(1) If* $p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$ *and* $q \leqslant p$*, then* $q \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$*.*

*(2) If* $p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$ *and* $p \Vdash_{\mathbb{P},\mathbf{M}} \psi(\tau_1, \ldots, \tau_n)$*, then* $p \Vdash_{\mathbb{P},\mathbf{M}} \big(\varphi(\tau_1, \ldots, \tau_n) \wedge \psi(\tau_1, \ldots, \tau_n)\big)$*.*

*Proof of Lemma 314:*

(1) Suppose that $p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$ and $q \leqslant p$. To show that $q \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$, we consider any filter $G$ which is both $\mathbb{P}$-generic over $\mathbf{M}$ and contains $q$. Since $G$ is a filter and $q \leqslant p$, it follows that $p \in G$. Also, from $p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$, it follows from the definition of the forcing relation that $\mathbf{M}[G] \models \varphi\big((\tau_1)_G, \ldots, (\tau_n)_G\big)$ holds.

Hence, we have shown that for all $G$, $\mathbb{P}$-generic over $\mathbf{M}$, such that $q \in G$, $\mathbf{M}[G] \models \varphi\big((\tau_1)_G, \ldots, (\tau_n)_G\big)$ holds which means — by definition — that $q \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$.

(2) We have

$$p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n) \text{ and } p \Vdash_{\mathbb{P},\mathbf{M}} \psi(\tau_1, \ldots, \tau_n)$$

$$\Longleftrightarrow$$

for all $G$ $\mathbb{P}$-generic over $\mathbf{M}$ such that $p \in G$, one has

$$\mathbf{M}[G] \models \varphi\big((\tau_1)_G, \ldots, (\tau_n)_G\big) \text{ and } \mathbf{M}[G] \models \psi\big((\tau_1)_G, \ldots, (\tau_n)_G\big)$$

$$\Longleftrightarrow$$

for all $G$ $\mathbb{P}$-generic over $\mathbf{M}$ such that $p \in G$, one has

$$\mathbf{M}[G] \models \Big(\varphi\big((\tau_1)_G, \ldots, (\tau_n)_G\big) \;\wedge\; \psi\big((\tau_1)_G, \ldots, (\tau_n)_G\big)\Big)$$

$$\Longleftrightarrow$$

$$p \Vdash_{\mathbb{P},\mathbf{M}} \big(\varphi(\tau_1, \ldots, \tau_n) \wedge \psi(\tau_1, \ldots, \tau_n)\big).$$

□ 314

## 15.2 Forcing from inside M

The idea now, is to define another notion of forcing, not inside **V** but inside **M** so that the two coincide. i.e., we want to define $\Vdash_*$ in **V** such that for all $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$ and $p \in \mathbb{P}$:

$$p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n) \Longleftrightarrow \big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

The definition of $\Vdash_*$ is done entirely inside **M**, by induction on the height of the formula $\varphi$ involved. Usually, constructions made by induction on the height formulas start from basic properties required at the atomic level and rather more involved ones when connectors and quantifiers come to play.

Here, as we will see, not only the whole construction is relatively tedious and cumbersome, but the definition of the relation $\Vdash_*$ is already difficult with atomic formulas — in particular for the equality — but gets easier with more complicated formulas. Anyhow, the results that such a construction will provide are definitely worth the effort.

For simplicity, we assume that the only connectors of our first order logic are " $\wedge$ " and " $\neg$ ", and " $\exists$ " is the only quantifier. Of course, one can get the definition of the definition of $\Vdash_*$ with the other connectors " $\vee$ " , " $\longrightarrow$ " , " $\longleftrightarrow$ " and " $\forall$ " by means of the usual equivalences:

- $(\varphi \vee \psi) \equiv \neg(\neg\varphi \wedge \neg\psi)$
- $(\varphi \longrightarrow \psi) \equiv \neg(\neg\neg\varphi \wedge \neg\psi)$
- $(\varphi \longleftrightarrow \psi) \equiv \big(\neg(\neg\neg\varphi \wedge \neg\psi) \wedge \neg(\neg\neg\psi \wedge \neg\varphi)\big)$
- $\forall x\ \varphi \equiv \neg\exists x\ \neg\varphi$.

We will see in Corollary 324 that if any two formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ satisfy

$$\vdash_c \forall x_1 \ldots \forall x_n \quad \big(\varphi(x_1, \ldots, x_n) \longleftrightarrow \psi(x_1, \ldots, x_n)\big),$$

then we have for any *c.t.m.* **M** of "**ZFC**", $\mathbb{P} \in \mathbf{M}$, $p \in \mathbb{P}$, and all $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$,

$$\big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}} \iff \big(p \Vdash_* \psi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

So, we could already state:

- $(\varphi \vee \psi) \equiv \neg(\neg\varphi \wedge \neg\psi)$
- $(\varphi \longrightarrow \psi) \equiv \neg(\varphi \wedge \neg\psi)$
- $(\varphi \longleftrightarrow \psi) \equiv \big(\neg(\varphi \wedge \neg\psi) \wedge \neg(\psi \wedge \neg\varphi)\big)$
- $\forall x\ \varphi \equiv \neg\exists x\ \neg\varphi$.

**Definition 315.** *Let* $\tau_1, \dots, \tau_n \in \mathbf{M}^{\mathbb{P}}$. *We let:*

- $p \Vdash_* \tau_1 = \tau_2$ *if and only if both*
  - *(1) for all* $(\pi_1, s_1) \in \tau_1$, *the following set is dense below* $p$:
  $$D_\alpha(\pi_1, s_1, \tau_2) = \Big\{q \in \mathbb{P} \,|\, q \leqslant s_1 \longrightarrow \exists(\pi_2, s_2) \in \tau_2 \;\; \big(q \leqslant s_2 \;\wedge\; q \Vdash_* \pi_1 = \pi_2\big)\Big\}$$
  - *(2) for all* $(\pi_2, s_2) \in \tau_2$, *the following set is dense below* $p$:
  $$D_\beta(\pi_2, s_2, \tau_1) = \Big\{q \in \mathbb{P} \,|\, q \leqslant s_2 \longrightarrow \exists(\pi_1, s_1) \in \tau_1 \;\; \big(q \leqslant s_1 \;\wedge\; q \Vdash_* \pi_2 = \pi_1\big)\Big\}$$
- $p \Vdash_* \tau_1 \in \tau_2$ *if and only if the following set is dense below* $p$:
$$\Big\{q \in \mathbb{P} \mid \exists(\pi, s) \in \tau_2 \;\; \Big(q \leqslant s \;\wedge\; q \Vdash_* \pi = \tau_1\Big)\Big\}$$
- $p \Vdash_* (\varphi \wedge \psi)$ *if and only if* $p \Vdash_* \varphi$ *and* $p \Vdash_* \psi$;
- $p \Vdash_* \neg\varphi$ *if and only if for all* $q \leqslant p$ , $q \nVdash_* \varphi$;
- $p \Vdash_* \exists x\, \varphi(x, \tau_1, \dots, \tau_n)$ *if and only if the following set is dense below* $p$:
$$\Big\{q \in \mathbb{P} \mid \exists \sigma \in \mathbf{M}^{\mathbb{P}} \;\; q \Vdash_* \varphi(\sigma, \tau_1, \dots, \tau_n)\Big\}.$$

The main idea behind this definition is to aim at proving the so-called *Truth Lemma* (see page 263) which states that given any formula $\varphi(x_1, \dots, x_n)$, any *c.t.m.* **M** of "**ZFC**", $\mathbb{P}$ any notion of forcing on **M**, any $\tau_1, \dots, \tau_n \in \mathbf{M}^{\mathbb{P}}$, and $G$ any filter $\mathbb{P}$-generic over **M**,

$$\mathbf{M}[G] \models \varphi\big(\,(\tau_1)_G\,, \dots, (\tau_n)_G\,\big) \iff \exists p \in G\big(p \Vdash_* \varphi(\tau_1, \dots, \tau_n)\big)^{\mathbf{M}}.$$

So, for instance for the definition of the membership relation ($p \Vdash_* \tau_1 \in \tau_2$) the *Truth Lemma* states that for all $G$ filter $\mathbb{P}$-generic over **M**, we have

$$\begin{aligned}\mathbf{M}[G] \models (\tau_1)_G \in (\tau_2)_G &\iff \exists p \in G\big(p \Vdash_* \tau_1 \in \tau_2\big)^{\mathbf{M}}\\ &\iff \exists p \in G \;\; \Big\{q \in \mathbb{P} \mid \exists(\pi, s) \in \tau_2 \;\; \Big(q \leqslant s \;\wedge\; q \Vdash_* \pi = \tau_1\Big)\Big\}\\ &\qquad \text{is dense below } p.\end{aligned}$$

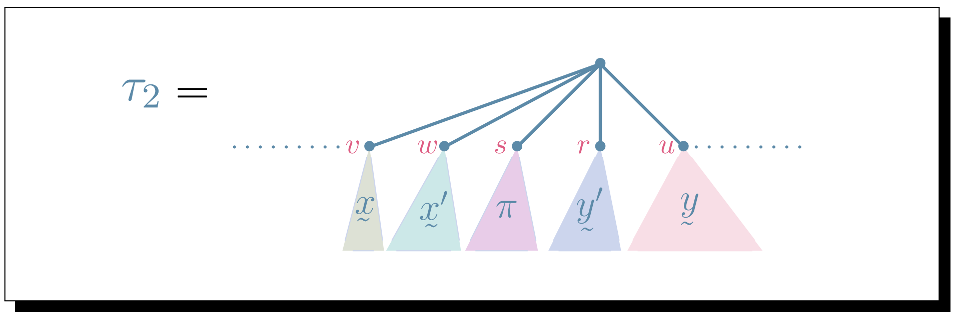
So, the definition of $p \Vdash_* \tau_1 \in \tau_2$ should be understood the following way:

($\Longleftarrow$) If $p \in G$ holds, then we would like to have $\mathbf{M}[G] \models (\tau_1)_G \in (\tau_2)_G$ to hold. This means that there should be some forcing condition $s$ and some $\pi \in \mathbf{M}^{\mathbb{P}}$ such that all following three conditions hold:

(1) $(\pi, s) \in \tau_2$ (2) $s \in G$ (3) $(\pi)_G = (\tau_1)_G$.

Let us study them in detail.

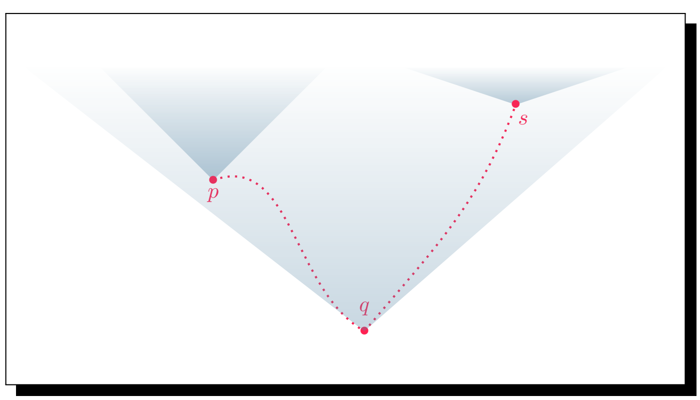
(1) $(\pi, s) \in \tau_2$:



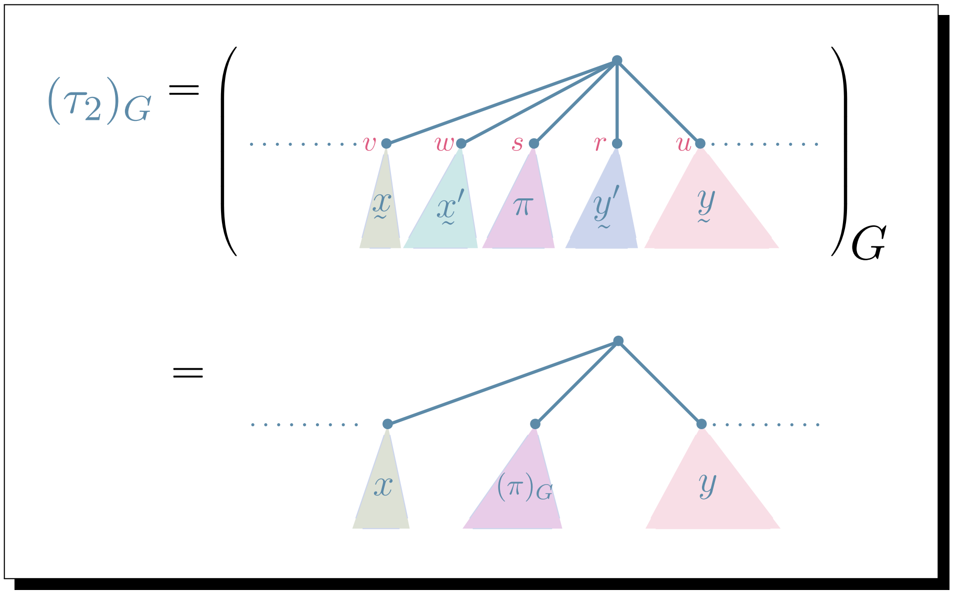

(2) $s \in G$: Since $p \in G$ and the following set $Q$ is dense below $p$:

$$Q = \left\{ q \in \mathbb{P} \mid \exists (\pi, s) \in \tau_2 \ \ \left( q \leqslant s \ \wedge \ q \Vdash_* \pi = \tau_1 \right) \right\}$$

we obtain $\{q \in Q \mid q \leqslant p\} \cap G \neq \varnothing$; so there exists some $q \in \mathbb{P}$ such that both $q \leqslant p$ and $q \in G$ holds. By construction of $Q$, there exists also some $s \geqslant q$ (hence $s \in G$ holds) such that both $(\pi, s) \in \tau_2$ and $q \Vdash_* \pi = \tau_1$.




(3) $(\pi)_G = (\tau_1)_G$:




Since $q \Vdash_* \pi = \tau_1$ and $q \in G$, we will have (since the proof will be by induction on the complexity[1] of the $\mathbb{P}$-names and the complexity of $\pi$ is smaller than the one of $\tau_1$) both

$$\mathbf{M}[G] \models (\pi)_G = (\tau_1)_G \text{ and } \mathbf{M}[G] \models (\pi)_G \in (\tau_2)_G\,;$$

hence we will end up with

$$\mathbf{M}[G] \models (\tau_1)_G \in (\tau_2)_G\,.$$

($\Longrightarrow$) This implication is

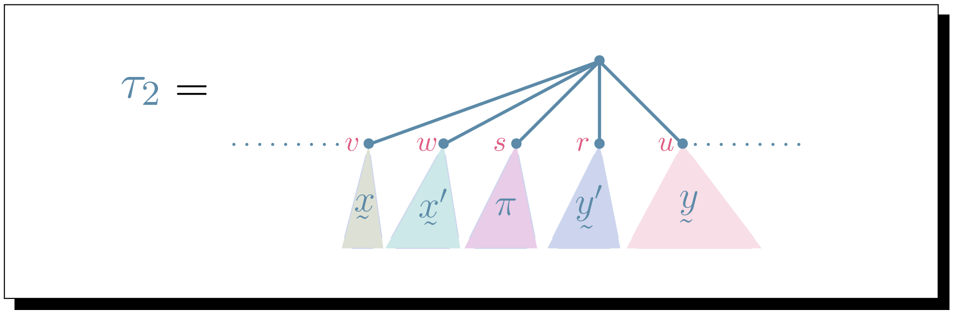
$$\mathbf{M}[G] \models (\tau_1)_G \in (\tau_2)_G \Longrightarrow \exists p \in G\big(p \Vdash_* \tau_1 \in \tau_2\big)^{\mathbf{M}}.$$

If $\mathbf{M}[G] \models (\tau_1)_G \in (\tau_2)_G$ holds, then there exists some $\mathbb{P}$-name $\pi$ together with some forcing condition $s \in G$ such that we have both following conditions satisfied:

(1) $(\pi, s) \in \tau_2$

(2) $\mathbf{M}[G] \models (\tau_1)_G = (\pi)_G$.

[1]See Definition 320 on page 259.




Since the complexity of $\pi$ is somehow smaller[2] than the one of $\tau_2$, one can apply the *Truth Lemma* and get some forcing condition $r \in G$ which satisfies: $\left(r \Vdash_* \tau_1 = \pi\right)^{\mathbf{M}}$.

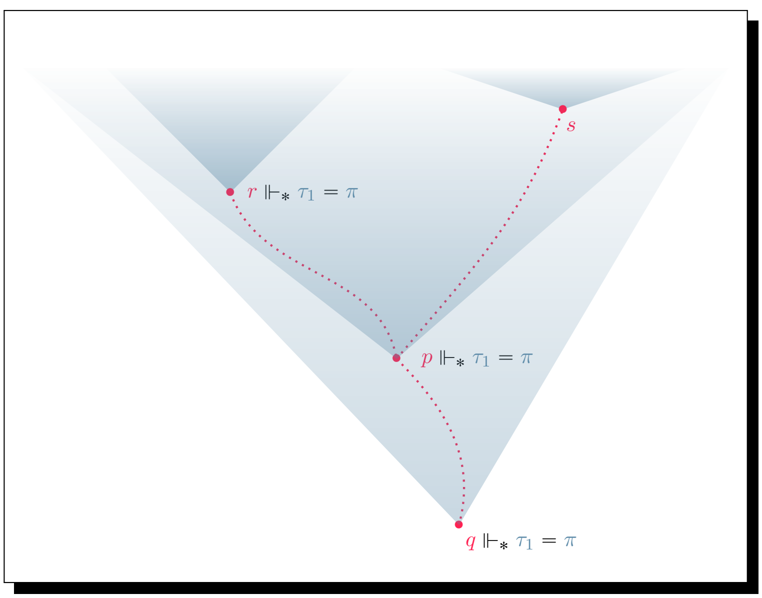
Since both $r \in G$ and $s \in G$, there exists some forcing condition $p \in G$ which satisfies both $p \leqslant r$ and $p \leqslant s$.

We will very soon — on page 253 — prove Lemma 317 which yields that the following are equivalent:

(1) $r \Vdash_* \tau_1 = \pi$;

(2) for all $t \leqslant r$, $t \Vdash_* \tau_1 = \pi$;

(3) the set $\left\{t \in \mathbb{P} \mid t \Vdash_* \tau_1 = \pi\right\}$ is dense below $r$.




So, it follows that $p \Vdash_* \tau_1 = \pi$ and also that every forcing condition $q \leqslant p$ satisfies both

[2]See Definition 320 on page 259.

(1) $q \leqslant s$ (2) $q \Vdash_* \tau_1 = \pi$.

Henceforth, the set

$$Q = \Big\{q' \in \mathbb{P} \mid \exists(\pi', s') \in \tau_2 \ \Big(q' \leqslant s' \ \wedge \ q' \Vdash_* \pi' = \tau_1\Big)\Big\}$$

is dense below $p$, which is the condition to fulfill in order to state that $p \Vdash_* \tau_1 \in \tau_2$ holds.

**Example 316.** *Notice that the empty set is a $\mathbb{P}$-name: it satisfies the requirements of Definition 298:*

- *it is a binary relation such that for all $(\sigma, p) \in \varnothing$, $\sigma$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.*
- *It is even a $\mathbb{P}$-name for the empty set, since $(\varnothing)_G = \varnothing$.*
- *Also, the canonical $\mathbb{P}$-name for the empty set is nothing but the empty set itself:*

$$\check{\varnothing} = \big\{(\check{\sigma}, \mathbb{1}) \mid \sigma \in \varnothing\big\} = \varnothing \quad \textit{and} \quad (\varnothing)_G = \varnothing.$$

*For every forcing condition $p$, and every $\mathbb{P}$-name $\tau$, the following three conditions are satisfied:*

*(1) $p \not\Vdash_* \tau \in \varnothing$ holds because the following set being empty, is definitely not dense below $p$:*

$$\Big\{q \in \mathbb{P} \mid \exists(\pi, s) \in \varnothing \ \Big(q \leqslant s \ \wedge \ q \Vdash_* \pi = \tau\Big)\Big\} = \varnothing.$$

*(2) $p \Vdash_* \varnothing = \varnothing$ holds because*

*(a) The statement: "for all $(\pi_1, s_1) \in \varnothing$, $D_\alpha(\pi_1, s_1, \varnothing)$ is dense below $p$" is of the form*

$$\forall(\pi_1, s_1) \ \Big((\pi_1, s_1) \in \varnothing \longrightarrow \text{"}D_\alpha(\pi_1, s_1, \varnothing) \text{ is dense below } p\text{"}\Big)$$

*and since $(\pi_1, s_1) \in \varnothing$ always fails, this statement is true.*

*(b) The statement "for all $(\pi_2, s_2) \in \varnothing$, the set $D_\beta(\pi_2, s_2, \varnothing)$ is dense below $p$" holds also for the same reason.*

*(3) $p \Vdash_* \check{\varnothing} \in \widecheck{\{\varnothing\}}$ holds because we have*

$$\check{\varnothing} = \varnothing \quad \textit{and} \quad \widecheck{\{\varnothing\}} = \big\{\,(\check{\sigma}, \mathbb{1}) \mid \sigma \in \{\varnothing\}\,\big\} = \big\{\,(\check{\varnothing}, \mathbb{1})\,\big\} = \big\{\,(\varnothing, \mathbb{1})\,\big\}$$

*and the following set is obviously dense below p:*

$$\begin{aligned}&\Big\{q\in\mathbb{P}\mid\exists(\pi,s)\in\widecheck{\{\varnothing\}}\ \ \Big(q\leqslant s\ \wedge\ q\Vdash_*\pi=\varnothing\Big)\Big\}\\=&\Big\{q\in\mathbb{P}\mid\Big(q\leqslant\mathbb{1}\ \wedge\ q\Vdash_*\varnothing=\varnothing\Big)\Big\}\\=&\ \mathbb{P}.\end{aligned}$$

**Lemma 317.** *Let $\mathbb{P}$ be a notion of forcing, and $p\in\mathbb{P}$. Let also $\varphi(x_1,\ldots,x_n)$ be any $\mathcal{L}_{\mathrm{ST}}$-formula whose free variables are among $x_1,\ldots,x_n$, and $\tau_1,\ldots,\tau_n\in\mathbf{V}^{\mathbb{P}}$.*
*The following are equivalent:*

*(1) $p\Vdash_*\varphi(\tau_1,\ldots,\tau_n)$;*

*(2) for all $r\leqslant p$, $r\Vdash_*\varphi(\tau_1,\ldots,\tau_n)$;*

*(3) the set $\big\{r\in\mathbb{P}\mid r\Vdash_*\varphi(\tau_1,\ldots,\tau_n)\big\}$ is dense below $p$.*

*Proof of Lemma* 317:

$(\mathbf{1})\Rightarrow(\mathbf{2})$ By induction on the height of $\varphi$.

$\boldsymbol{\varphi:\ x_1=x_2}$

Take any $r\leqslant p$, and suppose that $p\Vdash_*\tau_1=\tau_2$, which means that

$$\text{for all }(\pi_1,s_1)\in\tau_1,\ \ D_\alpha(\pi_1,s_1,\tau_2)\text{ is dense below }p.$$

Since $r\leqslant p$, $D_\alpha(\pi_1,s_1,\tau_2)$ is also dense below $r$. Analogously, for all $(\pi_2,s_2)\in\tau_2$, $D_\beta(\pi_2,s_2,\tau_1)$ is dense below $r$. Therefore $r\Vdash_*\tau_1=\tau_2$.

$\boldsymbol{\varphi:\ x_1\in x_2}$

Take any $r\leqslant p$, and suppose that $p\Vdash_*\tau_1\in\tau_2$, which means that the set

$$\Big\{q\in\mathbb{P}\mid\exists(\pi,s)\in\tau_2\ \ \Big(q\leqslant s\ \wedge\ q\Vdash_*\pi=\tau_1\Big)\Big\}$$

is dense below $p$. It follows that the same set is dense below $r$.

$\boldsymbol{\varphi:\ \exists x\ \psi(x,x_1,\ldots,x_n)}$

Take any $r\leqslant p$, and suppose that $p\Vdash_*\exists x\ \psi(x,\tau_1,\ldots,\tau_n)$, which means that the set

$$\Big\{q\in\mathbb{P}\mid\exists\sigma\in\mathbf{V}^{\mathbb{P}}\ \ q\Vdash_*\psi(\sigma,\tau_1,\ldots,\tau_n)\Big\}.$$

is dense below $p$. It follows that the same set is dense below $r$.

$\boldsymbol{\varphi}$ **:** $(\boldsymbol{\theta} \wedge \boldsymbol{\psi})$

Take any $r \leqslant p$, and suppose that $p \Vdash_* (\theta \wedge \psi)$, which means that both $p \Vdash_* \theta$ and $p \Vdash_* \psi$. So by induction hypothesis, one has $r \Vdash_* \theta$ and $r \Vdash_* \psi$, which comes down to $r \Vdash_* (\theta \wedge \psi)$.

$\boldsymbol{\varphi}$ **:** $\neg\boldsymbol{\psi}$

Take any $r \leqslant p$, and suppose that $p \Vdash_* \neg\psi$, which means that for all $q \leqslant p$ , $q \nVdash_* \psi$. So in particular, for all $q \leqslant r$ , $q \nVdash_* \varphi$; which means $r \nVdash_* \psi$.

**(2)** $\Rightarrow$ **(3)** is immediate.

**(3)** $\Rightarrow$ **(1)** By induction on the height of $\varphi$.

$\boldsymbol{\varphi}$ **:** $\boldsymbol{x_1 = x_2}$

We suppose the set $D = \left\{r \in \mathbb{P} \mid r \Vdash_* \tau_1 = \tau_2\right\}$ is dense below $p$. So, for all $(\pi_1, s_1) \in \tau_1$, $D_\alpha(\pi_1, s_1, \tau_2)$ is dense below $r$ for all $r \in D$. But since $D$ is dense below $p$, $D_\alpha(\pi_1, s_1, \tau_2)$ is dense below $p$ as well, and the same holds for $D_\beta(\pi_2, s_2, \tau_1)$. So, $p \Vdash_* \tau_1 = \tau_2$.

$\boldsymbol{\varphi}$ **:** $\boldsymbol{x_1 \in x_2}$

We suppose the set $D = \left\{r \in \mathbb{P} \mid r \Vdash_* \tau_1 \in \tau_2\right\}$ is dense below $p$. So, the set

$$\left\{q \in \mathbb{P} \mid \exists(\pi, s) \in \tau_2 \ \left(q \leqslant s \ \wedge \ q \Vdash_* \pi = \tau_1\right)\right\}$$

is dense below $r$ for all $r \in D$. Hence it is also dense below $p$, which yields $p \Vdash_* \tau_1 \in \tau_2$.

$\boldsymbol{\varphi}$ **:** $\exists \boldsymbol{x} \ \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{x_1}, \ldots, \boldsymbol{x_n})$

We suppose once again that the set $D = \left\{r \in \mathbb{P} \mid r \Vdash_* \exists x \varphi(x, \tau_1, \ldots, \tau_n)\right\}$ is dense below $p$. So, the set

$$\left\{q \in \mathbb{P} \mid \exists \sigma \in \mathbf{V}^{\mathbb{P}} \ \ q \Vdash_* \psi(\sigma, \tau_1, \ldots, \tau_n)\right\}.$$

is dense below each $r \in D$, which implies that it is dense below $p$. Henceforth, $p \Vdash_* \exists x \varphi(x, \tau_1, \ldots, \tau_n)$.

$\boldsymbol{\varphi}$ **:** $(\boldsymbol{\theta} \wedge \boldsymbol{\psi})$

We assume that the set $D = \left\{r \in \mathbb{P} \mid r \Vdash_* (\theta \wedge \psi)\right\}$ is dense below $p$. So, both sets

$$\left\{r \in \mathbb{P} \mid r \Vdash_* \theta\right\} \quad \text{and} \quad \left\{r \in \mathbb{P} \mid r \Vdash_* \psi\right\}$$

are dense below $p$. By induction hypothesis, this leads to $p \Vdash_* \theta$ and $p \Vdash_* \psi$, and finally to $p \Vdash_* (\theta \wedge \psi)$.

**$\varphi$ : $\neg\psi$**

We assume that the set $D = \{r \in \mathbb{P} \mid r \Vdash_* \neg\psi\}$ is dense below $p$ and proceed by contradiction. So, we suppose $p \nVdash_* \neg\psi$, which means that there exists $q \leqslant p$ such that $q \Vdash_* \psi$. By (1) $\Rightarrow$ (2) we see that for all $r \leqslant q$, $r \Vdash_* \psi$. Since $D$ is dense below $p$, it is also dense below $q$. Now, any $r \in D \cap \{s \in \mathbb{P} \mid s \leqslant q\}$ satisfies both $r \Vdash_* \neg\psi$ and $r \Vdash_* \psi$, a contradiction.

□ 317

**Proposition 318.** *Let $\mathbb{P}$ be a notion of forcing, and $p \in \mathbb{P}$. Let also $\varphi(x_1, \ldots, x_n)$ and $\psi(y_1, \ldots, y_k)$ be any $\mathcal{L}_{\mathrm{ST}}$-formulas whose free variables are among $x_1, \ldots, x_n$ and $y_1, \ldots, y_k$, respectively. Let $\tau_1, \ldots, \tau_n, \sigma_1, \ldots, \sigma_k \in \mathbf{V}^{\mathbb{P}}$. For any forcing condition $p \in \mathbb{P}$ we have*

*(1)*

$$p \Vdash_* \big(\varphi(\tau_1, \ldots, \tau_n) \vee \psi(\sigma_1, \ldots, \sigma_k)\big)$$
*if and only if*
$$\{q \in \mathbb{P} \mid q \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\} \cup \{q \in \mathbb{P} \mid q \Vdash_* \psi(\sigma_1, \ldots, \sigma_k)\} \quad \textit{is dense below } p$$

*(2)*

$$p \Vdash_* \big(\varphi(\tau_1, \ldots, \tau_n) \longrightarrow \psi(\sigma_1, \ldots, \sigma_k)\big)$$
*if and only if*
*for all $q \leqslant p$, if $q \Vdash_* \varphi(\tau_1, \ldots, \tau_n)$, then $q \Vdash_* \psi(\sigma_1, \ldots, \sigma_k)$*

*(3)*

$$p \Vdash_* \forall x_1\, \varphi(x_1, \tau_2, \ldots, \tau_n)$$
*if and only if*
*for all $\mathbb{P}$-names $\tau \in V^{\mathbb{P}}$, $p \Vdash_* \varphi(\tau, \tau_2, \ldots, \tau_n)$.*

*Proof of Lemma* 318:

(1) We have that $\big(\varphi(\tau_1, \ldots, \tau_n) \vee \psi(\sigma_1, \ldots, \sigma_k)\big) \equiv \neg\big(\neg\varphi(\tau_1, \ldots, \tau_n) \wedge \neg\psi(\sigma_1, \ldots, \sigma_k)\big)$. So we prove
$$p \Vdash_* \neg\big(\neg\varphi(\tau_1, \ldots, \tau_n) \wedge \neg\psi(\sigma_1, \ldots, \sigma_k)\big)$$
$$\Longleftrightarrow$$
$$\{q \in \mathbb{P} \mid q \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\} \cup \{q \in \mathbb{P} \mid q \Vdash_* \psi(\sigma_1, \ldots, \sigma_k)\} \quad \text{is dense below } p$$

($\Rightarrow$) Suppose that $p \Vdash_* \neg\big(\neg\varphi(\tau_1, \ldots, \tau_n) \wedge \neg\psi(\sigma_1, \ldots, \sigma_k)\big)$. Then for all $q \leqslant p$,
$$q \nVdash_* \big(\neg\varphi(\tau_1, \ldots, \tau_n) \wedge \neg\psi(\sigma_1, \ldots, \sigma_k)\big).$$

i.e. $q \not\Vdash_* \neg\varphi(\tau_1,\ldots,\tau_n)$ or $q \not\Vdash_* \neg\psi(\sigma_1,\ldots,\sigma_k)$. We show that the set of $r \leqslant p$ such that $r \Vdash_* \varphi(\tau_1,\ldots,\tau_n)$ or $r \Vdash_* \psi(\sigma_1,\ldots,\sigma_k)$ is dense below $p$. To see this, suppose that $q \leqslant p$. We have $q \not\Vdash_* \neg\varphi(\tau_1,\ldots,\tau_n)$ or $q \not\Vdash_* \neg\psi(\sigma_1,\ldots,\sigma_k)$. If $q \not\Vdash_* \neg\varphi(\tau_1,\ldots,\tau_n)$, then there exists $r \leqslant q$ such that $r \Vdash_* \varphi(\tau_1,\ldots,\tau_n)$. Similarly if $q \not\Vdash_* \neg\psi(\sigma_1,\ldots,\sigma_k)$, then there exists $r \leqslant q$ with $r \Vdash_* \psi(\sigma_1,\ldots,\sigma_k)$. In both cases we have found $r \leqslant q$ with $r \Vdash_* \varphi(\tau_1,\ldots,\tau_n)$ or $r \Vdash_* \psi(\sigma_1,\ldots,\sigma_k)$.

($\Leftarrow$) Suppose that $\{q \in \mathbb{P} \mid q \Vdash_* \varphi(\tau_1,\ldots,\tau_n)\} \cup \{q \in \mathbb{P} \mid q \Vdash_* \psi(\sigma_1,\ldots,\sigma_k)\}$ is dense below $p$. So, for all $r \leqslant p$ there exists $q \leqslant r$ such that $q \Vdash_* \varphi(\tau_1,\ldots,\tau_n)$ or $q \Vdash_* \psi(\sigma_1,\ldots,\sigma_k)$, and therefore $q \not\Vdash_* \neg\varphi(\tau_1,\ldots,\tau_n)$ or $q \not\Vdash_* \neg\psi(\sigma_1,\ldots,\sigma_k)$. Therefore, the set

$$\{q \in \mathbb{P} \mid q \not\Vdash_* \neg\varphi(\tau_1,\ldots,\tau_n) \ \text{ or } \ q \not\Vdash_* \neg\psi(\sigma_1,\ldots,\sigma_k)\}$$

is dense below $p$. Hence, the set

$$\big\{q \in \mathbb{P} \mid q \not\Vdash_* \big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)\big\}$$

is also dense below $p$. Thus, given any $r \leqslant p$, there exists some $q \leqslant r$ such that $q \not\Vdash_* \big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)$.

We distinguish between

$$(a) \quad q \not\Vdash_* \neg\big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)$$

and

$$(b) \quad q \Vdash_* \neg\big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)$$

(a) $q \not\Vdash_* \neg\big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)$ would lead to the existence of some $s \leqslant q \leqslant r \leqslant p$ with $s \Vdash_* \big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)$, and by Lemma 317 this would lead to all $s' \leqslant s$ satisfying $s' \Vdash_* \big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)$, contradicting the fact that the set

$$\big\{q \in \mathbb{P} \mid q \not\Vdash_* \big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)\big\}$$

is also dense below $p$. So, this case is impossible.

(b) So, the only possibility is that $q \Vdash_* \neg\big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)$ which shows that the set

$$\big\{q \in \mathbb{P} \mid q \Vdash_* \neg\big(\neg\varphi(\tau_1,\ldots,\tau_n) \ \wedge \ \neg\psi(\sigma_1,\ldots,\sigma_k)\big)\big\}$$

is dense below $p$. Hence, by Lemma 317, we obtain $p \Vdash_* \neg\big(\neg\varphi(\tau_1,\ldots,\tau_n) \wedge \neg\psi(\sigma_1,\ldots,\sigma_k)\big)$.

(2) We have that $\varphi(\tau_1,\ldots,\tau_n) \rightarrow \psi(\sigma_1,\ldots,\sigma_k) \equiv \neg\varphi(\tau_1,\ldots,\tau_n) \vee \psi(\sigma_1,\ldots,\sigma_k)$. By the previous point, $q \Vdash_* \neg\varphi(\tau_1,\ldots,\tau_n) \vee \psi(\sigma_1,\ldots,\sigma_k)$ if and only if

$$\{q \in \mathbb{P} \mid q \Vdash_* \neg\varphi(\tau_1,\ldots,\tau_n)\} \cup \{q \in \mathbb{P} \mid q \Vdash_* \psi(\sigma_1,\ldots,\sigma_k)\}$$

is dense below $p$. So we show that

$$\{q \in \mathbb{P} \mid q \Vdash_* \neg\varphi(\tau_1, \dots, \tau_n)\} \cup \{q \in \mathbb{P} \mid q \Vdash_* \psi(\sigma_1, \dots, \sigma_k)\} \text{ is dense below } p$$

$$\Longleftrightarrow$$

$$\text{for all } q \leqslant p, \text{ if } q \Vdash_* \varphi(\tau_1, \dots, \tau_n), \text{ then } q \Vdash_* \psi(\sigma_1, \dots, \sigma_k).$$

($\Rightarrow$) Take any $q \leqslant p$ such that $q \Vdash_* \varphi(\tau_1, \dots, \tau_n)$ holds. We show that the set

$$\{r \in \mathbb{P} \mid r \Vdash_* \psi(\sigma_1, \dots, \sigma_k)\}$$

is dense below $q$, which will guarantee that $q \Vdash_* \psi(\sigma_1, \dots, \sigma_k)$. Indeed, since

$$\left\{q' \in \mathbb{P} \mid q' \Vdash_* \neg\varphi(\tau_1, \dots, \tau_n)\right\} \cup \left\{q' \in \mathbb{P} \mid q' \Vdash_* \psi(\sigma_1, \dots, \sigma_k)\right\}$$

is dense below $p$, it is also dense below $q$. So, pick $r \leqslant q$ with

$$r \in \left\{q' \in \mathbb{P} \mid q' \Vdash_* \neg\varphi(\tau_1, \dots, \tau_n)\right\} \cup \left\{q' \in \mathbb{P} \mid q' \Vdash_* \psi(\sigma_1, \dots, \sigma_k)\right\}.$$

Notice that $r \in \{q' \in \mathbb{P} \mid q' \Vdash_* \neg\varphi(\tau_1, \dots, \tau_n)\}$ is impossible since $q \Vdash_* \varphi(\tau_1, \dots, \tau_n)$ and $r \leqslant q$ yield $r \Vdash_* \varphi(\tau_1, \dots, \tau_n)$. Therefore, one has

$$r \in \left\{q' \in \mathbb{P} \mid q' \Vdash_* \psi(\sigma_1, \dots, \sigma_k)\right\},$$

which shows that $\{r \in \mathbb{P} \mid r \Vdash_* \psi(\sigma_1, \dots, \sigma_k)\}$ is dense below $q$.

($\Leftarrow$) In order to show that $\{q \in \mathbb{P} \mid q \Vdash_* \neg\varphi(\tau_1, \dots, \tau_n)\} \cup \{q \in \mathbb{P} \mid q \Vdash_* \psi(\sigma_1, \dots, \sigma_k)\}$ is dense below $p$, consider any $r \leqslant p$. If $r \Vdash_* \neg\varphi(\tau_1, \dots, \tau_n)$ or $r \Vdash_* \psi(\sigma_1, \dots, \sigma_k)$ we are done. Otherwise, $r \not\Vdash_* \neg\varphi(\tau_1, \dots, \tau_n)$ and $r \not\Vdash_* \psi(\sigma_1, \dots, \sigma_k)$ yield there exists some $s \leqslant r$ such that $s \Vdash_* \varphi(\tau_1, \dots, \tau_n)$, hence $s \Vdash_* \psi(\sigma_1, \dots, \sigma_k)$ also holds from the assumption, which shows that

$$s \in \{q \in \mathbb{P} \mid q \Vdash_* \neg\varphi(\tau_1, \dots, \tau_n)\} \cup \{q \in \mathbb{P} \mid q \Vdash_* \psi(\sigma_1, \dots, \sigma_k)\}$$

and completes the proof that $\{q \in \mathbb{P} \mid q \Vdash_* \neg\varphi(\tau_1, \dots, \tau_n)\} \cup \{q \in \mathbb{P} \mid q \Vdash_* \psi(\sigma_1, \dots, \sigma_k)\}$ is dense below $p$.

(3) We have that $\forall v \varphi(v) \equiv \neg\exists v \neg\varphi(v)$. So we need to show

$$p \Vdash_* \neg\exists x_1 \neg\varphi(x_1, \tau_2, \dots, \tau_n)$$

$$\Longleftrightarrow$$

$$\text{for all } \mathbb{P}\text{-names } \tau \in V^{\mathbb{P}}, \ p \Vdash_* \varphi(\tau, \tau_2, \dots, \tau_n).$$

($\Rightarrow$) We suppose $p \Vdash_* \neg\exists x_1 \neg\varphi(x_1, \tau_2, \dots, \tau_n)$ and show that for each $\mathbb{P}$-name $\tau$, the set

$\{t \in \mathbb{P} \mid t \Vdash_* \varphi(\tau, \tau_2, \ldots, \tau_n)\}$ is dense below $p$.
So, pick any $q \leqslant p$. Since, $p \Vdash_* \neg\exists x_1 \neg\varphi(x_1, \tau_2, \ldots, \tau_n)$ we have $q \nVdash_* \exists x_1 \neg\varphi(x_1, \tau_2, \ldots, \tau_n)$, hence the set

$$\Big\{r \in \mathbb{P} \mid \exists \sigma \in \mathbf{V}^{\mathbb{P}} \;\; r \Vdash_* \neg\varphi(\sigma, \tau_2, \ldots, \tau_n)\Big\}.$$

is not dense below $q$. So there exists $r \leqslant q$ such that for all $\sigma \in \mathbf{V}^{\mathbb{P}}$ and all $s \leqslant r$ we have $s \nVdash_* \neg\varphi(\sigma, \tau_2, \ldots, \tau_n)$ which leads to the existence of some $t \leqslant s$ which satisfies $t \Vdash_* \varphi(\sigma, \tau_2, \ldots, \tau_n)$. So, for each $\mathbb{P}$-name $\sigma \in \mathbf{V}^{\mathbb{P}}$, we have found some $t \leqslant s \leqslant r \leqslant q \leqslant p$ which satisfies $t \Vdash_* \varphi(\sigma, \tau_2, \ldots, \tau_n)$, which shows that $\{t \in \mathbb{P} \mid t \Vdash_* \varphi(\sigma, \tau_2, \ldots, \tau_n)\}$ is dense below $p$, and ultimately, by Lemma 317, that $p \Vdash_* \varphi(\sigma, \tau_2, \ldots, \tau_n)$.

($\Leftarrow$) We assume for all $\mathbb{P}$-name $\tau \in V^{\mathbb{P}}$, we have $p \Vdash_* \varphi(\tau, \tau_2, \ldots, \tau_n)$. This implies that for all $\mathbb{P}$-name $\tau \in V^{\mathbb{P}}$ and all $q \leqslant p$ $q \Vdash_* \varphi(\tau, \tau_2, \ldots, \tau_n)$, hence $q \nVdash_* \neg\varphi(\tau, \tau_2, \ldots, \tau_n)$. Therefore, for each $q \leqslant p$, the set

$$\Big\{r \in \mathbb{P} \mid \exists \sigma \in \mathbf{V}^{\mathbb{P}} \;\; r \Vdash_* \neg\varphi(\sigma, \tau_2, \ldots, \tau_n)\Big\}.$$

is empty – hence not dense – below any $q \leqslant p$. So, $q \nVdash_* \exists x_1 \neg\varphi(x_1, \tau_2, \ldots, \tau_n)$ holds for each $q \leqslant p$, which precisely grants $p \Vdash_* \neg\exists x_1 \neg\varphi(x_1, \tau_2, \ldots, \tau_n)$.

□ 318

## 15.3 Connecting the Truth in $\mathbf{M}[G]$ to the Truth in $\mathbf{M}$

Providing we have access to the filter $G$, we show that we can go back and forth between the truth in $\mathbf{M}$ and the truth in $\mathbf{M}[G]$.

**Lemma 319.** *Let $\varphi(x_1, \ldots, x_n)$ be any $\mathcal{L}_{\text{ST}}$-formula, $\mathbf{M}$ any c.t.m. of "**ZFC**", $\mathbb{P}$ any notion of forcing on $\mathbf{M}$, $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$, and $G$ any filter $\mathbb{P}$-generic over $\mathbf{M}$.*

*(1) If $p \in G$ and $\big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}$, then*

$$\mathbf{M}[G] \models \varphi\big((\tau_1)_G, \ldots, (\tau_n)_G\big).$$

*(2) If $\mathbf{M}[G] \models \varphi\big((\tau_1)_G, \ldots, (\tau_n)_G\big)$, then there exists $p \in G$ such that*

$$\big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

Viewed from the perspective of the generic extension — in the sense that we start from picking elements in $\mathbf{M}[G]$ and find a name for them later on, as opposed to firstly starting with $\mathbb{P}$-names and secondly decoding them — this Lemma states that for all $G$ $\mathbb{P}$-generic over $\mathbf{M}$, and all sets $a_1, \ldots, a_n$ in $\mathbf{M}[G]$, we have

(1) If $p \in G$ and $\big(p \Vdash_* \varphi(\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)\big)^{\mathbf{M}}$, then

$$\mathbf{M}[G] \models \varphi\big(a_1, \ldots, a_n\big).$$

(2) If $\mathbf{M}[G] \models \varphi\big(a_1, \ldots, a_n\big)$, then there exists $p \in G$ such that

$$\big(p \Vdash_* \varphi(\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)\big)^{\mathbf{M}}.$$

Where $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ are $\mathbb{P}$-names in $\mathbf{M}$ such that $(\underset{\sim}{a}_1)_G = a_1, \ldots, (\underset{\sim}{a}_n)_G = a_n$.

**Definition 320.** *Given $\mathbb{P}$ a notion of forcing, and $\pi_1, \pi_2, \tau_1, \tau_2 \in \mathbf{V}^{\mathbb{P}}$, we define*

$$(\pi_1, \pi_2) \prec (\tau_1, \tau_2) \iff \pi_1 \in dom(\tau_1) \text{ and } \pi_2 \in dom(\tau_2).$$

Notice that this definition yields $\prec$ is well-founded since $rk(\pi_1) < rk(\tau_1)$ and $rk(\pi_2) < rk(\tau_2)$ both hold, therefore, $\min\big\{rk(\pi_1), rk(\pi_2)\big\} < \min\big\{rk(\pi_1), rk(\pi_2)\big\}$.

*Proof of Lemma 319*: We prove (1) and (2) simultaneously by induction on the height of $\varphi$.

$\boldsymbol{\varphi :\ x_1 = x_2}$ We prove (1) and (2) by $\prec$-induction.

(1) Let $p \in G$ be such that $(p \Vdash_* \tau_1 = \tau_2)^{\mathbf{M}}$, we want to show both

$$\mathbf{M}[G] \models (\tau_1)_G \subseteq (\tau_2)_G \quad \text{and} \quad \mathbf{M}[G] \models (\tau_2)_G \subseteq (\tau_1)_G.$$

We recall that

$$(\tau_1)_G = \{(\pi_1)_G \mid \exists s_1 \in G\ (\pi_1, s_1) \in \tau_1\}.$$

Let $(\pi_1, s_1) \in \tau_1$, and let us show that $(\pi_1)_G \in (\tau_2)_G$. To do so, we are reaching for some $s_2 \in G$ such that $(\pi_1, s_2) \in \tau_2$.

Since $p$ and $s_1$ are elements of the filter $G$, there exists $q \in G$ such that $q \leqslant p$ and $q \leqslant s_1$. The set $D_\alpha(\pi_1, s_1, \tau_2)$ is dense below $p$ and thus under $q$. By Lemma 297, one has $G \cap D_\alpha(\pi_1, s_1, \tau_2) \neq \varnothing$. Then take any $r \in G \cap D_\alpha(\pi_1, s_1, \tau_2) \neq \varnothing$. There thus exists $(\pi_2, s_2) \in \tau_2$ such that $r \leqslant s_2$ and $r \Vdash_* \pi_1 = \pi_2$. Moreover, since $r \in G$ and $r \leqslant s_2$, one obtains $s_2 \in G$. It follows that $\mathbf{M}[G] \models (\pi_2)_G \in (\tau_2)_G$.

We have $(\pi_1, \pi_2) \prec (\tau_1, \tau_2)$, $r \Vdash_* \pi_1 = \pi_2$, and $r \in G$. So, the induction hypothesis, gives $\mathbf{M}[G] \models (\pi_1)_G = (\pi_2)_G$.

Therefore, we have shown that $\mathbf{M}[G] \models (\pi_1)_G \in (\tau_2)_G$ holds for every $(\pi_1)_G \in (\tau_1)_G$, and so $\mathbf{M}[G] \models (\tau_1)_G \subseteq (\tau_2)_G$. The opposite inclusion is achieved in a similar fashion.

(2) Suppose that $\mathbf{M}[G] \models (\tau_1)_G = (\tau_2)_G$. Let

$$D = \{r \in \mathbb{P} \mid r \Vdash_* \tau_1 = \tau_2 \ \vee \ \psi_1(r) \ \vee \ \psi_2(r)\},$$

where

$$\psi_1(x) : \exists(\pi_1, s_1) \in \tau_1 \ \big(x \leqslant s_1 \ \wedge \ \forall(\pi_2, s_2) \in \tau_2 \ \forall q \leqslant s_2 \ \ (q \Vdash_* \pi_1 = \pi_2 \longrightarrow q \perp x)\big)$$

and

$$\psi_2(x) : \exists(\pi_2, s_2) \in \tau_2 \ \big(x \leqslant s_2 \ \wedge \ \forall(\pi_1, s_1) \in \tau_1 \ \forall q \leqslant s_1 \ \ (q \Vdash_* \pi_2 = \pi_1 \longrightarrow q \perp x)\big).$$

Let us show that $D$ is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$, if $p \Vdash_* \tau_1 = \tau_2$, then $p \in D$. Otherwise, there exists $(\pi_1, s_1) \in \tau_1$ such that $D_\alpha(\pi_1, s_1, \tau_2)$ is not dense below $p$, so there exists $(\pi_2, s_2) \in \tau_2$ such that $D_\beta(\pi_2, s_2, \tau_1)$ is not dense below $p$.

Suppose that there exists $(\pi_1, s_1) \in \tau_1$ such that $D_\alpha(\pi_1, s_1, \tau_2)$ is not dense below $p$, which means that there exists $r \leqslant p$ such that for all $q \leqslant r$, $q \notin D_\alpha(\pi_1, s_1, \tau_2)$. We show that $r$ satisfies $\psi_1$.

Let $q \leqslant r$, $q \notin D_\alpha(\pi_1, s_1, \tau_2)$, so $q \leqslant s_1$. Furthermore, for all $(\pi_2, s_2) \in \tau_2$, $q \nleqslant s_2$ ou $q \nVdash_* \pi_1 = \pi_2$.

For all $t \in \mathbb{P}$ and for all $(\pi_2, s_2) \in \tau_2$, if $t \leqslant s_2$ and $t \Vdash_* \pi_1 = \pi_2$ then $t \perp r$. Indeed, if this is not the case, there would exist $t' \leqslant r$ such that $t' \leqslant s_2$, $t' \Vdash_* \pi_1 = \pi_2$, but the last two properties assure us that $t' \in D_\alpha(\pi_1, s_1, \tau_2)$ which contradicts the definition of $r$. Therefore $r$ satisfies $\psi_1$. We reason in a similar manner if there exists $(\pi_2, s_2) \in \tau_2$ such that $D_\beta(\pi_2, s_2, \tau_1)$ is not dense below $p$. Hence, $D$ is dense in $\mathbb{P}$.

Let us now show that if $p \in G$, then $p$ does not satisfy neither $\psi_1$, nor $\psi_2$. Suppose towards contradiction that $p \in G$ and that $p$ satisfies $\psi_1$. Fix $(\pi_1, s_1) \in \tau_1$ such that:

$$p \leqslant s_1 \wedge \forall(\pi_2, s_2) \in \tau_2 \ \forall q \leqslant s_2 \ \ \big(q \Vdash_* \tau_1 = \tau_2 \rightarrow q \perp p\big).$$

We have $p \leqslant s_1$ and $p \in G$, so $s_1 \in G$. Hence $\mathbf{M}[G] \models (\pi_1)_G \in (\tau_1)_G$. Now, $\mathbf{M}[G] \models (\tau_1)_G = (\tau_2)_G$, so $\mathbf{M}[G] \models (\pi_1)_G \in (\tau_2)_G$. There thus exists $(\pi_2, s_2) \in \tau_2$ such that $(\pi_1)_G = (\pi_2)_G$.

By induction hypothesis, there exists $r \in G$ such that $r \Vdash_* \pi_1 = \pi_2$. It follows that there exists $q \leqslant r, s_2, p$ such that $q \Vdash_* \pi_1 = \pi_2$. But since $p$ satisfies $\psi_1$, from $q \leqslant s_2$ and $q \Vdash_* \pi_1 = \pi_2$ we deduce that $q \perp p$, but this contradicts the fact that $q \leqslant p$. The case of $\psi_2$ is analogous.

We can conclude by remarking that since $D$ is dense, there exists $p \in G \cap D$ such that $p \Vdash_* \tau_1 = \tau_2$.

$\boldsymbol{\varphi :\ x_1 \in x_2}$

(1) Suppose that there exists $p \in G$ such that $p \Vdash_* \tau_1 \in \tau_2$. The set

$$D = \big\{q \in \mathbb{P} \mid \exists(\pi_2, s_2) \in \tau_2 \quad \big(q \leqslant s_2 \ \wedge \ q \Vdash_* \tau_1 = \pi_2\big)\big\}$$

is thus dense below $p$. Since $G$ is $\mathbb{P}$-generic over $\mathbf{M}$, $G \cap D \neq \varnothing$. Let $q \in G \cap D$ and $(\pi_2, s_2) \in \tau_2$ be such that $q \leqslant s_2$ and $q \Vdash_* \tau_1 = \pi_2$. $G$ is a filter, so $s_2 \in G$, which in turn implies

$$\mathbf{M}[G] \models (\pi_2)_G \in (\tau_2)_G\,.$$

Furthermore, $q \in G$ and $q \Vdash_* \tau_1 = \pi_2$, so

$$\mathbf{M}[G] \models (\tau_1)_G = (\pi_2)_G\,.$$

Hence, $\mathbf{M}[G] \models (\tau_1)_G \in (\tau_2)_G$.

(2) Suppose that $\mathbf{M}[G] \models (\tau_1)_G \in (\tau_2)_G$. There thus exists $s_2 \in G$ such that $(\pi_2, s_2) \in \tau_2$ and $(\pi_2)_G = (\tau_1)_G$. Hence, by (2) for equality, there exists $q \in G$ such that $q \Vdash_* \pi_2 = \tau_1$. Since $G$ is a filter, there exists $p \in G$ such that $p \leqslant q$ and $p \leqslant s_2$. Since $p \leqslant q$ and $q \Vdash_* \pi_2 = \tau_1$, $p$ moreover verifies $p \Vdash_* \pi_2 = \tau_1$. The set

$$D = \big\{q \in \mathbb{P} \mid \exists(\pi_2, s_2) \in \tau_2 \quad \big(q \leqslant s_2 \ \wedge \ q \Vdash_* \tau_1 = \pi_2\big)\big\}$$

is then dense below $p$ since all $q' \leqslant p$ verify $q' \leqslant s_2$ and $q' \Vdash_* \tau_1 = \pi_2$. Hence $p \in G$ verifies $p \Vdash_* \tau_1 \in \tau_2$.

**$\varphi$: $(\varphi \wedge \psi)$**

(1) Suppose that there exists $p \in G$ such that $\big(p \Vdash_* (\varphi \wedge \psi)\big)^{\mathbf{M}}$. In particular, this means there exists $p \in G$ such that $(p \Vdash_* \varphi)^{\mathbf{M}}$ and $(p \Vdash_* \psi)^{\mathbf{M}}$ and by induction hypothesis, that $\mathbf{M}[G] \models \varphi$ and $\mathbf{M}[G] \models \psi$ both hold. Thus, $\mathbf{M}[G] \models (\varphi \wedge \psi)$ holds as well.

(2) Suppose that $\mathbf{M}[G] \models (\varphi \wedge \psi)$, so $\mathbf{M}[G] \models \varphi$ and $\mathbf{M}[G] \models \psi$. There thus exist $p, q \in G$ such that $(p \Vdash_* \varphi)^{\mathbf{M}}$ and $(q \Vdash_* \psi)^{\mathbf{M}}$. But since $G$ is a filter, there exists $r \in G$ such that $r \leqslant p$ and $r \leqslant q$, moreover such that $r$ verifies $(r \Vdash_* \varphi)^{\mathbf{M}}$ and $(r \Vdash_* \psi)^{\mathbf{M}}$. Hence, $\big(p \Vdash_* (\varphi \wedge \psi)\big)^{\mathbf{M}}$.

**$\varphi$: $\neg\varphi$**

(1) Suppose that there exists $p \in G$ such that $(p \Vdash_* \neg\varphi)^{\mathbf{M}}$. For the sake of contradiction, also suppose that $\mathbf{M}[G] \not\models \neg\varphi$. Then $\mathbf{M}[G] \models \varphi$, and so there exists $q \in G$ such that $(q \Vdash_* \varphi)^{\mathbf{M}}$. Since $G$ is a filter, there exists $r \in G$ such that $r \leqslant p$ and $r \leqslant q$. From $r \leqslant q$ and $(q \Vdash_* \varphi)^{\mathbf{M}}$, it follows that $(r \Vdash_* \varphi)^{\mathbf{M}}$. But $r \leqslant p$, so $(p \nVdash \neg\varphi)^{\mathbf{M}}$, which contradicts the assumptions we made on $p$.

(2) Suppose that $\mathbf{M}[G] \models \neg\varphi$. Let

$$D = \Big\{q \in \mathbb{P} \mid \big(q \Vdash_* \varphi\big)^{\mathbf{M}} \;\vee\; \big(q \Vdash_* \neg\varphi\big)^{\mathbf{M}}\Big\}.$$

The set $D$ is dense below $p \in \mathbb{P}$ in any case. Indeed, let $p \in \mathbb{P}$ and $q \leqslant p$, then we have two possible cases: either $(q \Vdash_* \neg\varphi)^{\mathbf{M}}$, and therefore $q \in D$, or there exists $r \leqslant q$ such that $(r \Vdash_* \varphi)^{\mathbf{M}}$ and $r \in D$.
Since for all $p \in \mathbb{P}$, $D$ is dense below $p$, $D \cap G \neq \varnothing$. Let $q \in D \cap G$, then either $(q \Vdash_* \neg\varphi)^{\mathbf{M}}$, and le the conclusion follows, or $(q \Vdash_* \varphi)^{\mathbf{M}}$. But the latter case is to exclude since it would imply that $\mathbf{M}[G] \models \varphi$.

$\boldsymbol{\exists x \quad \varphi(x, a_1, \ldots, a_n)}$ Let $\tau = (\tau_1, \ldots, \tau_n)$.

(1) Suppose that there exists $p \in G$ such that $\big(p \Vdash_* \exists x \quad \varphi(x, \tau)\big)^{\mathbf{M}}$. The set

$$D = \Big\{r \in \mathbb{P} \mid \exists \sigma \in \mathbf{M}^{\mathbb{P}} \big(r \Vdash_* \varphi(\sigma, \tau)\big)^{\mathbf{M}}\Big\}$$

is thus dense below $p$ and $D \cap G \neq \varnothing$. Let $q \in D \cap G$, there exists $\sigma \in \mathbf{M}^{\mathbb{P}}$ such that $\big(q \Vdash_* \varphi(\sigma, \tau)\big)^{\mathbf{M}}$. Hence $\mathbf{M}[G] \models \varphi\big((\sigma)_G, (\tau)_G\big)$. Therefore, $\mathbf{M}[G] \models \exists x \quad \varphi\big(x, (\tau)_G\big)$.

(2) Suppose that $\mathbf{M}[G] \models \exists x \quad \varphi\big(x, (\tau)_G\big)$. Let $(\sigma)_G$ be such that $\mathbf{M}[G] \models \varphi\big((\sigma)_G, (\tau)_G\big)$. By induction, there exists $p \in G$ such that $\big(p \Vdash_* \varphi(\sigma, \tau)\big)^{\mathbf{M}}$, so for all $r \leqslant p$, $\big(r \Vdash_* \varphi(\sigma, \tau)\big)^{\mathbf{M}}$. Thus

$$D = \Big\{r \in \mathbb{P} \mid \exists \sigma \in \mathbf{M}^{\mathbb{P}} \big(r \Vdash_* \varphi(\sigma, \tau)\big)^{\mathbf{M}}\Big\}$$

is dense below $p$ and it follows that $\big(p \Vdash_* \exists x \quad \varphi(x, \tau)\big)^{\mathbf{M}}$.

□ 319

At last, we are now able to prove the main result that connects the truth in $\mathbf{V}$ to the truth inside $\mathbf{M}$.

**Lemma 321.** *Let $\varphi(x_1, \ldots, x_n)$ be any $\mathcal{L}_{\mathrm{ST}}$-formula, $\mathbf{M}$ any c.t.m. of "**ZFC**", $\mathbb{P}$ any notion of forcing on $\mathbf{M}$, and $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$.*

*For all $p \in \mathbb{P}$,*

$$p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n) \Longleftrightarrow \big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

*Proof of Lemma 321:*

($\Longleftarrow$) Consider any $p \in \mathbb{P}$ such that $\big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}$. By Lemma 319 (1) for any[3] $G$

[3] Since $\mathbf{M}$ is a *c.t.m.* of "**ZFC**", such a $G$ $\mathbb{P}$-generic over $\mathbf{M}$ exists by Lemma 294.

$\mathbb{P}$-generic over $\mathbf{M}$ such that $p \in G$, one has $\mathbf{M}[G] \models \varphi(\tau_1, \ldots, \tau_n)$, which, by definition, is equivalent to $p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$.

($\Longrightarrow$) Fix $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$, and let

$$D = \left\{ r \in \mathbb{P} \mid \big(r \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}} \right\}.$$

$D$ is dense below $p$. Indeed, if this were not the case, there would exist $q \leqslant p$ such that for all $r \leqslant q$, $r \notin D$, i.e.,

$$\big(\forall r \leqslant q \quad r \nVdash \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

It would follow that $(q \Vdash_* \neg\varphi(\tau_1, \ldots, \tau_n))^{\mathbf{M}}$. By the reverse implication proved above, it would follow that $q \Vdash \neg\varphi(\tau_1, \ldots, \tau_n)$ and thus, for $G$ $\mathbb{P}$-generic over $\mathbf{M}$ with $q \in G$,

$$\mathbf{M}[G] \models \neg\varphi\big( (\tau_1)_G, \ldots, (\tau_n)_G \big)$$

would hold. But if $q \in G$, then $p \in G$ and having

$$\mathbf{M}[G] \models \varphi\big( (\tau_1)_G, \ldots, (\tau_n)_G \big)$$

would yield the desired contradiction.

□ 321

The next result is the main result of this chapter. It is really a theorem which builds on all the lemmas that were proved in this chapter. nevertheless, following the tradition of the "founding fathers", we do not call it a theorem, but a lemma.

However, its title — the "Truth Lemma" — should be enough to indicate that it is of major importance.

**The Truth Lemma.**

*Let $\varphi(x_1, \ldots, x_n)$ be any $\mathcal{L}_{\mathrm{ST}}$-formula, $\mathbf{M}$ any c.t.m. of "**ZFC**", $\mathbb{P}$ any notion of forcing on $\mathbf{M}$, and $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$.*

*For all $G$ $\mathbb{P}$-generic over $\mathbf{M}$,*

$$\mathbf{M}[G] \models \varphi\big( (\tau_1)_G, \ldots, (\tau_n)_G \big) \Longleftrightarrow \exists p \in G\big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

Viewed from the perspective of the generic extension — in the sense that we start from picking elements in $\mathbf{M}[G]$ and find a name for them later on, as opposed to beginning with $\mathbb{P}$-names — the Truth Lemma states that for all $G$ $\mathbb{P}$-generic over $\mathbf{M}$, and all sets $a_1, \ldots, a_n$ in $\mathbf{M}[G]$, we have

$$\mathbf{M}[G] \models \varphi\big(a_1, \ldots, a_n\big) \Longleftrightarrow \exists p \in G\big(p \Vdash_* \varphi(\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)\big)^{\mathbf{M}}.$$

*Proof of the Truth Lemma*: This is an immediate consequence of Lemmas 319 and 321.

□ Truth Lemma

Combining the Truth Lemma with Lemma 321 we obtain the following picture:

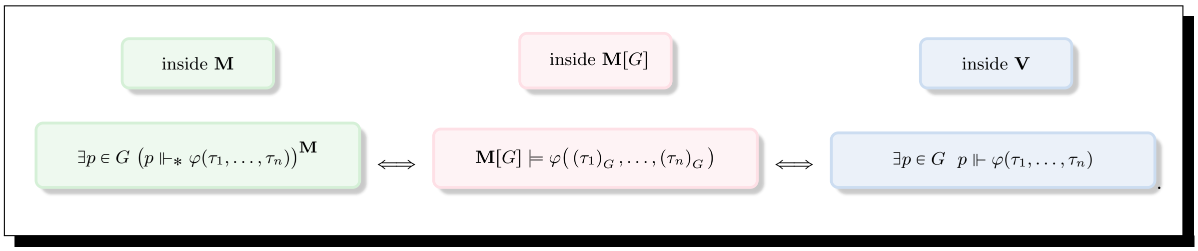



Figure 15.2: The connections between the forcing relations and the generic extension.

If the ground model is a *c.t.m.* of " **ZFC** ", the *forcing relation preserves all logical consequences*. This means that as soon as some forcing condition $p$ forces some formula $\varphi$, it also forces all the formulas that are deductible from $\varphi$. Namely,

**Proposition 323.** *Let $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$, be any $\mathcal{L}_{\text{ST}}$-formula, $\mathbf{M}$ any c.t.m. of "**ZFC** ", $\mathbb{P}$ any notion of forcing on $\mathbf{M}$, and $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$. For all $p \in \mathbb{P}$,*

$$\left.\begin{array}{c} \big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}} \\ \textit{and} \\ \vdash_c \forall x_1 \ldots \forall x_n\ \big(\varphi(x_1, \ldots, x_n) \longrightarrow \psi(x_1, \ldots, x_n)\big) \end{array}\right\} \Longrightarrow \big(p \Vdash_* \psi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

*Proof of Proposition 323*: Consider in $\mathbf{M}$ the following set:

$$Q = \big\{q \in \mathbb{P} \mid q \Vdash_* \psi(\tau_1, \ldots, \tau_n)\big\}.$$

We show that $Q$ is dense below $p$. Towards a contradiction, let us assume that there exists some $s \leqslant p$ such that for all $t \leqslant s$

$$t \nVdash_* \psi(\tau_1, \ldots, \tau_n).$$

This implies

$$s \Vdash_* \neg\psi(\tau_1, \ldots, \tau_n).$$

Since by Lemma 314 we have $s \leqslant p$ gives

$$s \Vdash_* \varphi(\tau_1, \ldots, \tau_n),$$

we end up with

$$s \Vdash_* \big(\varphi(\tau_1, \ldots, \tau_n) \wedge \neg\psi(\tau_1, \ldots, \tau_n)\big).$$

By Lemma 294 there exists some filter $G$ $\mathbb{P}$-generic over $\mathbf{M}$ such that $s \in G$. By the Truth Lemma, we have

$$\Big(s \in G \wedge \big(s \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}} \wedge \big(s \Vdash_* \neg\psi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}\Big)$$

implies

$$\mathbf{M}[G] \models \varphi(\tau_1, \ldots, \tau_n) \;\text{ and }\; \mathbf{M}[G] \models \neg\psi(\tau_1, \ldots, \tau_n)$$

Now, since

$$\vdash_c \forall x_1 \ldots \forall x_n \; \big(\varphi(x_1, \ldots, x_n) \longrightarrow \psi(x_1, \ldots, x_n)\big)$$

we have

$$\models \forall x_1 \ldots \forall x_n \; \big(\varphi(x_1, \ldots, x_n) \longrightarrow \psi(x_1, \ldots, x_n)\big)$$

and in particular

$$\mathbf{M}[G] \models \forall x_1 \ldots \forall x_n \; \big(\varphi(x_1, \ldots, x_n) \longrightarrow \psi(x_1, \ldots, x_n)\big)$$

which yields

$$\mathbf{M}[G] \models \big(\varphi(\tau_1, \ldots, \tau_n) \longrightarrow \psi(\tau_1, \ldots, \tau_n)\big)$$

By *modus ponens* this gives

$$\mathbf{M}[G] \models \psi(\tau_1, \ldots, \tau_n)$$

which yields the following contradiction

$$\mathbf{M}[G] \models \big(\psi(\tau_1, \ldots, \tau_n) \wedge \neg\psi(\tau_1, \ldots, \tau_n)\big).$$

So, we have shown that $Q$ is dense below $p$, and by Lemma 317 we obtain

$$\big(p \Vdash_* \psi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

□ 323

**Corollary 324.** *Let $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ be any logically equivalent $\mathcal{L}_{\text{ST}}$-formulas, $\mathbf{M}$ any c.t.m. of "**ZFC**", $\mathbb{P}$ any notion of forcing on $\mathbf{M}$, and $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$. For all $p \in \mathbb{P}$, we have*

$$\big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}} \iff \big(p \Vdash_* \psi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}.$$

We recall that $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are logically equivalent if they satisfy

$$\vdash_c \forall x_1 \ldots \forall x_n \; \big(\varphi(x_1, \ldots, x_n) \longleftrightarrow \psi(x_1, \ldots, x_n)\big).$$

*Proof of Corollary 324:* Immediate.

□ 324

From now on:

(1) if $\mathbf{M}$ is fixed, we will identify $p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\tau_1, \ldots, \tau_n)$ and $\big(p \Vdash_* \varphi(\tau_1, \ldots, \tau_n)\big)^{\mathbf{M}}$;

(2) in case both $\mathbb{P}$ and $\mathbf{M}$ are clear from the context, we will simply write $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$.

# Chapter 16

# ZFC within the Generic Extension and Cardinal Preservation

## 16.1 ZFC within the Generic Extension

This whole section is dedicated to proving, providing one starts with a ground model $\mathbf{M}$ that satisfies "**ZFC**", that the generic extension $\mathbf{M}[G]$ also satisfies "**ZFC**". Of, course, this statement should be understood backwards: whatever finite set of axioms $\Delta$ from **ZFC** we consider, we will end up with some generic extension $\mathbf{M}[G]$ that satisfies $\Delta$, providing we start from a ground model $\mathbf{M}$ that satisfies some (other) finite set $\Gamma$ of axioms from **ZFC**, where the relation between $\Gamma$ and $\Delta$ could be made explicit (but will never be).

**Theorem 325.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $(\mathbb{P}, \leqslant, \mathbb{1}) \in \mathbf{M}$ *be any partial order and* $G$ *be* $\mathbb{P}$*-generic over* $\mathbf{M}$*.*

$$\mathbf{M}[G] \text{ satisfies "}\mathbf{ZFC}\text{".}$$

This theorem really states that given any finite sub-theory $\Delta \subsetneq \mathbf{ZFC}$, there exists some finite sub-theory $\Gamma \subsetneq \mathbf{ZFC}$ such that in order to have $\mathbf{M}[G] \models \Delta$, it is enough to start from any *c.t.m.* $\mathbf{M}$ which satisfies $\mathbf{M} \models \Gamma$.

*Proof of Lemma* 325:

**Extensionality:** holds in $\mathbf{M}[G]$ since $\mathbf{M}[G]$ is transitive.

**Comprehension Schema:** We want to show that for all $\sigma, \pi_1, \ldots, \pi_n \in \mathbf{M}^{\mathbb{P}}$ and $\varphi(x, y_1, \ldots, y_n)$:

$$u = \left\{ z \in (\sigma)_G \mid \Big(\varphi\big(z, (\pi_1)_G, \ldots, (\pi_n)_G\big)\Big)^{\mathbf{M}[G]} \right\} \in \mathbf{M}[G].$$

We must find some $\tau \in \mathbf{M}^{\mathbb{P}}$ such that $u = (\tau)_G$. So, we set

$$\tau = \left\{(\theta, p) \in dom(\sigma) \times \mathbb{P} \mid p \Vdash_{\mathbb{P},\mathbf{M}} \left(\theta \in \sigma \wedge \varphi(\theta, \pi_1, \ldots, \pi_n)\right)\right\}.$$

We show that $(\tau)_G = u$.

$$\begin{aligned}
(\tau)_G &= \left\{ (\theta)_G \mid \exists p \in G \ \ (\theta, p) \in \tau\right\} \\
&= \left\{(\theta)_G \mid \theta \in dom(\sigma) \wedge \exists p \in G \ \ p \Vdash_{\mathbb{P},\mathbf{M}} \left(\theta \in \sigma \ \wedge \ \varphi(\theta, \pi_1, \ldots, \pi_n)\right)\right\} \\
&= \left\{(\theta)_G \mid \theta \in dom(\sigma) \ \wedge \ \left( (\theta)_G \in (\sigma)_G \ \wedge \ \varphi\left( (\theta)_G, (\pi_1)_G, \ldots, (\pi_n)_G\right)\right)^{\mathbf{M}[G]}\right\} \\
&= \left\{(\theta)_G \mid \theta \in dom(\sigma) \ \wedge \ (\theta)_G \in (\sigma)_G \ \wedge \ \left(\varphi\left( (\theta)_G, (\pi_1)_G, \ldots, (\pi_n)_G\right)\right)^{\mathbf{M}[G]}\right\} \\
&= u.
\end{aligned}$$

**Pairing:** We assume $\mathbf{M}$ is a *c.t.m.* of sufficiently enough finitely many formulas from "**ZFC**", so that given any $\mathbb{P}$-names $\tau, \sigma \in \mathbf{M}^{\mathbb{P}}$, we have[1] $\{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\} \in \mathbf{M}^{\mathbb{P}}$. Then we make use of the fact $\mathbb{1}$ belongs to $G$ to obtain:

$$\{(\sigma, \mathbb{1}), (\tau, \mathbb{1})\}_G = \{(\sigma)_G, (\tau)_G\} \in \mathbf{M}[G].$$

**Union:** Let $\sigma \in \mathbf{M}^{\mathbb{P}}$, to prove that $\bigcup (\sigma)_G \in \mathbf{M}[G]$, it is enough to show that there exists $\tau \in \mathbf{M}^{\mathbb{P}}$ such that $\bigcup (\sigma)_G \subseteq (\tau)_G$. We recall that

$$dom(\sigma) = \left\{\pi \in \mathbf{M}^{\mathbb{P}} \mid \exists p \in \mathbb{P} \ (\pi, p) \in \sigma\right\}.$$

We set

$$\tau = \bigcup dom(\sigma).$$

Since $\mathbf{M}$ is a *c.t.m.* of " a sufficiently large enough amount of axioms from **ZFC**," we have $\tau \in \mathbf{M}^{\mathbb{P}}$. Let $\pi \in dom(\sigma)$, then $\pi \subseteq \bigcup dom(\sigma) = \tau$, and thus $(\pi)_G \subseteq (\tau)_G$, which yields

$$\begin{aligned}
\bigcup (\sigma)_G &= \bigcup \left\{(\pi)_G \mid \exists p \in G \ (\pi, p) \in \sigma\right\} \\
&\subseteq \bigcup \left\{(\pi)_G \mid (\pi, p) \in \sigma\right\} \\
&= \left(\bigcup dom(\sigma)\right)_G \\
&= (\tau)_G.
\end{aligned}$$

[1] It is enough to guarantee, for instance, that $\mathbf{M}$ is closed under the class-function $(x, y) \mapsto \{(x, \mathbb{1}), (y, \mathbb{1})\}$.

**Infinity:** This axiom holds in $\mathbf{M}[G]$ since both $\omega \in \mathbf{M}$ and $\check{\omega} \in \mathbf{M}^{\mathbb{P}}$ are satisfied, and $\omega = (\check{\omega})_G \in \mathbf{M}[G]$.

**Power Set:** Let $\sigma \in \mathbf{M}^{\mathbb{P}}$, we must show that the set $\mathscr{P}\big((\sigma)_G\big) \cap \mathbf{M}[G]$ belongs to $\mathbf{M}[G]$. For this, it is enough to show that there exists $\tau \in \mathbf{M}^{\mathbb{P}}$ such that $\mathscr{P}\big((\sigma)_G\big) \cap \mathbf{M}[G] \subseteq (\tau)_G$, we then get the result by making use of both an instance of the comprehension schema and the axiom of extensionality.

We consider :

$$\begin{aligned} S &= \{\mu \in \mathbf{M}^{\mathbb{P}} \mid dom(\mu) \subseteq dom(\sigma)\} \\ &= \{\mu \in \mathbf{M}^{\mathbb{P}} \mid \mu \subseteq \big(dom(\sigma) \times \mathbb{P}\big)\} \\ &= \Big(\mathscr{P}\big(dom(\sigma) \times \mathbb{P}\big)\Big) \cap \mathbf{M}^{\mathbb{P}} \\ &= \Big(\mathscr{P}\big(dom(\sigma) \times \mathbb{P}\big)\Big)^{\mathbf{M}}. \end{aligned}$$

Notice that given any $b \in \mathscr{P}\big((\sigma)_G\big) \cap \mathbf{M}[G]$ and any $\mathbb{P}$-name $\underset{\sim}{b}$ for $b$, we have both

(1) the set $\underset{\sim}{b}' = \{(\theta, p) \in dom(\sigma) \times \mathbb{P} \mid p \Vdash_{\mathbb{P},\mathbf{M}} \theta \in \underset{\sim}{b}\}$ belongs to $S$;

(2) and

$$\begin{aligned} (\underset{\sim}{b}')_G &= \{(\theta)_G \in \mathbf{M}[G] \mid \theta \in dom(\sigma) \ \wedge\ \exists p \in G \ \ p \Vdash \theta \in \underset{\sim}{b}\} \\ &= \{(\theta)_G \in \mathbf{M}[G] \mid \theta \in dom(\sigma) \ \wedge\ (\theta)_G \in (\underset{\sim}{b})_G\} \\ &= \{(\theta)_G \in \mathbf{M}[G] \mid \theta \in dom(\sigma) \ \wedge\ (\theta)_G \in b\} \\ &= \{(\theta)_G \in \mathbf{M}[G] \mid (\theta)_G \in b\} \\ &= \{a \in \mathbf{M}[G] \mid a \in b\} \\ &= b. \end{aligned}$$

Assuming " **ZFC**, " contains enough axioms to guarantee that $S \times \{\mathbb{1}\}$ is some $\mathbb{P}$-name that belongs to $\mathbf{M}$, we set $\tau = S \times \{\mathbb{1}\}$ and we obtain:

$$\begin{aligned}
(\tau)_G &= (S \times \{\mathbb{1}\})_G \\
&= \{ (\mu)_G \mid \mu \in S \} \\
&= \{ (\mu)_G \mid \mu \in \mathbf{M}^{\mathbb{P}} \ \wedge \ \mu \subseteq \big(dom(\sigma) \times \mathbb{P}\big) \} \\
&\supseteq \Big\{ (\underset{\sim}{b}')_G \mid (\underset{\sim}{b})_G \in \mathscr{P}\big( (\sigma)_G \big) \cap \mathbf{M}[G] \Big\} \\
&= \Big\{ (\underset{\sim}{b})_G \mid (\underset{\sim}{b})_G \in \mathscr{P}\big( (\sigma)_G \big) \cap \mathbf{M}[G] \Big\} \\
&= \mathscr{P}\big( (\sigma)_G \big) \cap \mathbf{M}[G].
\end{aligned}$$

**Foundation:** This axiom holds in $\mathbf{M}[G]$ because $\mathbf{M}[G]$ is transitive and $\mathbf{M}$ satisfies the axiom of **Foundation** — which simply means that the axiom of **Foundation** belongs to "**ZFC**". To show this, we simply show that any infinite $\ni$-descending chain in $\mathbf{M}[G]$, would yield some other infinite $\ni$-descending chain in $\mathbf{M}$.

Notice that for all sets $a, b \in \mathbf{M}[G]$ with $\underset{\sim}{b}$ any $\mathbb{P}$-name such that $(\underset{\sim}{b})_G = b$, we have

$$\begin{aligned}
a \in b \in \mathbf{M}[G] &\Longrightarrow \exists \underset{\sim}{a} \in dom\,(\underset{\sim}{b}) \quad (\underset{\sim}{a})_G = a \\
&\Longrightarrow \exists \underset{\sim}{a}\ rk\,(\underset{\sim}{a}) < rk\,(\underset{\sim}{b})\,;
\end{aligned}$$

which induces

$$\Big(\exists (a_i)_{i\in\omega}\ \forall i \in \omega \ \ a_{i+1} \in a_i\Big)^{\mathbf{M}[G]} \Longrightarrow \Big(\exists (\underset{\sim}{a}_i)_{i\in\omega}\ \forall i \in \omega \ \ \underset{\sim}{a}_{i+1} \in dom\,(\underset{\sim}{a}_i)\Big)^{\mathbf{M}}$$

and equivalently

$$\begin{array}{ccc}
\neg\Big(\exists (\underset{\sim}{a}_i)_{i\in\omega}\ \forall i \in \omega \ \ \underset{\sim}{a}_{i+1} \in dom\,(\underset{\sim}{a}_i)\Big)^{\mathbf{M}} & \Longrightarrow & \neg\Big(\exists (a_i)_{i\in\omega}\ \forall i \in \omega \ \ a_{i+1} \in a_i\Big)^{\mathbf{M}[G]} \\
\| & & \| \\
\Big(\neg\exists (\underset{\sim}{a}_i)_{i\in\omega}\ \forall i \in \omega \ \ \underset{\sim}{a}_{i+1} \in dom\,(\underset{\sim}{a}_i)\Big)^{\mathbf{M}} & & \Big(\neg\exists (a_i)_{i\in\omega}\ \forall i \in \omega \ \ a_{i+1} \in a_i\Big)^{\mathbf{M}[G]}
\end{array}$$

Since

$$\Big(\neg\exists (\underset{\sim}{a}_i)_{i\in\omega}\ \forall i \in \omega \ \ \underset{\sim}{a}_{i+1} \in dom\,(\underset{\sim}{a}_i)\Big)^{\mathbf{M}}$$

holds, it follows that

$$\Big(\neg\exists (a_i)_{i\in\omega}\ \forall i \in \omega \ \ a_{i+1} \in a_i\Big)^{\mathbf{M}[G]}$$

holds as well.

**Replacement Schema:** for each formula $\varphi(x, y, z_1, \ldots, z_n)$, we want to prove that:

$$\forall z_1, \ldots, \forall z_n \in \mathbf{M}[G] \left( \begin{array}{c} \forall x \in \mathbf{M}[G]\ \exists! y \in \mathbf{M}[G]\ \ \big(\varphi(x, y, z_1, \ldots, z_n)\big)^{\mathbf{M}[G]} \\ \longrightarrow \\ \forall u \in \mathbf{M}[G]\ \exists v \in \mathbf{M}[G]\ \forall x \in u\ \exists y \in v\ \ \big(\varphi(x, y, z_1, \ldots, z_n)\big)^{\mathbf{M}[G]} \end{array} \right).$$

We fix $a_1 = (\underset{\sim}{a}_1)_G, \ldots, a_n = (\underset{\sim}{a}_n)_G$, and $u = (\underset{\sim}{u})_G$. Inside $\mathbf{M}$ we define:

$$\begin{aligned} &\mathbf{F} : dom(\underset{\sim}{u}) \times \mathbb{P} \to \mathbf{On} \\ &(\underset{\sim}{a}, p) \quad \to \begin{cases} \text{least } \alpha \in \mathbf{On} \text{ s.t. } \exists \underset{\sim}{b} \in \mathbf{M}^{\mathbb{P}} \cap \mathbf{V}_\alpha \ \ p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n) \\ 0 \text{ otherwise.} \end{cases} \end{aligned}$$

Since $\mathbf{M}$ satisfies the finitely many instances of the replacement schema our proof requires, there exists $\beta \in (\mathbf{On})^{\mathbf{M}}$ such that $\mathbf{F}\left[dom(\underset{\sim}{u}) \times \mathbb{P}\right] \subseteq \beta$. We set:

$$\eta = \left(\mathbf{M}^{\mathbb{P}} \cap \mathbf{V}_\beta\right) \times \{\mathbb{1}\} \in \mathbf{M}.$$

We assume

$$\forall x \in \mathbf{M}[G]\ \exists! y \in \mathbf{M}[G]\ \ \big(\varphi(x, y, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)\big)^{\mathbf{M}[G]}$$

and let $a \in u$. It follows that there exists some — unique — $b \in \mathbf{M}[G]$ such that

$$\big(\varphi(a, b, a_1, \ldots, a_n)\big)^{\mathbf{M}[G]}.$$

Therefore there exists $p \in G$ such that given any $\mathbb{P}$-names $\underset{\sim}{a} \in dom(\underset{\sim}{u})$ and $\underset{\sim}{b} \in \mathbf{M}^{\mathbb{P}}$ which satisfy $(\underset{\sim}{a})_G = a$ and $(\underset{\sim}{b})_G = b$, respectively, we have

$$p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)$$

It follows that there exists $\underset{\sim}{b}' \in \eta = \left(\mathbf{M}^{\mathbb{P}} \cap \mathbf{V}_\beta\right) \times \{\mathbb{1}\}$ such that

$$p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{a}, \underset{\sim}{b}', \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)$$

The Truth Lemma yields

$$\mathbf{M}[G] \models \varphi\Big(a, (\underset{\sim}{b}')_G, a_1, \ldots, a_n\Big).$$

Finally, by unicity, we obtain $b = (\underset{\sim}{b}')_G$, which shows that $b \in (\eta)_G$. Therefore $(\eta)_G$ satisfies

$$\left\{ b \in \mathbf{M}[G] \mid \exists a \in u\ \ \Big(\varphi\big(a, b, a_1, \ldots, a_n\big)\Big)^{\mathbf{M}[G]} \right\} \subseteq (\eta)_G.$$

**Choice:** In order to establish that $(\mathbf{AC})^{\mathbf{M}[G]}$ we show that any $A \in \mathbf{M}[G]$ can be well-ordered. To do so, it is enough to show that given any set $A \in \mathbf{M}[G]$, there exists some set $B \supseteq A$ such that $B \in \mathbf{M}[G]$ and $B$ can be well-ordered. (Indeed, the restriction of any well-ordering of $B$ to $A$ is some well-ordering of $A$). Now, in $\mathbf{M}[G]$, if we show that for some set $B$, there exist an ordinal $\alpha$ and a mapping $f : \alpha \xrightarrow{onto} B$, then $B$ can be well-ordered by: for all $b, b' \in B$,

$$b < b' \quad \text{if and only if} \quad \bigcap\{\beta \in \alpha \mid f(\beta) = b\} \in \bigcap\{\beta' \in \alpha \mid f(\beta') = b'\}.$$

Or to say it differently,

$$b < b' \quad \text{if and only if} \quad \min\{\beta \in \alpha \mid f(\beta) = b\} < \min\{\beta' \in \alpha \mid f(\beta') = b'\}.$$

Since we assume $\mathbf{M}$ is a *c.t.m.* of finitely many axioms from **ZFC**, we assume in particular that the axiom of choice (**AC**) is among those finitely many axioms. Thus, **AC** holds in $\mathbf{M}$.

Given any $A \in \mathbf{M}[G]$, we let $\underset{\sim}{A} \in \mathbf{M}^{\mathbb{P}}$ be some $\mathbb{P}$-name for $A$. i.e., it satisfies $(\underset{\sim}{A})_G = A$.

Inside $\mathbf{M}$, there exist some ordinal $\alpha$ and some mapping

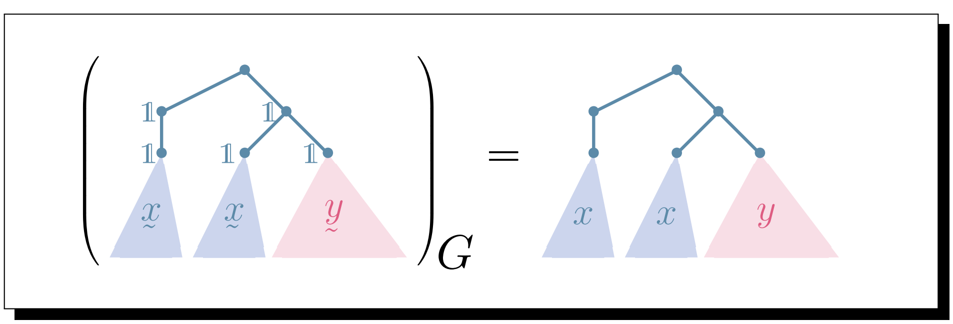
$$g : \alpha \xrightarrow{onto} dom(\underset{\sim}{A}).$$

We make use of the class-function $couple : \mathbf{M}^{\mathbb{P}} \times \mathbf{M}^{\mathbb{P}} \rightarrow \mathbf{M}^{\mathbb{P}}$ that was defined in Example 309. We recall that

$$couple(\underset{\sim}{x}, \underset{\sim}{y}) = \left\{ \left( \{(\underset{\sim}{x}, \mathbb{1})\}, \mathbb{1} \right), \left( \{(\underset{\sim}{x}, \mathbb{1}), (\underset{\sim}{y}, \mathbb{1})\} \right\}$$

as shown in the picture below, providing $(\underset{\sim}{x})_G = x$ and $(\underset{\sim}{y})_G = y$, we have

$$\left( couple(\underset{\sim}{x}, \underset{\sim}{y}) \right)_G = (x, y).$$



We consider the $\mathbb{P}$-name $\underset{\sim}{f} \in \mathbf{M}^{\mathbb{P}}$ defined by

$$\underset{\sim}{f} = \left\{ \left( couple(\check{\beta}, g(\beta)), \mathbb{1} \right) \in \mathbf{M}^{\mathbb{P}} \times \{\mathbb{1}\} \mid \beta \in \alpha \right\}$$

We then have

$$\begin{aligned} f = \left(\underset{\sim}{f}\right)_G &= \left\{ \left( couple(\check{\beta}, g(\beta)) \right)_G \in \mathbf{M}[G] \mid \left(\beta \in \alpha\right)^{\mathbf{M}[G]} \right\} \\ &= \left\{ \left( (\check{\beta})_G, \left(g(\beta)\right)_G \right) \in \mathbf{M}[G] \mid \beta \in \alpha \right\} \\ &= \left\{ \left( \beta, \left(g(\beta)\right)_G \right) \in \mathbf{M}[G] \mid \beta \in \alpha \right\}. \end{aligned}$$

Clearly, $f$ is some mapping from $\alpha$ to some set

$$B = ran(f) = \left\{ \left(g(\beta)\right)_G \mid \beta \in \alpha \right\} = \left\{ (\sigma)_G \mid \sigma \in dom(\underset{\sim}{A}) \right\}$$

that belongs to $\mathbf{M}[G]$ and, by construction, satisfies $A \subseteq \left\{ (\sigma)_G \mid \sigma \in dom(\underset{\sim}{A}) \right\} = B$.

It remains to show

$$f : \alpha \xrightarrow{onto} B$$

To show that $f$ is onto, it suffices to notice that for every $b \in B$, there exist some $\sigma \in dom(\underset{\sim}{A})$ and some $\beta < \alpha$ such that $b = (\sigma)_G$. Since $g : \alpha \xrightarrow{onto} dom(\underset{\sim}{A})$ is surjective, there exists some $\beta < \alpha$ such that $g(\beta) = \sigma$. Thus we finally obtain:

$$b = (\sigma)_G = \left(g(\beta)\right)_G = f(\beta).$$

This shows that $f : \alpha \xrightarrow{onto} B$, which yields the existence of a well-ordering of $B$, and since $A \subseteq B$ holds, the existence of a well-ordering of $A$ as well.

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## 16.2 A First Attempt to Deny CH

As a start, we try to apply our knowledge of the generic extensions and propose a notion of forcing $\mathbb{P} \in \mathbf{M}$, where $\mathbf{M}$ is a *c.t.m.* of "**ZFC**", such that for every $G$ that is $\mathbb{P}$-generic over $\mathbf{M}$ one has

$$\mathbf{M}[G] \models \neg\mathbf{CH}.$$

As we will later see, this first attempt will hit the target, but in order for us to be able to realize that, we will need to discuss the notion of "cardinal preservation".

**Example 326.**
*We let $\mathbb{P}$ be the following notion of forcing, and $\mathbf{M}$ be a c.t.m. of "**ZFC**" with $\mathbb{P} \in \mathbf{M}$:*

$$\mathbb{P} = \left\{ f \ : \ \left(\omega_2\right)^{\mathbf{M}} \times \omega \to \{0,1\} \mid dom(f) \ \text{ is finite} \right\}$$

*with*

$$f \leqslant g \iff f \supseteq g$$

*and*

$$\mathbb{1} = \varnothing.$$

*Let $G$ be $\mathbb{P}$-generic over $\mathbf{M}$ and $\mathscr{F} = \bigcup G$. From Exercise 293 we already know that*

*(1) $\mathscr{F}$ is a function (see Exercise 293):*

$$\mathscr{F} \ : \ \left(\omega_2\right)^{\mathbf{M}} \times \omega \longrightarrow \{0,1\}.$$

*(2) Given any $p \in \mathbb{P}$ and any ordinal $\alpha < \left(\omega_2\right)^{\mathbf{M}}$ and any integer $n$ such that $(\alpha, n) \notin dom(p)$, one has $r = p \cup \left\{\left((\alpha,n),0\right)\right\}$ and $q = p \cup \left\{\left((\alpha,n),1\right)\right\}$ satisfy*

$$q \leqslant p \ \wedge \ r \leqslant p \ \wedge \ q \perp r$$

*hence, by Lemma 295, $G \notin \mathbf{M}$.*

*(3) By Lemma 307, $G \in \mathbf{M}[G]$, hence $\mathscr{F} \in \mathbf{M}[G]$.*

*For $\alpha < \beta < \left(\omega_2\right)^{\mathbf{M}}$, we consider:*

$$D_{\alpha,\beta} = \left\{ p \in \mathbb{P} \mid \exists n < \omega \ \ \left((\alpha,n) \in dom(p) \ \ \wedge \ \ (\beta,n) \in dom(p) \ \ \wedge \ \ p(\alpha,n) \neq p(\beta,n)\right) \right\}.$$

*We show that $D_{\alpha,\beta}$ is dense in $\mathbb{P}$. Indeed, let $q \in \mathbb{P}$, since $dom(q)$ is finite, there exists $n \in \omega$ such that $(\alpha,n)$ and $(\beta,n)$ do not belong to $dom(q)$. Set*

$$p = q \cup \left\{ \left((\alpha,n),0\right), \left((\beta,n),1\right) \right\},$$

*to obtain $p \leqslant q$ and $p \in D_{\alpha,\beta}$, which shows that $D_{\alpha,\beta}$ is dense in $\mathbb{P}$.*

*We also have $D_{\alpha,\beta}$ belongs to $\mathbf{M}$ (any $\alpha < \beta < \left(\omega_2\right)^{\mathbf{M}}$ and, since $G$ is $\mathbb{P}$-generic over $\mathbf{M}$, for all $\alpha < \beta < \left(\omega_2\right)^{\mathbf{M}}$, we also have:*

$$D_{\alpha,\beta} \cap G \neq \varnothing.$$

*Thus there exist $p \in G$ and $n \in \omega$ such that*

$$p(\alpha,n) \neq p(\beta,n).$$

*It follows that for all* $\alpha < \beta < \left(\omega_2\right)^{\mathbf{M}}$*, there exists an integer* $n$ *such that*

$$\mathscr{F}(\alpha, n) \neq \mathscr{F}(\beta, n).$$

*For each ordinal* $\alpha < \left(\omega_2\right)^{\mathbf{M}}$*, we consider the following subset of the integers:*

$$a_\alpha = \{n < \omega \mid \mathscr{F}(\alpha, n) = 1\}.$$

*For all* $\alpha < \beta < \left(\omega_2\right)^{\mathbf{M}}$*, since there exists* $n \in \omega$ *such that* $\mathscr{F}(\alpha, n) \neq \mathscr{F}(\beta, n)$*, we have*

$$a_\alpha \neq a_\beta.$$

*It follows that there exist at least* $\left(\omega_2\right)^{\mathbf{M}}$*-many different subsets of* $\omega$ *in* $\mathbf{M}[G]$*.*

In the Example above, a question remains:

$$\text{what is the cardinality of } \left(\omega_2\right)^{\mathbf{M}} \text{ inside } \mathbf{M}[G] \text{ ?}$$

i.e.,

$$\text{what is } \left|\left(\omega_2\right)^{\mathbf{M}}\right|^{\mathbf{M}[G]} \text{ ?}$$

In order to succeed in our attempt, we would like two things:

(1) to claim $\mathbf{M}[G] \models 2^{\aleph_0} \geqslant \aleph_2$, and

(2) to carefully be able to determine whether or not $\left(\omega_2\right)^{\mathbf{M}} = \left(\omega_2\right)^{\mathbf{M}[G]}$ holds.

In order to answer these questions, we need to investigate the collapse of cardinal numbers that may occur during the move from $\mathbf{M}$ to $\mathbf{M}[G]$. In particular, since the ordinals of $\mathbf{M}$ and $\mathbf{M}[G]$ are the same, we would like to know of some conditions which guarantee that the ordinals that are cardinals in $\mathbf{M}$ still remain cardinals in $\mathbf{M}[G]$.

## 16.3 Cardinal Preservation

We recall from Definition 288, that given $\mathbf{M}$ any *c.t.m.* of "**ZFC**", $(\mathbb{P} \leqslant, \mathbb{1}) \in \mathbf{M}$ any notion of forcing on $\mathbf{M}$,

$$\mathscr{A} \subseteq \mathbb{P} \text{ is a (strong) } \textit{antichain} \iff \forall p \in \mathscr{A}\ \forall q \in \mathscr{A}\ \left(p \neq q \longrightarrow p \perp q\right).$$

**Definition 327.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $\mathbb{P} \in \mathbf{M}$ *any notion of forcing on* $\mathbf{M}$*, and* $\left(\lambda \textit{ is a cardinal}\right)^{\mathbf{M}}$*, we say*

$\mathbb{P}$ *has the* $\lambda$-chain condition *— or* $\mathbb{P}$ *is* $\lambda$*-c.c. —*

$$\Longleftrightarrow$$

*in* $\mathbf{M}$*, every antichain of* $\mathbb{P}$ *has cardinality strictly less than* $\lambda$*.*

*We say that* $\mathbb{P}$ *is c.c.c. if* $\mathbb{P}$ *is* $\aleph_1$*-c.c..*

$\lambda$*-chain condition* is the wording commonly adopted. However, the correct formulation should rather be $\lambda$*-antichain condition*, or even $\lambda$*-strong antichain condition*.

**Definition 328.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $\mathbb{P} \in \mathbf{M}$ *any notion of forcing on* $\mathbf{M}$*, and* $(\lambda \text{ is a cardinal})^{\mathbf{M}}$*.*

$\mathbb{P}$ *preserves cardinals* $\geqslant \lambda$ *(respectively* $\leqslant \lambda$*)*

$$\Longleftrightarrow$$

*for all* $G$ $\mathbb{P}$*-generic over* $\mathbf{M}$*,* $\mathbf{M}$ *and* $\mathbf{M}[G]$ *have the same cardinals* $\geqslant \lambda$ *(respectively* $\leqslant \lambda$*).*

The following theorem gives an explicit condition on the poset which guarantees that the cardinals above some threshold are preserved.

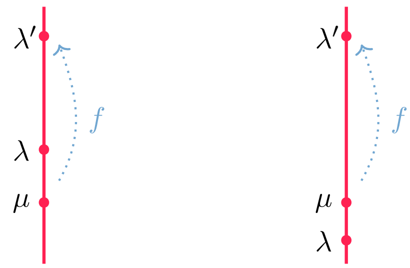
**Theorem 329.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $\mathbb{P} \in \mathbf{M}$ *any notion of forcing on* $\mathbf{M}$*.*

$$\left.\begin{array}{c} (\lambda \text{ is a regular cardinal})^{\mathbf{M}} \\ \text{and} \\ (\mathbb{P} \text{ is } \lambda\text{-c.c.})^{\mathbf{M}} \end{array}\right\} \Longrightarrow \mathbb{P} \text{ preserves cardinals } \geqslant \lambda.$$

In particular, if $\mathbf{M} \models \lambda = \aleph_1$, we have $\mathbf{M} \models$ "$\aleph_1$ is regular" because we assume **AC** is part of the finitely many axioms that $\mathbf{M}$ satisfies. Therefore, given any $\mathbb{P} \in \mathbf{M}$ which satisfies $\mathbf{M} \models \mathbb{P}$ is -c.c.c. ($\mathbb{P}$ is $\aleph_1$-c.c.), we have $\mathbb{P}$ preserves all cardinals $\geqslant \aleph_1$. Since $\aleph_0$ and all finite cardinals are all absolute for transitive classes, we have that $\mathbf{M}$ and $\mathbf{M}[G]$ have exactly the same cardinals.

*Proof of Theorem 329*: Let $G$ be $\mathbb{P}$-generic over $\mathbf{M}$. Towards a contradiction, we suppose there exists $\lambda' \geqslant \lambda$ a cardinal (in $\mathbf{M}$) such that $\lambda'$ is collapsed (in $\mathbf{M}[G]$) down to some ordinal $\mu < \lambda'$. So, there exists, inside $\mathbf{M}[G]$, a mapping from $\mu$ onto $\lambda'$:

$$f : \mu \xrightarrow{\text{onto}} \lambda'$$




We let $\underset{\sim}{f} \in \mathbf{M}^{\mathbb{P}}$ satisfy $f = \left(\underset{\sim}{f}\right)_G$. By the Truth Lemma, there exists $p_0 \in G$ such that:

$$p_0 \Vdash \underset{\sim}{f} : \check{\mu} \xrightarrow{onto} \check{\lambda}'.$$

Since $\mathbf{M}$ and $\mathbf{M}[G]$ have same ordinal numbers (Lemma 312), we define inside $\mathbf{M}$:

$$\begin{aligned} \mathbf{F} : \mu &\longrightarrow \mathscr{P}(\lambda') \\ \alpha &\longmapsto \left\{\beta < \lambda' \mid \exists q \leqslant p_0 \;\; q \Vdash \underset{\sim}{f}(\check{\alpha}) = \check{\beta}\right\}. \end{aligned}$$

We show:

$$\mathbf{M}[G] \models \;\forall \alpha < \mu \;\; f(\alpha) \in \mathbf{F}(\alpha).$$

For this, we let

$$\mathbf{M}[G] \models f(\alpha) = \beta.$$

By the Truth Lemma, there exists $q_0 \in G$ such that : $q_0 \Vdash \underset{\sim}{f}(\check{\alpha}) = \check{\beta}$.

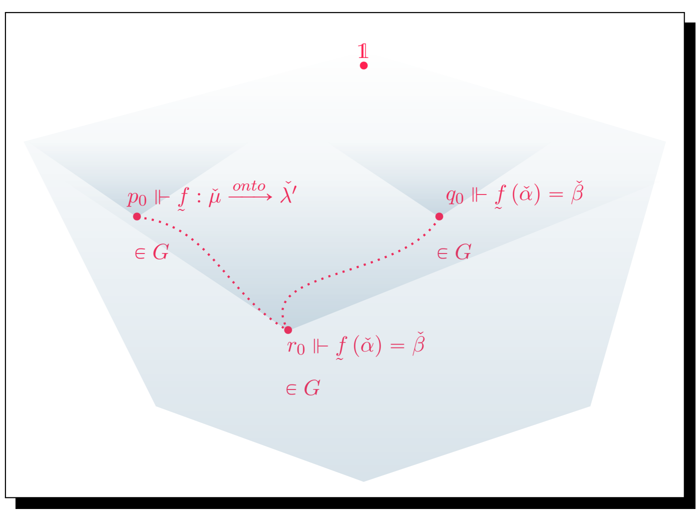



Figure 16.1: $p_0 \geqslant r_0 \leqslant q_0$ with $p_0, q_0, r_0 \in G$.

Therefore, there exists $r_0 \in G$ such that $r_0 \leqslant q_0$, $r_0 \leqslant p_0$ and $r_0 \Vdash \underset{\sim}{f}(\check{\alpha}) = \check{\beta}$, hence $\beta \in \mathbf{F}(\alpha)$ and thus, for all $\alpha < \mu$, $f(\alpha) \in \mathbf{F}(\alpha)$.

We now show that for all $\alpha < \mu$, $(|\mathbf{F}(\alpha)| < \lambda)^{\mathbf{M}}$. Let $\alpha < \mu$ and $\beta$ in $\mathbf{F}(\alpha)$. With the help of **AC** — which holds in **M** — we map $\beta$ to some $q_\beta \in \mathbb{P}$ which satisfies both

(1) $q_\beta \leqslant p_0$ (2) $q_\beta \Vdash \underset{\sim}{f}(\check{\alpha}) = \check{\beta}$.

We notice that the underlying mapping: $\begin{array}{rcl}\mathbf{H} : \mathbf{F}(\alpha) & \xrightarrow{1-1} & \mathbb{P} \\ \beta & \longmapsto & q_\beta\end{array}$ is 1-1.

This relies on the fact that not only do we have $\beta \neq \beta' \implies q_\beta \neq q_{\beta'}$, but we even have $\beta \neq \beta' \implies q_\beta \perp q_{\beta'}$.
To show this, let us assume towards a contradiction, that there exist $\beta \neq \beta'$ with $q_\beta$ and $q_{\beta'}$ compatible. Then there would also exist some $q \in \mathbb{P}$ such that $q \leqslant q_\beta$ and $q \leqslant q_{\beta'}$, hence:

(1) $q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f} : \check{\mu} \xrightarrow{onto} \check{\lambda}'$ (2) $q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{\alpha}) = \check{\beta}$ (3) $q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{\alpha}) = \check{\beta}'$.

Then, in **V**, we could get some filter $J$ which contains $q$ and is $\mathbb{P}$-generic over **M**. By the Truth Lemma, this would lead to some generic extension $\mathbf{M}[J]$ that would satisfy:

(1) $\mathbf{M}[J] \models f : \mu \xrightarrow{onto} \lambda'$ (2) $\mathbf{M}[J] \models f(\alpha) = \beta$ (3) $\mathbf{M}[J] \models f(\alpha) = \beta'$,

which leads to $\mathbf{M}[J] \models \beta = \beta'$, contradicting our hypotheses about $\beta$ and $\beta'$.
So, not only have we shown that **H** is injective, but we have also shown that $\mathbf{H}[\mathbf{F}(\alpha)]$ is an antichain. Now, since $\mathbb{P}$ is $\lambda$-c.c., we obtain

$$\Big(|\mathbf{H}[\mathbf{F}(\alpha)]| < \lambda\Big)^{\mathbf{M}}.$$

and since **H** is 1-1,

$$\Big(|\mathbf{F}(\alpha)| < \lambda\Big)^{\mathbf{M}}.$$

<u>Inside **M**</u>:
We define $S \subseteq \lambda'$ by

$$\begin{aligned} S &= \bigcup ran(F) \\ &= \bigcup \{\mathbf{F}(\alpha) \mid \alpha < \mu\}. \end{aligned}$$

For each $\alpha < \mu$, since we showed that $f(\alpha) \in \mathbf{F}(\alpha)$, we have $f(\alpha) \in S$, and since $f : \mu \xrightarrow{onto} \lambda'$, we have

$$f[\mu] = \lambda' \subseteq S.$$

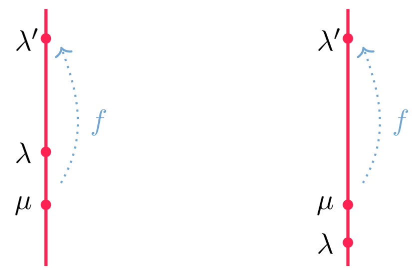
Since both $S \subseteq \lambda'$ and $\lambda' \subseteq S$ hold, we obtain

$$S = \lambda'.$$

For each $\alpha < \mu$, we set

$$|\mathbf{F}(\alpha)| = \lambda_\alpha < \lambda.$$

We distinguish between $\mu \geqslant \lambda$ and $\mu < \lambda$ in order to show that in any case, $|S| < \lambda'$ holds (which will contradict $S = \lambda'$).




- If $\mu \geqslant \lambda$, then $|S| \leqslant |\mu| \cdot \lambda = \max\{|\mu|, \lambda\} = |\mu| \leqslant \mu < \lambda'$.
- If $\mu < \lambda$, then since $\lambda$ is regular, we have $cof(\lambda) = \lambda$, hence for any family $(\nu_\alpha)_{\alpha<\mu}$ of cardinals $\nu_\alpha < \lambda$, we have

  $$\sup\{\nu_\alpha \mid \alpha < \mu\} < \lambda.$$

  By induction on $\alpha < \mu$ we set

  $$\nu_\alpha = \sup\left(\{\nu_\xi \mid \xi < \alpha\} \cup \{\omega, \lambda_\alpha\}\right)$$

  By induction on $\alpha$, we see that $(\nu_\alpha)_{\alpha<\mu}$ is an increasing sequence of infinite cardinal numbers that satisfies for each $\alpha < \mu$:

  $$\begin{aligned}\left|\bigcup\{\mathbf{F}(\xi) \mid \xi \leqslant \alpha\}\right| &\leqslant \left|\bigcup\{\{\xi\} \times \mathbf{F}(\xi) \mid \xi \leqslant \alpha\}\right| \\ &\leqslant \nu_\alpha.\end{aligned}$$

  This is why we have

  $$\begin{aligned}|S| &= |\bigcup ran(F)| \\ &= \left|\bigcup\{\mathbf{F}(\alpha) \mid \alpha < \mu\}\right| \\ &\leqslant \left|\bigcup\{\{\alpha\} \times \mathbf{F}(\alpha) \mid \alpha < \mu\}\right| \\ &\leqslant \sup\{\nu_\alpha \mid \alpha < \mu\} \\ &< \lambda.\end{aligned}$$

So, in both cases we have $|S| < \lambda'$ which contradict $S = \lambda'$.

□ 329

We already have a condition on posets — being $\lambda$-c.c. for some $\lambda$ regular — which guarantees that cardinals above a certain threshold are preserved. We now propose another condition on posets which guarantees the same type of preservation not above but below the same kind of threshold. The first condition relied on sizes of antichains, this new one deals with sizes of chains.

**Definition 330.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*",* $\mathbb{P} \in \mathbf{M}$ *any notion of forcing, and* $\big(\lambda \textit{ is a cardinal}\big)^{\mathbf{M}}$.

$$\mathbb{P} \textit{ is } \lambda\text{-closed}$$

$$\Longleftrightarrow$$

*for all* $\gamma < \lambda$ *and decreasing sequence* $(p_\xi)_{\xi<\gamma}$ *from* $\mathbb{P}$*, there exists* $p \in \mathbb{P}$ *s.t.* $p \leqslant p_\xi$ *(any* $\xi < \gamma$*).*

**Theorem 331.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*",* $\mathbb{P} \in \mathbf{M}$ *be any notion of forcing.*

$$\textit{If } (\lambda \textit{ is a cardinal})^{\mathbf{M}} \textit{ and } (\mathbb{P} \textit{ is } \lambda\textit{-closed})^{\mathbf{M}}\textit{, then } \mathbb{P} \textit{ preserves all cardinals} \leqslant \lambda.$$

Before proving this theorem, we need some easy preliminary result.

**Lemma 332.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*",* $\mathbb{P} \in \mathbf{M}$ *be any notion of forcing, and* $p \in \mathbb{P}$ *be any forcing condition. Let also* $\varphi(x, x_1, \ldots, x_n)$ *be any* $\mathcal{L}_{\text{ST}}$*-formula, and* $\underset{\sim}{b}, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n \in \mathbf{M}^{\mathbb{P}}$.

$$\textit{If } p \Vdash_{\mathbb{P},\mathbf{M}} \exists x \, \big(x \in \underset{\sim}{b} \;\wedge\; \varphi(x, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)\big)\textit{, then there exists } q \leqslant p \textit{ and } \underset{\sim}{c} \in dom(\underset{\sim}{b}) \textit{ s.t.}$$
$$q \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{c}, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n).$$

*Proof of Lemma 332:* Let $G$ be $\mathbb{P}$-generic over $\mathbf{M}$ such that $p \in \mathbf{M}[G]$ and set $b = (\underset{\sim}{b})_G, a_1 = (\underset{\sim}{a}_1)_G =, \ldots, a_n = (\underset{\sim}{a}_n)_G$. We have:

$$\mathbf{M}[G] \models \exists x \, \big(x \in b \;\wedge\; \varphi\,(x, a_1, \ldots, a_n)\big);$$

therefore there exists — by the very definition of $b$ — some $c = (\underset{\sim}{c})_G$ with $\underset{\sim}{c} \in dom\,(\underset{\sim}{b})$ such that

$$\mathbf{M}[G] \models \big(c \in b \;\wedge\; \varphi\,(c, a_1, \ldots, a_n)\big).$$

By the Truth Lemma, there exists $p' \in G$ such that

$$p' \Vdash_{\mathbb{P},\mathbf{M}} \varphi\left(\underset{\sim}{c}, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n\right).$$

Since both $p$ and $p'$ belong to $G$, there exists $q \in G$ such that both $q \leqslant p$ and $q \leqslant p'$ hold. This yields

$$q \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{c}, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n).$$

$\square$ 332

*Proof of Theorem* 331: We simply show the following which will immediately give the result: For all $\mu < \lambda$, for all $\xi \in \mathbf{On}$, for all $f : \mu \xrightarrow{onto} \xi$,

$$\text{if } f \in \mathbf{M}[G], \text{ then } f \in \mathbf{M}.$$

We assume $f = \left(\underset{\sim}{f}\right)_G \in \mathbf{M}[G]$, and let $p_0 \in G$ such that $p_0 \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f} : \check{\mu} \xrightarrow{onto} \check{\xi}$. We set

$$D = \left\{ p \in \mathbb{P} \mid \left( \exists \breve{g}\ \forall \alpha < \mu\ \exists! x\ \left( couple(\check{\alpha}, x) \in \breve{g}\ \wedge\ p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{\alpha}) = x \right) \right)^{\mathbf{M}} \right\}.$$

which we summarize as

$$D = \left\{ p \in \mathbb{P} \mid \exists \breve{g} \in \mathbf{M}\ \forall \alpha < \mu\ \ p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{\alpha}) = \breve{g}(\check{\alpha}) \right\}.$$

We show that $D$ is dense below $p_0$. For this purpose, we define both a $\leqslant_{\mathbb{P}}$-decreasing sequence $(p_\alpha)_{\alpha<\mu}$ and a sequence of ordinals $(\xi_\alpha)_{\alpha<\mu}$ such that for all $\alpha < \beta < \mu$:

$$p_\beta \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}\,(\check{\alpha}) = \check{\xi}_\alpha.$$

The definition is by recursion on $\alpha < \mu$. At each step $\alpha$, both $(p_\zeta)_{\zeta \leqslant \alpha}$ and $(\xi_\zeta)_{\zeta<\alpha}$ are defined. In particular, all $\xi_\zeta$ are defined at successor level (even for for $\zeta$ limit).

**$\alpha$ := 0:** Nothing needs to be defined at this stage, since only $p_0$ is required and it is already defined.

**$\alpha$ := $\alpha$ + 1:** we define $p_{\alpha+1} \leqslant p_\alpha$ and $\xi_\alpha$. By Construction, we have $p_\alpha \leqslant p_0$, hence

$$p_\alpha \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f} : \check{\mu} \xrightarrow{onto} \check{\xi}.$$

Since $\alpha < \mu$, it follows that

$$p_\alpha \Vdash_{\mathbb{P},\mathbf{M}} \exists x \in \check{\xi}\ \ \underset{\sim}{f}\left(\check{\alpha}\right) = x.$$

By definition, $\check{\xi} = \{(\check{\eta}, \mathbb{1}) \mid \eta < \xi\}$ and $dom\left(\check{\xi}\right) = \{\check{\eta} \mid \eta < \xi\}$. From Lemma 332, there exists $p_{\alpha+1} \leqslant p_\alpha$ and $\xi_\alpha < \xi$ such that

$$p_{\alpha+1} \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}\,(\check{\alpha}) = \check{\xi}_\alpha.$$

**$\alpha$ limit:** assuming the decreasing sequence $(p_\zeta)_{\zeta<\alpha}$ has been constructed, since $\mathbb{P}$ is $\lambda$-closed, there exists $p_\alpha$ which is below every $p_\zeta$. Since $\alpha$ is limit, there is no other condition on $p_\alpha$ to satisfy and there is no ordinal of the form $\xi_\zeta$ to define.

Since $\alpha < \mu < \lambda$ and $\mathbb{P}$ is $\lambda$-closed, there exists some $p_\mu \in \mathbb{P}$ such that for all $\alpha < \mu$ we have

$$p_\mu \leqslant p_\alpha \quad \text{and} \quad p_\mu \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{\alpha}) = \check{\xi}_\alpha.$$

Inside $\mathbf{M}$, we set $g(\alpha) = \xi_\alpha$ so that we have

$$p_\mu \leqslant p_0 \quad \text{and} \quad \forall \alpha < \mu \;\; p_\mu \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{\alpha}) = \check{g}(\check{\alpha})$$

which shows that $p_\mu$ belongs to $D$ and completes the proof that $D$ is dense below $p_0$.

Now, since $p_0 \in G$, we have $D \cap G \neq \varnothing$. For any $q \in D \cap G$, by the very definition of $D$ there exists $g \in \mathbf{M}$ such that

$$\forall \alpha < \mu \;\; q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{\alpha}) = \check{g}(\check{\alpha}).$$

Thus, we have for all $\alpha < \mu$,

$$\mathbf{M}[G] \models \left(\underset{\sim}{f}(\alpha)\right)_G = \left(\check{g}(\check{\alpha})\right)_G$$

i.e.,

$$\mathbf{M}[G] \models f(\alpha) = g(\alpha).$$

which shows that $f = g \restriction \mu$ which belongs to $\mathbf{M}$. So, finally we obtain $f \in \mathbf{M}$.

□ 331

# Chapter 17

# Independence of CH

In this chapter, we will prove that if the theory **ZF** is consistent, so is the theory $\mathbf{ZF} + 2^{\aleph_0} = \aleph_2$. Since we already know that if **ZF** is consistent, so is $\mathbf{ZF} + 2^{\aleph_0} = \aleph_1$ (see Theorem 278), this new result will show that $2^{\aleph_0} = \aleph_1$ is independent from **ZF**. i.e., if **ZF** is consistent, then

- $\mathbf{ZF} \nvdash_c 2^{\aleph_0} = \aleph_1$
- $\mathbf{ZF} \nvdash_c 2^{\aleph_0} \neq \aleph_1$.

The same result holds for **ZFC** as well. i.e., if **ZFC** is consistent, then

- $\mathbf{ZFC} \nvdash_c 2^{\aleph_0} = \aleph_1$
- $\mathbf{ZFC} \nvdash_c 2^{\aleph_0} \neq \aleph_1$.

In the next chapter, we will also have similar results for **AC** instead of **CH**. i.e., if **ZF** is consistent, then

- $\mathbf{ZF} \nvdash_c \mathbf{AC}$
- $\mathbf{ZF} \nvdash_c \neg\mathbf{AC}$.

Moreover, many more independence results can be obtained by applying forcing techniques. We only illustrate the by a few samples, but many more can be found in the litterature.

## 17.1 Forcing $2^{\aleph_0} = \aleph_2$

We go back to the poset that was introduced in Example 326:

$$\mathbb{P}_{\omega_2} = \left\{ f : (\omega_2)^{\mathbf{M}} \times \omega \longrightarrow 2 \mid |dom(f)| < \omega \right\}.$$

In order to conclude that, when we forced with this notion of forcing, the generic extension satisfied $2^{\aleph_0} = \aleph_2$ we needed to make sure that cardinals were preserved. This is precisely what this section will establish: simply by proving that $\mathbb{P}_{\omega_2}$ has the c.c.c..

As a preliminary, we need to prove some purely combinatorial result.

**Lemma 333.** *Let $F$ be any family of finite sets such that $|F| = \aleph_1$. There exist $F' \subseteq F$ and $r$ finite such that:*

- $|F'| = \aleph_1$;
- *for all* $a, b \in F'$, $a \cap b = r$.

*$F'$ is called a* $\Delta$-system.

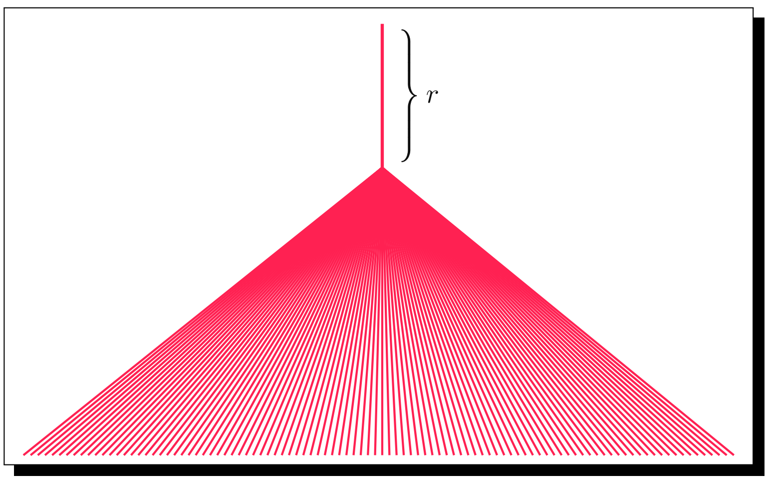



Figure 17.1: Some $\Delta$-system $F'$, where $\forall a, b \in F' \;\; a \cap b = r$.

*Proof of Lemma 333:* Since $F$ is some family of finite sets, and $|F| = \aleph_1$, there exists an integer $n$ and a subset of $F$ with cardinality $\aleph_1$ only containing sets of cardinality $n$. So without loss of generality we may assume that for all $a \in F$, $|a| = n$. The proof then goes by induction on $n$.

If $n = 1$, then $F' = F$ and $r = \varnothing$ works.

We now suppose that the property holds for $n$ and show that it also holds for $n + 1$. So we let $F$ be such that for all $a \in F$, $|a| = n + 1$. We then distinguish between two cases.

(1) There exists $x$ such that $|F_x| = \aleph_1$ where $F_x = \{a \in F \mid x \in a\}$. We then set $F_0 = \{a \smallsetminus \{x\} \mid a \in F_x\}$. We obtain $|F_0| = \aleph_1$ and for all $a \in F_0$, $|a| = n$. By induction hypothesis, there exists $F_0' \subseteq F_0$ and $r_0$ such that $|F_0'| = \aleph_1$ and for all $a, b \in F_0'$, $a \cap b = r_0$. We then set

$$F' = \left\{a_0 \cup \{x\} \mid a \in F_0'\right\} \text{ and } r = r_0 \cup \{x\}.$$

Notice that we have $|F'| = \aleph_1$ and for all $a, b \in F'$, $a \cap b = r$.

(2) For every $x$, $|F_x| < \aleph_1$. We then define by induction on $\xi$ a sequence $(a_\xi)_{\xi<\omega_1}$ of two by two disjoint elements of $F$. We start with $a_0$ being any element, and for each $\beta < \omega_1$, we choose $a_\beta$ such that for all $\xi < \beta$, $a_\xi \cap a_\beta = \varnothing$. We can do so, for otherwise there would exist some (least) $\beta < \omega_1$ such that no $a \in F$ satisfies that for all $\xi < \beta$, $a_\xi \cap a = \varnothing$. We would then be able to define the following mapping:

$$\begin{aligned} f : (F \setminus \{a_\xi \mid \xi < \beta\}) &\longrightarrow \beta \\ a \qquad\qquad &\longmapsto \xi \text{ such that } a \cap a_\xi \neq \varnothing. \end{aligned}$$

Since $|F \setminus \{a_\xi \mid \xi < \beta\}| = \aleph_1$ and $|\beta| < \aleph_1$, there exists $\xi < \beta$ such that $\left|f^{-1}[\xi]\right| = \aleph_1$. But $a_\xi$ is finite and there are $\aleph_1$-many elements in $F$ that have one element in common with $a_\xi$, therefore there exists $x \in a_\xi$ such that $|F_x| = \aleph_1$, which contradicts the hypothesis. This guarantees the existence of the sequence $(a_\xi)_{\xi<\omega_1}$. We then finally set $F' = \{a_\xi \mid \xi < \omega_1\}$ and $r = \varnothing$.

□ 333

The following notion of forcing was introduced in Example 326.

**Definition 334.** *The notion of forcing* $(\mathbb{P}_{\omega_2}, \leqslant, \mathbb{1})$ *is defined by*

*(1)* $\mathbb{P}_{\omega_2} = \left\{f : (\omega_2)^{\mathbf{M}} \times \omega \to \{0,1\} \mid dom(f) \text{ is finite}\right\}$

*(2)* $f \leqslant f' \iff f \supseteq f'$

*(3)* $\mathbb{1} = \varnothing$

**Lemma 335.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***" and* $(\mathbb{P}_{\omega_2}, \leqslant, \mathbb{1}) \in \mathbf{M}$.

$$\mathbb{P}_{\omega_2} \text{ has the c.c.c..}$$

*Proof of Lemma 335:* Towards a contradiction, we suppose that $\mathscr{A}$ is an antichain in $\mathbb{P}_{\omega_2}$ with cardinality $\aleph_1$. We set $F = \{dom(p) \mid p \in \mathscr{A}\}$. Since in $\mathbb{P}_{\omega_2}$ there exist only finitely many different functions over any finite fixed domain, we necessarily have that $|F| = \aleph_1$. By Lemma 333, there exists some $\Delta$-*system* $F' \subseteq F$ such that $|F'| = \aleph_1$ and $r$ finite such that for any two different $a, b \in F'$, $a \cap b = r$.
We let $\{p_\alpha \mid \alpha < \omega_1\} \subseteq \mathscr{A}$ be a subset of $\mathscr{A}$ such that for any $\alpha < \omega_1$, $dom(p_\alpha) \in F'$.
Since $\mathscr{A}$ is an antichain for any two different $\alpha, \alpha' < \omega_1$, we have $p_\alpha \perp p_{\alpha'}$, hence

$$p_\alpha \restriction r \neq p_{\alpha'} \restriction r.$$

It follows that the mapping

$$\begin{aligned} \aleph_1 &\xrightarrow{1-1} 2^r \\ \alpha &\longmapsto p_\alpha \restriction r \end{aligned}$$

is injective which is impossible because $2^r$ is finite (recall $r$ is finite).

□ 335

**Corollary 336.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $\mathbb{P}_{\omega_2} \in \mathbf{M}$*, and* $G$ $\mathbb{P}$*-generic over* $\mathbf{M}$*.*

$$\mathbb{P}_{\omega_2} \textit{ preserves all cardinals.}$$

*i.e., for all* $\alpha \in \mathbf{On}$*,*

$$\aleph_\alpha{}^{\mathbf{M}[G]} = \aleph_\alpha{}^{\mathbf{M}}.$$

*Proof of Corollary 336:* By Lemma 335, $\mathbb{P}_{\omega_2}$ has the countable chain condition (c.c.c.). By Theorem 329, it preserves all cardinals $\geqslant \aleph_1$. Moreover, by absoluteness, $\aleph_0{}^{\mathbf{M}} = \aleph_0{}^{\mathbf{M}[G]}$ and also for each integer $n$, $n^{\mathbf{M}} = n^{\mathbf{M}[G]}$. So, $\mathbb{P}_{\omega_2}$ preserves all cardinals.

□ 336

This Corollary guarantees that forcing with $\mathbb{P}_{\omega_2}$ yields at least $(\omega_2)^{\mathbf{M}}$-many different subsets of $\omega$ in $\mathbf{M}[G]$. Now, we know from Corollary 336 that in $\mathbf{M}[G]$ there are at least $(\omega_2)^{\mathbf{M}[G]}$-many different subsets of $\omega$. Therefore, $\mathbf{M}[G] \models 2^{\aleph_0} \geqslant \aleph_2$. Some more work is still required to show that $\mathbf{M}[G] \models 2^{\aleph_0} = \aleph_2$. Namely, we are going to show that $\mathbf{M}[G] \models 2^{\aleph_0} \leqslant \aleph_2$. This will be done by obtaining a bound on the size of $\mathscr{P}(\lambda)$ in $\mathbf{M}[G]$ that depends on some properties of the notion of forcing $\mathbb{P}$.

**Lemma 337.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $\mathbb{P} \in \mathbf{M}$ *any notion of forcing.*

*If* $(\mathbb{P}$ *has the c.c.c.*$)^{\mathbf{M}}$*,* $(\lambda$ *is an infinite cardinal* $)^{\mathbf{M}}$*, and* $G$ *is* $\mathbb{P}$*-generic over* $\mathbf{M}$*. Then*

$$|\mathscr{P}(\lambda)|^{\mathbf{M}[G]} \leqslant \left(|\mathbb{P}|^\lambda\right)^{\mathbf{M}}.$$

*Proof of Lemma 337:* Let $X \in \mathscr{P}(\lambda)^{\mathbf{M}[G]}$. Inside $\mathbf{M}[G]$, we choose a $\mathbb{P}$-name $\underset{\sim}{X}$ in $\mathbf{M}$ such that $(\underset{\sim}{X})_G = X$.

<u>Inside $\mathbf{M}$</u> we define:

$$\begin{aligned} f_{\underset{\sim}{X}} : \lambda &\longrightarrow \mathscr{P}(\mathbb{P}) \\ \alpha &\longmapsto \mathscr{A}_\alpha \text{ some maximal antichain in } \{p \in \mathbb{P} \mid p \Vdash \check{\alpha} \in \underset{\sim}{X}\}. \end{aligned}$$

We define inside $\mathbf{M}[G]$:

$$\begin{aligned}\mathbf{F}: \mathscr{P}(\lambda) &\longrightarrow \left({}^{\lambda}\{\text{antichains of } \mathbb{P}\}\right)^{\mathbf{M}} \\ X &\longmapsto f_{\underset{\sim}{X}}.\end{aligned}$$

We show that $\mathbf{F}$ is 1-1:

For this we let $X, X' \in \mathscr{P}(\lambda)^{\mathbf{M}[G]}$ be distinct and $\underset{\sim}{X}$, $\underset{\sim}{X}'$ such that $(\underset{\sim}{X})_G = X$ and $(\underset{\sim}{X}')_G = X'$. Pick $\alpha \in (X \smallsetminus X') \cup (X' \smallsetminus X)$. By symmetry, we assume $\alpha \in X \smallsetminus X'$. By the Truth Lemma, there exists $p_0 \in G$ such that

$$p_0 \Vdash \left(\check{\alpha} \in \underset{\sim}{X} \ \wedge \ \check{\alpha} \notin \underset{\sim}{X}'\right).$$

Since $f_{\underset{\sim}{X}}(\alpha) = \mathscr{A}_\alpha$ is some maximal antichain, there exists $p \in \mathscr{A}_\alpha$ such that $p_0$ and $p$ are compatible. Moreover, $p_0$ is not compatible with any element from $\mathscr{A}'_\alpha$. Indeed, if $p_0$ were compatible with some $p' \in A'_\alpha$ there would exist $r$ such that $r \leqslant p_0$, $r \leqslant p'$ and

$$r \Vdash \check{\alpha} \notin \underset{\sim}{X}' \qquad \text{and} \qquad r \Vdash \check{\alpha} \in \underset{\sim}{X}'.$$

A contradiction since

$$r \Vdash \check{\alpha} \notin \underset{\sim}{X}' \iff r \Vdash \neg\check{\alpha} \in \underset{\sim}{X}' \iff \forall t \leqslant r \ \ t \nVdash \check{\alpha} \in \underset{\sim}{X}' \Longrightarrow r \nVdash \check{\alpha} \in \underset{\sim}{X}'.$$

Since $p_0$ is compatible with some element from $f_{\underset{\sim}{X}}(\alpha) = \mathscr{A}_\alpha$ but no element from $f_{\underset{\sim}{X}'}(\alpha) = \mathscr{A}'_\alpha$, we conclude that $\mathscr{A}_\alpha \neq \mathscr{A}'_\alpha$ and $f_{\underset{\sim}{X}} \neq f_{\underset{\sim}{X}'}$, which shows that $\mathbf{F}$ is 1-1; which in turn gives

$$|\mathscr{P}(\lambda)|^{\mathbf{M}[G]} \leqslant \left|{}^{\lambda}\{\text{antichains of } \mathbb{P}\}\right|^{\mathbf{M}}.$$

Since $\mathbb{P}$ has the c.c.c., every antichain is countable. So, we have

$$|\{\text{antichains of } \mathbb{P}\}|^{\mathbf{M}} \leqslant \left(|\mathbb{P}|^{\aleph_0}\right)^{\mathbf{M}}.$$

Finally, an easy computation gives the result:

$$|\mathscr{P}(\lambda)|^{\mathbf{M}[G]} \leqslant \left|{}^{\lambda}\{\text{antichains of } \mathbb{P}\}\right|^{\mathbf{M}} \leqslant \left|\left(|\mathbb{P}|^{\aleph_0}\right)^{\lambda}\right|^{\mathbf{M}} = \left||\mathbb{P}|^{\aleph_0\cdot\lambda}\right|^{\mathbf{M}} \leqslant \left(|\mathbb{P}|^{\lambda}\right)^{\mathbf{M}}.$$

□ 337

**Corollary 338** (Cohen)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "* **ZFC + CH** *",* $\mathbb{P} = \left(\mathbb{P}_{\omega_2}\right)^{\mathbf{M}}$*, and* $G$ *be* $\mathbb{P}$*-generic over* $\mathbf{M}$*.*

$$\left(2^{\aleph_0} = \aleph_2\right)^{\mathbf{M}[G]}.$$

Notice that we start with a ground model $\mathbf{M}$ that satisfies **CH**. i.e., $(2^{\aleph_0} = \aleph_1)^{\mathbf{M}}$.

*Proof of Corollary 338:* By Lemma 335, $\mathbb{P}_{\omega_2}$ has the c.c.c., so by Lemma 337, we have:

$$|\mathscr{P}(\omega)|^{\mathbf{M}[G]} \leqslant \left(|\mathbb{P}_{\omega_2}|^{\aleph_0}\right)^{\mathbf{M}}.$$

Notice that

$$f \in \mathbb{P}_{\omega_2} \iff f \text{ is some finite function } : \omega_2 \times \omega \to \{0,1\}.$$

So formally,

$$f \in \mathbb{P}_{\omega_2} \iff \Big(f \subseteq \big((\omega_2 \times \omega) \times \{0,1\}\big) \text{ and } f \text{ is finite }\Big),$$

hence,

$$|\mathbb{P}_{\omega_2}|^{\mathbf{M}} = \aleph_2^{\mathbf{M}}.$$

In $\mathbf{M}$, $\omega_2$ is regular since **AC** is satisfied. So, any function from $\omega$ into $\omega_2$ is indeed some mapping from $\omega$ into some $\alpha < \omega_2$, and certainly $|\alpha| \leqslant \aleph_1$. So, in $\mathbf{M}$:

$$|{}^{\omega}\alpha| \leqslant \aleph_1^{\aleph_0}.$$

Since $\mathbf{M}$ is a *c.t.m.* of " $\mathbf{ZFC} + \mathbf{CH}$" we have $\left(\aleph_1 = 2^{\aleph_0}\right)^{\mathbf{M}}$. Thus, in $\mathbf{M}$, for every $\alpha < \omega_2$ we also have:

$$|{}^{\omega}\alpha| \leqslant \aleph_1^{\aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \aleph_1.$$

When $\alpha$ varies over $\omega_2$, we obtain:

$$\aleph_2 \leqslant \aleph_2^{\aleph_0} = \left|{}^{\aleph_0}\aleph_2\right| = \left|\bigcup_{\alpha<\omega_2} {}^{\omega}\alpha\right| \leqslant \left|\left\{(\alpha, f) \mid \alpha < \omega_2 \ \wedge \ f \in {}^{\omega}\alpha\right\}\right| \leqslant \aleph_2 \cdot \aleph_1 = \aleph_2.$$

So, we have shown

$$|\mathscr{P}(\omega)|^{\mathbf{M}[G]} \leqslant \left(|\mathbb{P}_{\omega_2}|^{\aleph_0}\right)^{\mathbf{M}} = \aleph_2{}^{\mathbf{M}}.$$

In Example 326 we obtained

$$\aleph_2{}^{\mathbf{M}} \leqslant |\mathscr{P}(\omega)|^{\mathbf{M}[G]}.$$

Altogether, this gives

$$|\mathscr{P}(\omega)|^{\mathbf{M}[G]} = \aleph_2{}^{\mathbf{M}}.$$

i.e.,

$$\left(2^{\aleph_0}\right)^{\mathbf{M}[G]} = \aleph_2{}^{\mathbf{M}}.$$

In Corollary 336 we proved that $\mathbb{P}_{\omega_2}$ preserves all cardinals, so in particular

$$\aleph_2{}^{\mathbf{M}} = \aleph_2{}^{\mathbf{M}[G]}$$

which finally leads to

$$\left(2^{\aleph_0} = \aleph_2\right)^{\mathbf{M}[G]}.$$

$\square$ 338

## 17.2 Reflecting back on forcing $2^{\aleph_0} = \aleph_2$

Formally, in order to show

$$cons(\mathbf{ZFC}) \longrightarrow cons(\mathbf{ZFC} + 2^{\aleph_0} = \aleph_2)$$

we proceeded by contraposition and proved:

$$\neg cons(\mathbf{ZFC} + 2^{\aleph_0} = \aleph_2) \longrightarrow \neg cons(\mathbf{ZFC}).$$

For this purpose, we supposed there exist axioms $\varphi_1, \ldots, \varphi_n$ in $\mathbf{ZFC} + 2^{\aleph_0} = \aleph_2$ such that:

$$\varphi_1, \ldots, \varphi_n \vdash_c \bot.$$

One can then determine within "**ZFC**" — in advance and *independently of* $\mathbf{M}$ — some other formulas $\psi_1, \ldots, \psi_k$ in "**ZFC**" such that if $\mathbf{M}$ is a model of $\psi_1, \ldots, \psi_k$ and $G$ is $\mathbb{P}_{\omega_2}$-generic over $\mathbf{M}$, then $\mathbf{M}[G]$ is a model of $\varphi_1, \ldots, \varphi_n$. We add to $\psi_1, \ldots, \psi_k$ other formulas $\psi_{k+1}, \ldots, \psi_l$ which enable us to prove other results such as the ones on cardinal preservations, on $\Delta$-systems, or on absoluteness, etc.

Then, we work in **ZFC**:

$$\mathbf{ZFC} \vdash_c \left( \left. \begin{array}{c} \text{“ } \exists \mathbf{M} \text{ a } c.t.m. \text{ s.t. } \Big( \{\psi_1, \ldots, \psi_l\} \Big)^{\mathbf{M}} \text{ ”} \\ \wedge \\ \text{“ } \exists G \ \ (\mathbb{P}_{\omega_2})^{\mathbf{M}}\text{-generic over } \mathbf{M} \text{ ”} \end{array} \right\} \Longrightarrow \exists \mathbf{M}[G] \ \Big( \{\varphi_1, \ldots, \varphi_n\} \Big)^{\mathbf{M}[G]} \right)$$

Since,

$$\mathbf{ZFC} \vdash_c \text{“ } \exists \mathbf{M} \text{ a } c.t.m. \text{ s.t. } \Big( \{\psi_1, \ldots, \psi_l\} \Big)^{\mathbf{M}} \text{ ” } \wedge \text{ “ } \exists G \ \ (\mathbb{P}_{\omega_2})^{\mathbf{M}}\text{-generic over } \mathbf{M} \text{ ”}.$$

by *modus ponens* follows,

$$\mathbf{ZFC} \vdash_c \exists \mathbf{M}[G] \ \Big( \{\varphi_1, \ldots, \varphi_n\} \Big)^{\mathbf{M}[G]}.$$

or more generally,

$$\mathbf{ZFC} \vdash_c \underbrace{\exists N \ \Big( \{\varphi_1, \ldots, \varphi_n\} \Big)^{N}}_{\psi}.$$

Since $\varphi_1, \dots, \varphi_n \vdash_c \bot$,

$$\mathbf{ZFC} \vdash_c \underbrace{\neg \exists N \left( \{\varphi_1, \dots, \varphi_n\} \right)^N}_{\neg\psi}.$$

Therefore,

$$\mathbf{ZFC} \vdash_c \bot.$$

As put by Kenneth Kunen: *"The inelegant part of this argument is that the procedure of finding $\psi_1, \dots, \psi_l$, although straightforward, completely effective, and finitistically valid, is also very tedious"* [21, p. 233]

## 17.3 Forcing $2^{\aleph_0} = \aleph_{\alpha+1}$

The same argument, *mutatis mutandis* yields, for any ordinal $\alpha$, the equiconsistency of **ZFC** and **ZFC** + $2^{\aleph_0} = \aleph_{\alpha+1}$.

**Definition 339.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***". Given any* $\alpha \in \mathbf{On}$*, we let* $\left(\mathbb{P}_{\aleph_\alpha}, \leqslant, \mathbb{1}\right)$ *be*

*(1)* $\mathbb{P}_{\aleph_\alpha} = \left\{ f \ : \ (\aleph_\alpha)^{\mathbf{M}} \times \omega \to \{0,1\} \mid dom(f) \text{ is finite} \right\}$

*(2)* $f \leqslant f' \iff f \supseteq f'$

*(3)* $\mathbb{1} = \varnothing$

We first need to show that $\mathbb{P}_{\aleph_\alpha}$ has the c.c.c. which will guarantee that all cardinals are preserved.

**Lemma 340.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $0 < \alpha \in \mathbf{On}$ *and* $\mathbb{P}_{\aleph_\alpha} \in \mathbf{M}$.

$$\mathbb{P}_{\aleph_\alpha} \text{ has the c.c.c..}$$

*Proof of Lemma 340:* *Mutatis mutandis*, identical to the proof of Lemma 335. Towards a contradiction, we suppose that $\mathscr{A}$ is an antichain in $\mathbb{P}_{\aleph_\alpha}$ with cardinality $\aleph_1$. We set

$$F = \{dom(p) \mid p \in \mathscr{A}\}.$$

Since in $\mathbb{P}_{\aleph_\alpha}$ there exist only finitely many different functions over any finite fixed domain, we necessarily have that $|F| = \aleph_1$. By Lemma 333, there exists some $\Delta$-*system* $F' \subseteq F$ such that $|F'| = \aleph_1$ and $r$ finite such that for any two different $a, b \in F'$, $a \cap b = r$. We let $(p_\alpha)_{\alpha<\omega_1}$ be a sequence of elements of $\mathscr{A}$ such that for any $\alpha < \omega_1$, $dom(p_\alpha) \in F'$. For $\alpha < \alpha' < \omega_1$, we have

$p_\alpha \perp p_{\alpha'}$, hence $p_\alpha \restriction r \neq p_{\alpha'} \restriction r$. It follows that the mapping

$$\begin{aligned} \aleph_1 &\xrightarrow{1-1} 2^r \\ \alpha &\longmapsto p_\alpha \restriction r \end{aligned}$$

is injective which is impossible because $2^r$ is finite (recall $r$ is finite).

□ 340

**Theorem 341** (Cohen)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "* $\mathbf{ZFC{+}GCH}$ *" and* $0 < \alpha \in \mathbf{On}$. *If* $\mathbb{P} = \left(\mathbb{P}_{\aleph_{\alpha+1}}\right)^{\mathbf{M}}$ *and* $(\mathbf{GCH})^{\mathbf{M}}$*, then for all* $G$ $\mathbb{P}$*-generic over* $\mathbf{M}$*,*

$$\left(2^{\aleph_0} = \aleph_{\alpha+1}\right)^{\mathbf{M}[G]}.$$

*Proof of Lemma* 341: By Lemma 117 $\aleph_{\alpha+1}$ is regular and by Lemma 340 $\mathbb{P}_{\aleph_{\alpha+1}}$ has the c.c.c., thus $\mathbb{P}_{\aleph_{\alpha+1}}$ preserves all cardinals. By Lemma 337, we have

$$|\mathscr{P}(\omega)|^{\mathbf{M}[G]} \leqslant \left(\left|\mathbb{P}_{\aleph_{\alpha+1}}\right|^{\aleph_0}\right)^{\mathbf{M}}.$$

So, we need to compute $\left(\left|\mathbb{P}_{\aleph_{\alpha+1}}\right|^{\aleph_0}\right)^{\mathbf{M}}$. i.e., $\left|{}^{\aleph_0}\mathbb{P}_{\aleph_{\alpha+1}}\right|^{\mathbf{M}}$.

Inside $\mathbf{M}$, one has

- $\left|\mathbb{P}_{\aleph_{\alpha+1}}\right| = \aleph_{\alpha+1}$;
- since $\mathbf{M}$ is a *c.t.m.* of " $\mathbf{ZFC} + \mathbf{GCH}$ " we have $2^{\aleph_\alpha} = \aleph_{\alpha+1}$;
- $\aleph_{\alpha+1}$ is regular since $\mathbf{AC}$ is satisfied. So, any function from $\omega$ into $\aleph_{\alpha+1}$ is indeed some mapping from $\omega$ into some $\xi < \omega_{\alpha+1}$, and certainly $|\xi| \leqslant \aleph_\alpha$. So,

$$|{}^{\omega}\xi| \leqslant \aleph_\alpha^{\aleph_0}.$$

- The mapping $\chi$ that associates to each function $f$ from $\aleph_0$ into $\aleph_\alpha$, its characteristic function $\chi_f : \aleph_0 \times \aleph_\alpha \longrightarrow \{0,1\}$ is 1-1. Thus

$${\aleph_\alpha}^{\aleph_0} \leqslant 2^{\aleph_0 \cdot \aleph_\alpha} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

- So we obtain:

$$\begin{aligned}\aleph_{\alpha+1} &\leqslant \aleph_{\alpha+1}^{\aleph_0}\\ &= \left|{}^{\aleph_0}\aleph_{\alpha+1}\right|\\ &= \left|\bigcup_{\xi<\aleph_{\alpha+1}} \xi^{\aleph_0}\right|\\ &\leqslant \left|\left\{\left(\xi, f\right) \mid \xi < \aleph_{\alpha+1} \;\wedge\; f \in {}^{\omega}\xi\right\}\right|\\ &\leqslant \aleph_{\alpha+1} \cdot \aleph_{\alpha}^{\aleph_0}\\ &\leqslant \aleph_{\alpha+1} \cdot \aleph_{\alpha+1}\\ &= \aleph_{\alpha+1}\end{aligned}$$

  which yields

$$\left|\mathbb{P}_{\aleph_{\alpha+1}}\right|^{\aleph_0} = \aleph_{\alpha+1}^{\aleph_0} = \aleph_{\alpha+1}.$$

Thus, by applying Lemma 337 we obtain:

$$|\mathscr{P}(\omega)|^{\mathbf{M}[G]} \leqslant \left(\left|\mathbb{P}_{\aleph_{\alpha+1}}\right|^{\aleph_0}\right)^{\mathbf{M}} = (\aleph_{\alpha+1})^{\mathbf{M}} = (\aleph_{\alpha+1})^{\mathbf{M}[G]}.$$

i.e.,

$$\Big(|\mathscr{P}(\omega)| \leqslant \aleph_{\alpha+1}\Big)^{\mathbf{M}[G]}.$$

For the other inequality, we set $\mathscr{F} = \bigcup G$ and notice that

(1) $\mathscr{F}$ is a function (see Exercise 293):

$$\mathscr{F} \;:\; \left(\aleph_{\alpha+1}\right)^{\mathbf{M}} \times \omega \longrightarrow \{0,1\}.$$

(2) $G \notin \mathbf{M}$ since by Lemma 295, given any $p \in \mathbb{P}_{\aleph_{\alpha+1}}$ and any integer $n \notin dom(p)$, one has $r = p \cup \{(n,0)\}$ and $q = p \cup \{(n,1)\}$ satisfy

$$q \leqslant p \;\wedge\; r \leqslant p \;\wedge\; q \perp r.$$

(3) $G \in \mathbf{M}[G]$ (see Lemma 307), hence $\mathscr{F} \in \mathbf{M}[G]$.

For $\alpha < \beta < \left(\aleph_{\alpha+1}\right)^{\mathbf{M}}$, we consider:

$$D_{\alpha,\beta} = \Big\{p \in \mathbb{P} \mid \exists n < \omega \;\; \big((\alpha,n) \in dom(p) \;\wedge\; (\beta,n) \in dom(p) \;\wedge\; p(\alpha,n) \neq p(\beta,n)\big)\Big\}.$$

$D_{\alpha,\beta}$ is dense in $\mathbb{P}$ because given any $q \in \mathbb{P}$, since $dom(q)$ is finite, there exists $n \in \omega$ such that $(\alpha,n)$ and $(\beta,n)$ do not belong to $dom(q)$, thus the following forcing condition $p \leqslant q$ belongs to

$D_{\alpha,\beta}$.

$$p = q \cup \Big\{\big((\alpha, n), 0\big), \big((\beta, n), 1\big)\Big\}.$$

Since each $D_{\alpha,\beta}$ is dense and belongs to $\mathbf{M}$, and $G$ is $\mathbb{P}$-generic over $\mathbf{M}$,

$$D_{\alpha,\beta} \cap G \neq \varnothing.$$

Thus there exists $p \in G$ and $n \in \omega$ such that $p(\alpha, n) \neq p(\beta, n)$. It follows that for all $\alpha < \beta < \left(\aleph_{\alpha+1}\right)^{\mathbf{M}}$, there exists an integer $n$ such that

$$\mathscr{F}(\alpha, n) \neq \mathscr{F}(\beta, n).$$

For each ordinal $\alpha < \left(\aleph_{\alpha+1}\right)^{\mathbf{M}}$, we consider

$$X_\alpha = \{n < \omega \mid \mathscr{F}(\alpha, n) = 1\}.$$

If $\alpha < \beta < \left(\aleph_{\alpha+1}\right)^{\mathbf{M}}$, since there exists $n \in \omega$ such that $\mathscr{F}(\alpha, n) \neq \mathscr{F}(\beta, n)$, we have

$$X_\alpha \neq X_\beta.$$

It follows that there exist at least $\left(\aleph_{\alpha+1}\right)^{\mathbf{M}}$-many subsets of $\omega$ in $\mathbf{M}[G]$. Thus,

$$|\mathscr{P}(\omega)|^{\mathbf{M}[G]} \geqslant \left(\aleph_{\alpha+1}\right)^{\mathbf{M}} = \left(\aleph_{\alpha+1}\right)^{\mathbf{M}[G]}.$$

i.e.,

$$\Big(\,|\mathscr{P}(\omega)| \geqslant \aleph_{\alpha+1}\Big)^{\mathbf{M}[G]}.$$

Finally, we have shown

$$\Big(\aleph_{\alpha+1} \leqslant |\mathscr{P}(\omega)| \leqslant \aleph_{\alpha+1}\Big)^{\mathbf{M}[G]}$$

which yields

$$\Big(2^{\aleph_0} = \aleph_{\alpha+1}\Big)^{\mathbf{M}[G]}.$$

□ 341

# Chapter 18

# Independence of AC

## 18.1 Notions of Forcing and Automorphisms

We shift our attention to the axiom of choice and intend to prove:

$$\mathbf{ZFC} \vdash_c cons\big(\mathbf{ZFC}\big) \longrightarrow cons\big(\mathbf{ZF} + \neg\mathbf{AC}\big).$$

We will do this by first forcing from a ground model $\mathbf{M}$ which satisfies "**ZFC**". This will provide us with a generic extension $\mathbf{M}[G]$ which will also satisfy "**ZFC**" as shown by Theorem 325. So, there is no chance we get a model in which the axiom of choice fails this way. However, we will consider a submodel of the generic extension for which we will be able to prove that it denies the axiom of choice.

**Definition 342.** *Let* $\mathbf{M}$ *be a c.t.m. of "***ZFC***" and* $(\mathbb{P}, \leqslant, \mathbb{1})$ *a partial order over* $\mathbf{M}$.

*Any mapping* $\pi : \mathbb{P} \longrightarrow \mathbb{P}$ *is an* automorphism *of* $\mathbb{P}$ *if*

- $\pi$ *is a bijection;*
- $\forall p \in \mathbb{P}\ \forall q \in \mathbb{P}\ \ \big(p \leqslant q \ \longleftrightarrow\ \pi(p) \leqslant \pi(q)\big)$;
- $\pi(\mathbb{1}) = \mathbb{1}$.

**Lemma 343.** *Let* $\mathbf{M}$ *be a c.t.m. of "***ZFC***" and* $(\mathbb{P}, \leqslant, \mathbb{1})$ *a partial order over* $\mathbf{M}$. *If* $\pi \in \mathbf{M}$ *is an automorphism of* $\mathbb{P}$, *then*

$$G \text{ is } \mathbb{P}\text{-generic over } \mathbf{M} \iff \pi[G] \text{ is } \mathbb{P}\text{-generic over } \mathbf{M}.$$

*Proof of Lemma* 343:

($\Longrightarrow$) In order to prove that $\pi[G]$ is $\mathbb{P}$-generic over $\mathbf{M}$, we first show that $\pi[G]$ is a filter over $\mathbb{P}$. We have

(1) Given any $\pi(p), \pi(q) \in \pi[G]$, since $G$ is a filter, there exists $r \in G$ such that $r \leqslant p$ and $r \leqslant q$. Since $\pi$ is a automorphism, we have $\pi(r) \in \pi[G]$, together with

$$\pi(r) \leqslant \pi(p) \text{ and } \pi(r) \leqslant \pi(q).$$

(2) If $\pi(p) \in \pi[G]$ and $\pi(p) \leqslant \pi(q)$, then

$$p \leqslant q \longleftrightarrow \pi(p) \leqslant \pi(q)$$

holds, which yields $q \in G$ (since $G$ is a filter), hence $\pi(q) \in \pi[G]$.

(3) $\mathbb{1} = \pi(\mathbb{1})$, thus $\mathbb{1} \in \pi[G]$.

We now check that $\pi[G]$ satisfies the density clause:

$$\text{For every } D \in \mathbf{M} \text{ which is dense in } \mathbb{P}, \;\; \pi[G] \cap D \neq \varnothing.$$

It suffices to show that $\pi^{-1}[D]$ is dense in $\mathbb{P}$, since

$$\pi[G] \cap D = \pi\big[G \cap \pi^{-1}[D]\big].$$

Let $p \in \mathbb{P}$, $D$ is dense, so there exists $r \leqslant \pi(p)$ such that $r \in D$; hence $\pi^{-1}(r) \leqslant p$ and $\pi^{-1}(r) \in \pi^{-1}[D]$, which shows that $\pi^{-1}[D]$ is dense.

So we have $\pi^{-1}[D] \cap G \neq \varnothing$ and thus $D \cap \pi[G] \neq \varnothing$. $\pi[G]$ is thus $\mathbb{P}$-generic over $\mathbf{M}$.

($\Longleftarrow$) The proof of the reverse implication is simply addressed by replacing $\pi$ by $\pi^{-1}$.

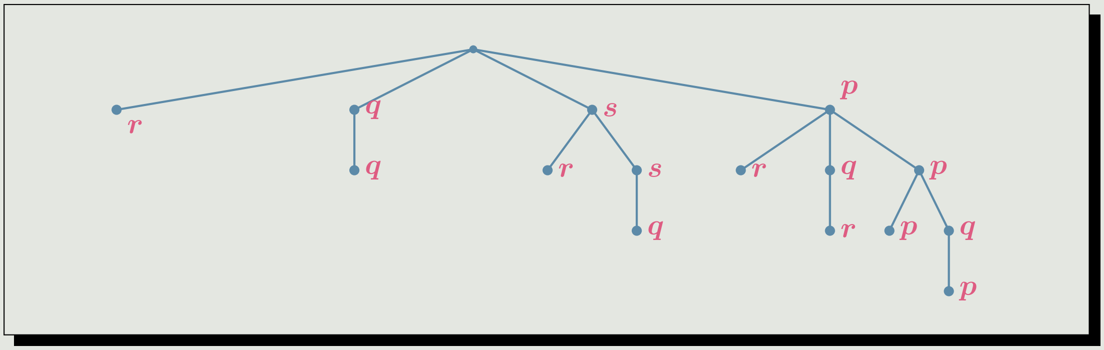
□ 343

**Definition 344.** *Let* $\mathbf{M}$ *be a c.t.m. of "***ZFC***",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *a partial order over* $\mathbf{M}$ *and* $\pi : \mathbb{P} \longrightarrow \mathbb{P}$. *By transfinite recursion, we define*

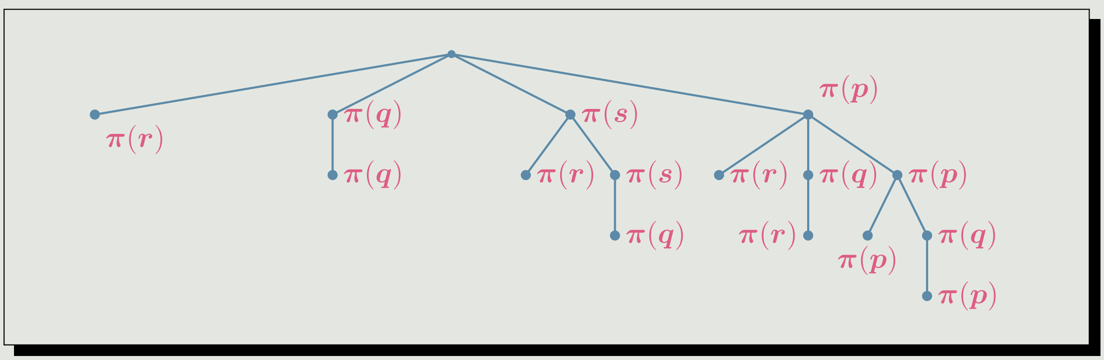
$$\begin{aligned} \tilde{\pi} : \mathbf{M}^{\mathbb{P}} &\longrightarrow \mathbf{M}^{\mathbb{P}} \\ \tau &\longmapsto \{(\tilde{\pi}(\sigma), \pi(p)) \mid (\sigma, p) \in \tau\}. \end{aligned}$$

**Example 345.**

*The* $\mathbb{P}$*-name* $\tau$*:*




*The* $\mathbb{P}$*-name* $\tilde{\pi}(\tau)$*:*




We show that the image of a $\mathbb{P}$-generic filter over $\mathbf{M}$ by an automorphism of $\mathbb{P}$ yields exactly the same generic extension as the original filter.

**Lemma 346.** *Let* $\mathbf{M}$ *be a c.t.m. of "***ZFC***",* $\mathbb{P} \in \mathbf{M}$ *be a notion of forcing,* $G$ *be* $\mathbb{P}$*-generic over* $\mathbf{M}$*, and* $\pi \in \mathbf{M}$ *be an automorphism of* $\mathbb{P}$*.*

$$\mathbf{M}\big[\pi[G]\big] = \mathbf{M}[G].$$

*Proof of Lemma 346:* Notice first that, for all $\tau \in \mathbf{M}^{\mathbb{P}}$, we have

$$(\tau)_G = \big(\tilde{\pi}(\tau)\big)_{\pi[G]}.$$

(Indeed, given any $b \in \mathbf{M}\big[\pi[G]\big]$, and $\tau \in \mathbf{M}^{\mathbb{P}}$ such that $b = (\tau)_{\pi[G]}$ we have $b = (\tau)_{\pi[G]} =$

$\left(\tilde{\pi}^{-1}(\tau)\right)_{G}$.)

This yields

$$\mathbf{M}\big[\pi[G]\big] \subseteq \mathbf{M}[G].$$

For the reverse inclusion, we make use of Lemma 311 which stated that if $\mathbf{N}$ is a transitive model of "**ZFC**" with $\mathbf{M} \subseteq \mathbf{N}$ such that $G \in \mathbf{N}$, then $\mathbf{M}[G] \subseteq \mathbf{N}$. We notice that

(1) $\mathbf{M}\big[\pi[G]\big]$ is transitive.

(2) $\mathbf{M} \subseteq \mathbf{M}\big[\pi[G]\big]$;

(3) $G = \pi^{-1}\big[\pi[G]\big] \in \mathbf{M}\big[\pi[G]\big]$.

This gives

$$\mathbf{M}[G] \subseteq \mathbf{M}\big[\pi[G]\big]$$

which yields

$$\mathbf{M}\big[\pi[G]\big] = \mathbf{M}[G].$$

□ 346

**Lemma 347.** *Let* $\mathbf{M}$ *be a c.t.m. of "*__ZFC__*" and* $\mathbb{P}$ *a notion of forcing over* $\mathbf{M}$*. Let also* $\varphi(x_1, \dots, x_n)$ *be any* $\mathcal{L}_{\mathrm{ST}}$*-formula. If* $\pi \in \mathbf{M}$ *is an automorphism of* $\mathbb{P}$*, then*

*(1) for all* $x \in \mathbf{M}$*,* $\tilde{\pi}(\check{x}) = \check{x}$*;*

*(2) for all* $\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_n \in \mathbf{M}^{\mathbb{P}}$*, and* $p \in \mathbb{P}$*,*

$$p \Vdash \varphi\left(\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_n\right) \Longleftrightarrow \pi(p) \Vdash \varphi\big(\tilde{\pi}(\underset{\sim}{a}_1), \dots, \tilde{\pi}(\underset{\sim}{a}_n)\big).$$

*Proof of Lemma* 347: The proof of (1) is immediate. For (2), we write $a_1, \dots, a_n$ for $\left(\underset{\sim}{a}_1\right)_G, \dots, (\underset{\sim}{a}_n)_G$, repsectively. By using Lemmas 343 and 346, we have

$$
\begin{aligned}
p \Vdash \varphi\left(\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n\right) &\iff \text{for all } G\ \mathbb{P}\text{-generic over } \mathbf{M} \text{ with } p \in G \\
&\qquad \mathbf{M}[G] \models \varphi\left(a_1, \ldots, a_n\right) \\
&\iff \text{for all } G\ \mathbb{P}\text{-generic over } \mathbf{M} \text{ with } p \in G \\
&\qquad \mathbf{M}[G] \models \varphi\Big(\left(\tilde{\pi}\left(\underset{\sim}{a}_1\right)\right)_{\pi[G]}, \ldots, \left(\tilde{\pi}\left(\underset{\sim}{a}_n\right)\right)_{\pi[G]}\Big) \\
&\iff \text{for all } G\ \mathbb{P}\text{-generic over } \mathbf{M} \text{ with } p \in G \\
&\qquad \mathbf{M}\big[\pi[G]\big] \models \varphi\Big(\left(\tilde{\pi}\left(\underset{\sim}{a}_1\right)\right)_{\pi[G]}, \ldots, \left(\tilde{\pi}\left(\underset{\sim}{a}_n\right)\right)_{\pi[G]}\Big) \\
&\iff \text{for all } \pi[G]\ \mathbb{P}\text{-generic over } \mathbf{M} \text{ with } \pi(p) \in \pi[G] \\
&\qquad \mathbf{M}\big[\pi[G]\big] \models \varphi\Big(\left(\tilde{\pi}\left(\underset{\sim}{a}_1\right)\right)_{\pi[G]}, \ldots, \left(\tilde{\pi}\left(\underset{\sim}{a}_n\right)\right)_{\pi[G]}\Big) \\
&\iff \pi(p) \Vdash \varphi\big(\tilde{\pi}(\underset{\sim}{a}_1), \ldots, \tilde{\pi}(\underset{\sim}{a}_n)\big).
\end{aligned}
$$

□ 347

## 18.2 Hereditarily Ordinal Definable Sets

**Definition 348.** *Given any set $A$,*

*(1)* $\mathbf{OD}(A)$ *is defined by*

$$b \in \mathbf{OD}(A)$$

$$\iff$$

*for some $\mathcal{L}_{\mathrm{ST}}$-formula $\varphi(x, x_1, \ldots, x_n, y_1, \ldots, y_k, y_{k+1})$, ordinals $\alpha, \alpha_1, \ldots \alpha_n$ and $a_1, \ldots, a_k \in A$.*

$$b = \left\{ z \in \mathbf{V}_\alpha \mid \Big(\varphi(z, \alpha_1, \ldots \alpha_n, a_1, \ldots, a_k, A)\Big)^{\mathbf{V}_\alpha} \right\}.$$

*(2)* $\mathbf{HOD}(A)$ *is defined by*

$$b \in \mathbf{HOD}(A)$$

$$\iff$$

$b \in \mathbf{OD}(A)$ *and the transitive closure of $b$ is included in* $\mathbf{OD}(A)$.

**Theorem 349.** *Let $A$ be an arbitrary set.*

$$(\mathbf{ZF})^{\mathbf{HOD}(A)}.$$

*Proof of Lemma* 349: The proof is identical to the proof of $(\mathbf{ZF})^{\mathbf{HOD}}$ — see exercises sheet.

□ 349

## 18.3 Forcing ¬AC

This section is entirely dedicated to "constructing" a model of **ZF** in which the axiom of choice fails. Namely,

**Theorem 350.**

$$\mathbf{ZFC} \vdash_c cons(\mathbf{ZFC}) \longrightarrow cons(\mathbf{ZF} + \neg\mathbf{AC}).$$

*Proof of Theorem* 350: To do so, we prove that given $\mathbf{M}$ any *c.t.m.* of "**ZFC**" with $\mathbb{P}_{\aleph_0} \in \mathbf{M}$, if $G$ is $\mathbb{P}_{\aleph_0}$-generic over $\mathbf{M}$, then there exists a set $A \in \mathbf{M}[G]$ such that:

$$\mathbf{M}[G] \models (\neg\mathbf{AC})^{\mathbf{HOD}(A)}.$$

Or, to say it differently,

$$\Big((\neg\mathbf{AC})^{\mathbf{HOD}(A)}\Big)^{\mathbf{M}[G]}.$$

We start by forcing with

$$\mathbb{P}_{\aleph_0} = \big\{f : \omega \times \omega \longrightarrow \{0,1\} \mid dom(f) \text{ finite}\big\}.$$

Given any $G$ $\mathbb{P}_{\aleph_0}$-generic over $\mathbf{M}$, we have $\mathscr{F} = \bigcup G$ satisfies

$$\mathscr{F} : \omega \times \omega \to \{0,1\}.$$

Let

$$a_k = \big\{n < \omega \mid \mathscr{F}(k,n) = 1\big\} \qquad \text{and} \qquad A = \big\{a_k \mid k < \omega\big\}.$$

We have $A \in \mathbf{M}[G]$, and $A \notin \mathbf{M}$ for otherwise, one could recover from $A$ some filter $\pi[G]$ for some automorphism $\pi$ of $\mathbb{P}$. This would yield $\pi[G] \in \mathbf{M}$, henceforth $\mathbf{M}\big[\pi[G]\big] = \mathbf{M}[G] = \mathbf{M}$ which yields $G \in \mathbf{M}$ which would contradict Lemma 295 since given any $p \in \mathbb{P}_{\aleph_0}$, there exists $q, r \in \mathbb{P}_{\aleph_0}$ such that $q, r \leqslant p$ and $q \perp r$.

Also, since

$$D_{n,m} = \big\{p \in \mathbb{P}_{\aleph_0} \mid \exists k \leqslant \omega \ \ p(n,k) \neq p(m,k)\big\}$$

is dense in $\mathbb{P}_{\aleph_0}$, for all integers $n \neq m$, we have $a_n \neq a_m$, which shows that $A$ is infinite.

Inside $\mathbf{M}[G]$, we verify that $A$ is an element of $\mathbf{HOD}(A)$:

(1) $A \in \mathbf{OD}(A)$ since it is definable from itself.

(2) If $x \in A$, then $x$ is definable from itself, so $x \in \mathbf{OD}(A)$. If $y \in x \in A$, then $y \in \omega$, hence $y \in \mathbf{OD} \subseteq \mathbf{OD}(A)$. So, the transitive closure of $A$ is included in $\omega \cup A \subseteq \mathbf{OD}(A)$.

Now, for each $n \in \omega$, we define *canonical* $\mathbb{P}_{\aleph_0}$-names $\underset{\sim}{a}_n$ and $\underset{\sim}{A}$ for, $a_n$ and $A$ respectively:

$$\underset{\sim}{a}_n = \{(\check{m}, p) \mid p(n,m) = 1\}.$$

and

$$\underset{\sim}{A} = \{(\underset{\sim}{a}_n, \mathbb{1}) \mid n < \omega\},$$

so that we have

$$(\underset{\sim}{a}_n)_G = a_n \quad \text{and} \quad (\underset{\sim}{A})_G = A.$$

We let

$$\mathbf{N} = \big(\mathbf{HOD}(A)\big)^{\mathbf{M}[G]}.$$

Towards a contradiction, we suppose $\big(\mathbf{AC}\big)^{\mathbf{N}}$.

So, $\mathbf{N}$ satisfies that the set $A$ can be well-ordered. In particular, there exists some mapping $f : A \xrightarrow{1-1} \mathbf{On}$. Since $f \in \mathbf{N}$, we have in particular

$$\big(f \in \mathbf{OD}(A)\big)^{\mathbf{M}[G]}$$

and therefore $f$ is definable in $\mathbf{M}[G]$ with parameters $\alpha_1, \ldots, \alpha_n \in \mathbf{On}$, $a_1, \ldots, a_k \in A$ and $A$. Let $a_{k+1} \in A$ and $\alpha \in \mathbf{On}$ such that $f(a_{k+1}) = \alpha$. The set $a_{k+1}$ is definable in $\mathbf{M}[G]$ with parameters $\alpha, \alpha_1, \ldots, \alpha_n \in \mathbf{On}$, $a_1, \ldots, a_k \in A$ and $A$ and some $\mathcal{L}_{\mathrm{ST}}$-formula $\varphi$:

$$\Big(\text{“ } a_{k+1} \text{ is the only } x \text{ such that } \underbrace{\varphi(x, \alpha, \alpha_1, \ldots, \alpha_n, a_1, \ldots, a_k, A)}_{\text{“} f(x)=\alpha \text{”}} \text{ ”}\Big)^{\mathbf{M}[G]}.$$

So, by the Truth Lemma, there exists $r \in G$ such that

$$\big(r \Vdash \text{“} \underset{\sim}{a}_{k+1} \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k, \underset{\sim}{A}) \text{”}\big)^{\mathbf{M}}.$$

We then consider

$$D = \Big\{q \in \mathbb{P}_{\aleph_0} \mid \exists l > k+1 \ \ \big(q \Vdash \text{“} \underset{\sim}{a}_l \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k, \underset{\sim}{A}) \text{”}\big)^{\mathbf{M}}\Big\}.$$

We have $D \in \mathbf{M}$ and we still need to show

**Claim 351.**

*$D$ is dense below $r$.*

*Proof of Claim 351:* Given any $q \leqslant r$, since $dom(r)$ is finite,

$$\exists l > k+1 \quad \forall i < \omega \quad (l,i) \notin dom(q)\,.$$

Then, we consider the permutation $\rho : \omega \times \omega \longrightarrow \omega \times \omega$ defined for all $i < \omega$ by

- $\rho(k+1,i) = (l,i)$;
- $\rho(l,i) = (k+1,i)$;
- $\rho(n,i) = (n,i)$ (any $n \notin \{l, k+1\}$).

This permutation induces the automorphism $\pi : \mathbb{P}_{\aleph_0} \longrightarrow \mathbb{P}_{\aleph_0}$ defined for all $p \in \mathbb{P}_{\aleph_0}$ by

- $dom\big(\pi(p)\big) = \rho\big[dom(p)\big]$
- $\pi(p)\big(\rho(n,m)\big) = p(n,m)$.

We denote by $\tilde{\pi}$ its extension to $\mathbf{M}^{\mathbb{P}}$. We have:

For $i \notin \{k+1, l\}$:

$$\begin{aligned}
\tilde{\pi}(\underset{\sim}{a}_i) &= \big\{\big(\tilde{\pi}(\check{m}), \pi(p)\big) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \mid p(i,m) = 1\big\} \\
&= \big\{\big(\check{m}, \pi(p)\big) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \mid \pi(p)(i,m) = 1\big\} \\
&= \big\{\big(\check{m}, q\big) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \mid q(i,m) = 1\big\} \\
&= \underset{\sim}{a}_i;
\end{aligned}$$

$$\begin{aligned}
\tilde{\pi}(\underset{\sim}{a}_{k+1}) &= \big\{\big(\tilde{\pi}(\check{m}), \pi(p)\big) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \mid p(k+1,m) = 1\big\} \\
&= \big\{\big(\check{m}, \pi(p)\big) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \mid \pi(p)(l,m) = 1\big\} \\
&= \big\{\big(\check{m}, q\big) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \mid q(l,m) = 1\big\} \\
&= \underset{\sim}{a}_l;
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\pi}(\underset{\sim}{a}_l) &= \big\{\big(\tilde{\pi}(\check{m}), \pi(p)\big) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \mid p(l,m) = 1\big\} \\
&= \big\{\big(\check{m}, \pi(p)\big) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \mid \pi(p)(k+1,m) = 1\big\} \\
&= \big\{\big(\check{m}, q\big) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \mid q(k+1,m) = 1\big\} \\
&= \underset{\sim}{a}_{k+1}.
\end{aligned}$$

So, we also have

$$\begin{aligned}\tilde{\pi}(\underset{\sim}{A}) &= \{(\tilde{\pi}(\underset{\sim}{a}_n), \pi(\mathbb{1})) \mid n < \omega\}\\ &= \{(\tilde{\pi}(\underset{\sim}{a}_n), \mathbb{1}) \mid n < \omega\}\\ &= \{(\underset{\sim}{a}_n, \mathbb{1}) \mid n < \omega\}\\ &= \underset{\sim}{A}.\end{aligned}$$

In $\mathbf{M}$, since $q \leqslant r$ and $r \Vdash$ "$\underset{\sim}{a}_{k+1}$ is the only $x$ such that $\varphi(x, \check{\alpha}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k, \underset{\sim}{A})$", we also have

$$q \Vdash \text{"}\underset{\sim}{a}_{k+1} \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k, \underset{\sim}{A})\text{"}$$

then, since $\pi : \mathbb{P} \to \mathbb{P}$ is an automorphism, we also have

$$\pi(q) \Vdash \text{"}\tilde{\pi}(\underset{\sim}{a}_{k+1}) \text{ is the only } x \text{ such that } \varphi\big(x, \tilde{\pi}(\check{\alpha}), \tilde{\pi}(\check{\alpha}_1), \ldots, \tilde{\pi}(\check{\alpha}_n), \tilde{\pi}(\underset{\sim}{a}_1), \ldots, \tilde{\pi}(\underset{\sim}{a}_k), \tilde{\pi}(\underset{\sim}{A})\big)\text{"},$$

hence

$$\pi(q) \Vdash \text{"}\underset{\sim}{a}_l \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k, \underset{\sim}{A})\text{"}.$$

But $q$ is not defined over $l$, so $\pi(q)$ is not defined over $k+1$ and for all integers $i \notin \{k+1, l\}$ and $m < \omega$, we have $q(i, m) = \pi(q)(i, m)$. Therefore, $q$ and $\pi(q)$ are compatible and $s = q \cup \pi(q)$ satisfies both $s \leqslant q$ and $s \leqslant \pi(q)$, and also

$$s \Vdash \text{"}\underset{\sim}{a}_l \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k, \underset{\sim}{A})\text{"},$$

which shows that $s \in D$, and completes the proof that $D$ is dense below $r$.

□ 351

Finally, since $G$ is $\mathbb{P}_{\aleph_0}$-generic over $\mathbf{M}$, one has $D \cap G \neq \varnothing$, but any $q \in D \cap G$ yields that there exists $l > k+1$ such that

$$\big(q \Vdash \text{"}\underset{\sim}{a}_l \text{ is the only } x \text{ such that } \varphi(x, \check{\alpha}, \check{\alpha}_1, \ldots, \check{\alpha}_n, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_k, \underset{\sim}{A})\text{"}\big)^{\mathbf{M}}$$

By the Truth Lemma, this gives

$$\Big(\text{"}a_l \text{ is the only } x \text{ such that } \varphi(x, \alpha, \alpha_1, \ldots, \alpha_n, a_1, \ldots, a_k, A)\text{"}\Big)^{\mathbf{M}[G]}.$$

which contradicts the uniqueness of $a_{k+1}$ in $\mathbf{M}[G]$.

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