# Part VI

# ZF without the Axiom of Choice

# Chapter 19

# Cardinality Revisited

With the axiom of choice out of hand, we cannot use the notion of the cardinality of a set $A$ as it was defined when we had the axiom of choice at hand. The reason if that if the least ordinal $\alpha$ such that there exists a bijection $A \xrightarrow{bij.} \alpha$ always exists when $A$ can be well-ordered — because the order type of this well-ordering yields at least one ordinal which satisfies $A \xrightarrow{bij.} \alpha$, so the class of all ordinals that are equipotent to $A$ being non-empty admits a minimal element. This may not be the case when deprived of the axiom of choice. For instance, as we will see in Section 22.2 Theorem 379, one can force the set of reals to lack any well-ordering at all.

## 19.1 Injections and Surjections Revisited

We first introduce some notations for the existence of an injection or a surjection.

**Notation 352** (**ZF**)**.** *Given any sets* $A, B$*, we write*

- $A \overset{1-1}{\lesssim} B$ *whenever there exists some injective mapping* $f : A \xrightarrow{1-1} B$*;*
- $A \overset{1-1}{\not\lesssim} B$ *whenever* $A \overset{1-1}{\lesssim} B$ *does not hold;*
- $A \overset{onto}{\lesssim} B$ *whenever there exists some surjective mapping* $f : B \xrightarrow{onto} A$*;*
- $A \overset{onto}{\not\lesssim} B$ *whenever* $A \overset{onto}{\lesssim} B$ *does not hold.*

The following definition of being infinite is known as "Dedkind-infinite". A set is "Dedkind-infinite" if one can inject the set of integers into it, and "Dedkind-finite" otherwise.

**Definition 353.** *Let* $A$ *be any set.*

$$A \text{ is Dedekind-infinite if } \omega \overset{1-1}{\lesssim} A.$$

*(*$A$ *is Dedekind-finite if* $\omega \overset{1-1}{\not\lesssim} A$*.)*

We will see later[1] that — unless inconsistent — **ZF** does not prove that every *Dedekind-finite*-set is finite. But of course **ZFC** proves that every set can be well-ordered, therefore $\omega \overset{1-1}{\not\lesssim} A$ only holds when $A$ is finite.

**Lemma 354.**

$$\mathbf{ZF} \vdash_c \Big(\mathbf{AC} \longrightarrow \forall A\, \forall B\ \big(A \overset{onto}{\lesssim} B \longrightarrow A \overset{1-1}{\lesssim} B\big)\Big)$$

*Proof of Lemma 354*:
The result is trivial when $A$ is empty, for $B$ must be empty as well. So, we assume $A$ and $B$ are non-empty. Since $A \overset{onto}{\lesssim} B$, take any $g : B \xrightarrow{onto} A$ and form $\{g^{-1}(a) \mid a \in A\}$ which is a non-empty set of non-empty sets. By **AC**, one obtains a choice function $c$ which for each $a \in A$ provides a *unique* $c(a) \in B$ such that $c(a) \in g^{-1}(a)$. By construction, $c : A \xrightarrow{1-1} B$ witnesses that $A \overset{1-1}{\lesssim} B$.

□ 354

**Corollary 355.** *Given any sets* $A, B$,

$$\mathbf{ZFC} \vdash_c \Big(\big(A \overset{1-1}{\lesssim} B \wedge B \overset{onto}{\lesssim} A\big) \longrightarrow A \simeq B\Big).$$

*Proof of Corollary 355*: Immediate from Lemma 354 and Cantor-Schröder-Bernstein Theorem (page 57).

□ 355

However, as we will see later, $A \overset{onto}{\lesssim} B \Longrightarrow A \overset{1-1}{\lesssim} B$ may fail in the absence of the axiom of choice. Nonetheless, we have this equivalence between the axiom of choice and the existence of inverses of surjections.

**Lemma 356.**

$$\mathbf{ZF} \vdash_c \big(\mathbf{AC} \ \longleftrightarrow\ \forall A\, \forall B\ \forall g : B \xrightarrow{onto} A\ \ \exists f : A \xrightarrow{1-1} B\quad g \circ f = id\big)$$

*Proof of Lemma 356*:

($\Longrightarrow$) Given any family $(A_i)_{i \in I}$ of non-empty disjoint sets, we obtain a choice function $f : I \to \bigcup_{i \in I} A_i$ by letting $g : \bigcup_{i \in I} A_i \xrightarrow{onto} I$ be defined as $g(a) = i$ iff $a \in A_i$ and $f : I \xrightarrow{1-1} \bigcup_{i \in I} A_i$ be any function such that $g \circ f = id$ — which guarantees that $f(i) \in A_i$ holds for every $i \in I$.

[1]Such a result can be found in Claim 380 on page 347

($\Longleftarrow$) The result is trivial when $A$ is empty, for $B$ must be empty as well. So, we assume $A$ and $B$ are non-empty. Since $g : B \xrightarrow{onto} A$, form $\{g^{-1}(a) \mid a \in A\}$ which is a non-empty set of non-empty sets. By **AC**, one obtains a choice function $f$ which for each $a \in A$ provides a unique $f(a) \in g^{-1}(a)$. By construction, $f : A \xrightarrow{1-1} B$ and $g \circ f = id$ both hold.

□ 356

**Lemma 357 (ZF).** *Given any non-empty sets $A$ and $B$,*

*(1) if there exists $f : A \xrightarrow{1-1} B$, then there exists $g : B \xrightarrow{onto} A$,*

*(2) if there exists $f : A \xrightarrow{1-1} B$, then there exists $g : \mathcal{P}(A) \xrightarrow{1-1} \mathcal{P}(B)$.*

*Proof of Lemma* 357:

(1) Assume $f : A \xrightarrow{1-1} B$, then take any element $a' \in A$ and define $g : B \xrightarrow{onto} A$ by $g(x) = a'$ if $x \notin f[A]$, and $g(x) = a$ if $f(a) = x$. The fact that $f$ is 1-1 guarantees that $g$ is onto.

(2) Given $f : A \xrightarrow{1-1} B$, define $g : \mathcal{P}(A) \xrightarrow{1-1} \mathcal{P}(B)$ by $g(C) = f[C]$.

□ 357

## 19.2 Hartogs' Lemma

Without the axiom of choice, there may be sets that do not inject into any ordinal number. But, for any set, there is always some ordinal which does not injects into that set.

**Hartogs' Lemma (ZF).** *Given any set $A$, there exists some ordinal $\alpha$ such that*

$$\alpha \not\lesssim A.$$

*Proof of Hartogs' Lemma*: We consider the following set:

$$\mathcal{W} = \{(B, <_B) \subseteq A \times \mathscr{P}(A \times A) \mid (B, <_B) \textit{ is a well-ordering}\}.$$

Notice that this set is non-empty since the empty ordering $(\varnothing, \varnothing)$ belongs to $\mathcal{W}$. We then consider the class-function $\mathbf{F} : \mathcal{W} \to \mathbf{On}$ defined by

$$\mathbf{F}\big((B, <_B)\big) = \text{ the unique ordinal } \beta \text{ s.t. } (\beta, \in_\beta) \simeq (B, <_B).$$

We set

$$\alpha = \sup\Big\{\mathbf{F}\big((B,<_B)\big)+1 \mid (B,<_B)\in\mathcal{W}\Big\}.$$

It turns out that $\alpha \overset{1-1}{\not\lesssim} A$ holds; for otherwise if we let $f:\alpha \xrightarrow{1-1} A$ and set

$$B = f[\alpha] \text{ and } <_B = \big\{\big(f(\gamma),f(\delta)\big) \mid \gamma<\delta<\alpha\big\},$$

we then obtain $(B,<_B)\in\mathcal{W}$, hence $\alpha\in\mathbf{F}[\mathcal{W}]$, contradicting $\alpha > \mathbf{F}\big((B,<_B)\big)$.

□ Hartogs' Lemma

## 19.3 Cardinals without the axiom of choice

**Definition 359** (**ZF**)**.** *Given any set $A$, we define the cardinal of $A$ — denoted by $|A|$ — by*

$$|A| = \{B\in\mathbf{V}_{\alpha+1} \mid B\simeq A\}$$

*where $\alpha$ is the least ordinal such that there exists some $B\in\mathbf{V}_{\alpha+1}$ that satisfies $B\simeq A$.*

**Notation 360** (**ZF**)**.** *Given any set $A$, we by denote $\alpha_{|A|}$ the least ordinal such that there exists some $B\in\mathbf{V}_{\alpha+1}$ that satisfies $B\simeq A$.*

With this definition we notice that

**Lemma 361** (**ZF**)**.** *Given any non-empty sets $A$ and $A'$,*

*(1) $|A|$ is a set;*

*(2) $|A|=|A'| \iff A\simeq A'$;*

*(3) $|A|=|A'| \iff A\overset{1-1}{\lesssim}A'$ and $A'\overset{1-1}{\lesssim}A$.*

*Proof of Lemma 361:*

(1) Obvious.

(2) (⟹) If $|A|=|A'|$, then we have the following equality between non-empty sets

$$\Big\{B\in\mathbf{V}_{\alpha_{|A|}+1} \mid B\simeq A\Big\} = \Big\{B\in\mathbf{V}_{\alpha_{|A'|}+1} \mid B\simeq A'\Big\}$$

which yields $\alpha_{|A|} = \alpha_{|A'|}$, hence

$$\left\{B \in \mathbf{V}_{\alpha_{|A|}+1} \mid B \simeq A\right\} = \left\{B \in \mathbf{V}_{\alpha_{|A|}+1} \mid B \simeq A'\right\}$$

which leads to $A \simeq A'$.

**($\Longleftarrow$)** If $A \simeq A'$, then $A' \simeq B$ holds for every $B$ such that $A \simeq B$. Therefore $A' \simeq B$ holds for all $B \in |A| = \left\{B \in \mathbf{V}_{\alpha_{|A|}+1} \mid B \simeq A\right\}$ which yields $\alpha_{|A'|} \leqslant \alpha_{|A|}$. By symmetry, one also has $\alpha_{|A|} \leqslant \alpha_{|A'|}$, thus $\alpha_{|A|} = \alpha_{|A'|}$, which leads to $|A| = |A'|$.

(3) This is immediate via Cantor-Schröder-Bernstein Theorem (page 57).

□ 361

# Chapter 20

# About $\mathbb{R}$ without the axiom of choice

## 20.1 Variations on the Reals

In this chapter, we will not really be interested at the reals as an algebraic structure nor a topological structure. We will concentrate on the reals as a set which is equipotent to the power set of the integers. This is the reason why we first recall the following relations:

**Lemma 362 (ZF).**
$$\mathbb{R} \simeq {}^{\omega}\mathbb{R} \simeq {}^{\omega}\omega \simeq {}^{\omega}({}^{\omega}\omega) \simeq {}^{\omega}2 \simeq {}^{\omega}({}^{\omega}2).$$

*Proof of Lemma* 362:

(1) $\mathbb{R} \simeq {}^{\omega}\omega \simeq {}^{\omega}2$;

We recall ${}^{\omega}\omega = \big\{ f\omega \to \omega \big\}$ and ${}^{\omega}2 = \big\{ f : \omega \to \{0,1\} \big\}$.

By Cantor-Schröder-Bernstein Theorem (page 57), we only need to show $\mathbb{R} \overset{1-1}{\lesssim} {}^{\omega}\omega \overset{1-1}{\lesssim} {}^{\omega}2 \overset{1-1}{\lesssim} \mathbb{R}$

**$\mathbb{R} \overset{1-1}{\lesssim} {}^{\omega}\omega$:** assume every real $r$ is written in base 10 as

- in case $0 \leqslant r$:

$$r = +\, e_0\, e_1\, e_2 \ldots e_k,\ d_0\, d_1\, d_2\, d_3 \ldots d_n\, d_{n+1}\, d_{n+2} \ldots\ldots$$

- in case $r < 0$:

$$r = -\, e_0\, e_1\, e_2 \ldots e_k,\ d_0\, d_1\, d_2\, d_3 \ldots d_n\, d_{n+1}\, d_{n+2} \ldots\ldots$$

where

(a) $k$ is finite,

(b) for each $i \leqslant k$ and each $j \in \omega$, $e_i, d_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(c) $e_0 = 0 \Longrightarrow k = 0$,

(d) $\langle d_j / j \in \omega \rangle$ satisfies $\forall j\, \exists j' > j\ d_j \neq 9$. i.e., it is not ultimately constant with value 9. This means for instance that the real $0,23999999999999\ldots$ is rather represented by $+0,24000000000000\ldots$ and the integer $-3$ by $-3,00000000000\ldots$.

We describe the following mapping $f : \mathbb{R} \xrightarrow{1-1} {}^{\omega}\omega$ by

- If $r = +\, e_0\, e_1\, e_2 \ldots e_k,\ d_0\, d_1\, d_2\, d_3 \ldots d_n\, d_{n+1}\, d_{n+2} \ldots\ldots$, then

$$f(r) = \langle 8, 1+e_0, 1+e_1, \ldots, 1+e_k, 0, 1+d_0, 1+d_1, \ldots, 1+d_n, 1+d_{n+1}, \ldots \rangle$$

- If $r = +, e_0, e_1, e_2 \ldots e_k, d_0, d_1, d_2, d_3 \ldots d_n, d_{n+1}, d_{n+2} \ldots\ldots$, then

$$f(r) = \langle 9, 1+e_0, 1+e_1, \ldots, 1+e_k, 0, 1+d_0, 1+d_1, \ldots, 1+d_n, 1+d_{n+1}, \ldots \rangle$$

**${}^{\omega}\omega \overset{1-1}{\lesssim} {}^{\omega}2$:** we define $g : {}^{\omega}\omega \xrightarrow{1-1} {}^{\omega}2$ by

$$g\big(\langle a_i / i \in \omega \rangle\big) = 1\,\underbrace{0\ldots0}_{a_0}\,1\,\underbrace{0\ldots0}_{a_1}\,1\,\underbrace{0\ldots0}_{a_2}\,1\ldots$$

**${}^{\omega}2 \overset{1-1}{\lesssim} \mathbb{R}$:** we define $h : {}^{\omega}2 \xrightarrow{1-1} \mathbb{R}$ by

$$g\big(\langle a_i / i \in \omega \rangle\big) = 0, a_0\, a_1\, a_2\ \ldots\ a_n\, a_{n+1}\ \ldots\ldots$$

(2) ${}^{\omega}\mathbb{R} \simeq {}^{\omega}({}^{\omega}\omega) \simeq {}^{\omega}({}^{\omega}2)$.

It is enough to show that whenever $A \overset{1-1}{\lesssim} B$ holds for non-empty sets $A$ and $B$, then ${}^{\omega}A \overset{1-1}{\lesssim} {}^{\omega}B$ holds as well. So, given any $f : A \xrightarrow{1-1} B$, define $h : {}^{\omega}A \xrightarrow{1-1} {}^{\omega}B$ by

$$h\big(\langle a_i / i \in \omega \rangle\big) = \langle f(a_i) / i \in \omega \rangle\,.$$

(3) ${}^{\omega}2 \simeq {}^{\omega}({}^{\omega}2)$.

${}^{\omega}2 \overset{1-1}{\lesssim} {}^{\omega}({}^{\omega}2)$ is obvious. We show ${}^{\omega}({}^{\omega}2) \overset{1-1}{\lesssim} {}^{\omega}2$ by providing $f : {}^{\omega}({}^{\omega}2) \xrightarrow{1-1} {}^{\omega}2$ defined by

$$f\Big(\Big\langle\, \langle a_{i,j} / j < \omega \rangle\ i < \omega \Big\rangle\Big) = \langle b_k / k < \omega \rangle$$

where $b_k = a_{i,j}$ iff $k = \frac{(i+j)(i+j+1)}{2} + i$.

Notice that the mapping $(i,j) \mapsto \frac{(i+j)(i+j+1)}{2} + i$ is a bijection between $\omega \times \omega$ and $\omega$.

□ 362

**Lemma 363 (ZF).**

$$\omega_1 \overset{onto}{\lesssim} {}^{\omega}2.$$

*Proof of Lemma 363:* We construct $f : {}^{\omega}2 \xrightarrow{onto} \omega_1$.

(1) we define a mapping $\ulcorner \quad \urcorner : \omega \times \omega \xrightarrow{1-1} \omega$ by $\ulcorner n, m \urcorner = 2^{n+1} \cdot 3^{m+1}$.

(2) For each $s = \langle a_i / i \in \omega \rangle \in {}^{\omega}2$ we set

- if $\underbrace{\exists i \, \forall j \geqslant i \; a_j = 0}_{s \text{ contains finitely many } 1}$, then $f(s) = i$ for the least such $i$;
- if $\underbrace{\forall i \, \exists j \geqslant i \; a_j = 1}_{s \text{ contains infinitely many } 1}$, then
  - if $\exists i \, \forall n \, \forall m \; (a_i = 1 \wedge \ulcorner n, m \urcorner \neq i)$, then $f(s) = 0$
  - if $\forall i \, \exists n \, \exists m \; (a_i = 1 \longrightarrow \ulcorner n, m \urcorner = i)$, then
    - if $\Big(\omega, \big\{(n, m) \mid a_{\ulcorner n,m \urcorner} = 1\big\}\Big)$ is not a well-ordering, then $f(s) = 0$;
    - if $\Big(\omega, \big\{(n, m) \mid a_{\ulcorner n,m \urcorner} = 1\big\}\Big)$ is a well-ordering, then $f(s) = \alpha$ where $\alpha$ is the unique ordinal isomorphic to $\Big(\omega, \big\{(n, m) \mid a_{\ulcorner n,m \urcorner} = 1\big\}\Big)$. Notice that $\alpha \in \omega_1$ since $\alpha$ is countable.

    To show that $f$ is onto, it is enough to show that for every infinite countable ordinal $\alpha$ there exists some $s \in {}^{\omega}2$ such that $f(s) = \alpha$. For this, notice that $\alpha$ being countable, any bijection $h : \omega \xleftrightarrow{bij.} \alpha$ induces a well-ordering on $\omega$ of type $\alpha$. Namely, $(\omega, <_\alpha)$ where $<_\alpha = \{(n, m) \in \omega \times \omega \mid h(n) < h(m)\}$.
    By construction, $s = \langle a_i / i \in \omega \rangle \in {}^{\omega}2$ defined by $a_i = 1$ iff there exists $(n, m) \in {<_\alpha}$ such that $\ulcorner n, m \urcorner = i$.

□ 363

We will see later that it is consistent with **ZF** to have $\omega_1 \overset{1-1}{\not\lesssim} {}^{\omega}2$. This means, if **ZF** is consistent, there exists a model of **ZF** in which there exists some surjection from ${}^{\omega}2$ to $\omega_1$, but no injection from $\omega_1$ to ${}^{\omega}2$. i.e.,

$$\omega_1 \overset{onto}{\lesssim} {}^{\omega}2 \quad \text{but} \quad \omega_1 \overset{1-1}{\not\lesssim} {}^{\omega}2.$$

Nice examples of such models where $\omega_1 \overset{1-1}{\not\lesssim} {}^{\omega}2$ holds are given by those where the set of reals is a countable union of countable sets (see Section 22.1).

**Notation 364.** *Given any sets $A$ and $B$, the disjoint union of $A$ and $B$ is*

$$A \mathbin{\dot\cup} B := \big(A \times \{0\}\big) \cup \big(B \times \{1\}\big).$$

**Lemma 365** (**ZF**).

$${}^{\omega}2 \mathbin{\dot\cup} \omega_1 \overset{onto}{\lessapprox} {}^{\omega}2.$$

*Proof of Lemma 365:* We construct $f : {}^{\omega}2 \xrightarrow{onto} {}^{\omega}2 \mathbin{\dot\cup} \omega_1$. From Lemma 363, we are granted with a mapping $f' : {}^{\omega}2 \xrightarrow{onto} \omega_1$. Given any $s = \langle a_i / i \in \omega \rangle \in {}^{\omega}2$ we define $f(s)$ as follows:

- if $a_0 = 0$, then $f(s) = \langle a_{i+1}/i \in \omega \rangle$;
- if $a_0 = 1$, then $f(s) = f'\big(\langle a_{i+1}/i \in \omega \rangle\big)$.

□ 365

## 20.2 Outcomes of $\mathbb{R}$ as a Countable Union of Countable Sets

**Proposition 366** (**ZF**). *If $\mathbb{R}$ is a countable union of countable sets, then*

$$\omega_1 \overset{1-1}{\not\lessapprox} {}^{\omega}2.$$

*Proof of Proposition 366:* Notice first that by Lemma 362 we have

$$\mathbb{R} \simeq {}^{\omega}\mathbb{R} \simeq {}^{\omega}\omega \simeq {}^{\omega}({}^{\omega}\omega) \simeq {}^{\omega}2 \simeq {}^{\omega}({}^{\omega}2).$$

Thus, the assumption is equivalent to saying that any of these sets is a countable union of countable sets. So, we assume that ${}^{\omega}({}^{\omega}2)$ is a countable union of countable sets. i.e., there exists $(\mathcal{G}_n)_{n<\omega}$ where for each integer $n$, $\mathcal{G}_n$ is non-empty, countable and

$${}^{\omega}({}^{\omega}2) = \bigcup_{n<\omega} \mathcal{G}_n.$$

First, we set

$$\begin{aligned}\mathscr{H}_n &= \bigcup_{\mathscr{S}\in\mathcal{G}_n} \mathscr{S}[\omega] \\ &= \bigcup \{\mathscr{S}[\omega] \subseteq {}^{\omega}2 \mid \mathscr{S} \in \mathcal{G}_n\} \\ &= \{\mathscr{S}(k) \in {}^{\omega}2 \mid \mathscr{S} \in \mathcal{G}_n \ \wedge \ k \in \omega\} \\ &= \{s \in {}^{\omega}2 \mid \exists \mathscr{S} \in \mathcal{G}_n \ \exists k < \omega \ \mathscr{S}(k) = s\}\,.\end{aligned}$$

We first establish that for each integer $n$ we have

$$\mathscr{H}_n \overset{1-1}{\lesssim} \omega.$$

Since $\mathcal{G}_n$ is a non-empty countable set, we take any $g : \mathcal{G}_n \xrightarrow{1-1} \omega$ and construct

$$\begin{aligned}\mathscr{I} : \mathscr{H}_n &\xrightarrow{1-1} \omega \\ s \quad &\mapsto \quad \mathscr{I}(s) = \tfrac{(i+j)(i+j+1)}{2} + i\end{aligned}$$

(1) $i$ is the least integer such that

$$s \in \big(g^{-1}(i)\big)[\omega]$$

i.e., there exists $\mathscr{S} \in \mathcal{G}_n$ with $g(\mathscr{S}) = i$, and there exists some $k < \omega$ $\mathscr{S}(k) = s$;

(2) $j$ is the least such that $\big(g^{-1}(i)\big)(j) = s$.

Towards a contradiction, we then assume that $\omega_1 \overset{1-1}{\lesssim} {}^{\omega}2$ holds, so that there exists some injective mapping $f : \omega_1 \xrightarrow{1-1} {}^{\omega}2$.

For each integer $n$, we define

$$\begin{aligned}\alpha_n &= \bigcap f^{-1}[\mathscr{H}_n] \\ &= \min\{\alpha \in \omega_1 \mid f(\alpha) \notin \mathscr{H}_n\}\,.\end{aligned}$$

We then define, by diagonalization, some mapping which will yield a contradiction:

$$\begin{aligned}h : \omega &\longrightarrow {}^{\omega}2 \\ n \ &\mapsto \ f(\alpha_n).\end{aligned}$$

By its very definition, $h \in {}^{\omega}({}^{\omega}2) = \bigcup_{n<\omega} \mathcal{G}_n$, hence for some integer $n$ we have $h \in \mathcal{G}_n$. We then consider $h(n) \in {}^{\omega}2$, and discuss whether $h(n)$ belongs to $\mathscr{H}_n$ or not:

- $h(n) \in \mathscr{H}_n$ holds since $h \in \mathcal{G}_n$ and $\mathscr{H}_n = \bigcup_{\mathscr{S}\in\mathcal{G}_n} \mathscr{S}[\omega]$;

- $h(n) \notin \mathscr{H}_n$ holds too since $h(n) = f(\alpha_n)$ and $f(\alpha) \notin \mathscr{H}_n$.

This contradiction shows that no injective mapping $f : \omega_1 \xrightarrow{1-1} {}^{\omega}2$ exists; namely $\omega_1 \overset{1-1}{\not\lesssim} {}^{\omega}2$.

□ 366

**Corollary 367 (ZF).** *If $\mathbb{R}$ is a countable union of countable sets, then there exists some partition $\mathcal{R}$ of $\mathbb{R}$ such that $\mathbb{R} \lneqq \mathcal{R}$ which stands for*

$$\mathbb{R} \overset{1-1}{\lesssim} \mathcal{R} \quad \textit{and} \quad \mathcal{R} \overset{1-1}{\not\lesssim} \mathbb{R}.$$

*Proof of Corollary 367:* We first show that this statement is equivalent to the existence of some partition $\mathcal{C}$ of ${}^{\omega}2$ such that ${}^{\omega}2 \lneqq \mathcal{C}$, i.e.,

$${}^{\omega}2 \overset{1-1}{\lesssim} \mathcal{C} \quad \text{and} \quad \mathcal{C} \overset{1-1}{\not\lesssim} {}^{\omega}2.$$

Indeed, if $\mathbb{R} \lneqq \mathcal{R}$ holds, then take any $f : \mathbb{R} \xleftrightarrow{bij.} {}^{\omega}2$ and define $\mathcal{C} = \{f[p] \mid p \in \mathcal{R}\}$. Clearly $\mathcal{C}$ is a partition of ${}^{\omega}2$ that satisfies $\mathcal{R} \simeq \mathcal{C}$, which yields ${}^{\omega}2 \lneqq \mathcal{C}$ since one has ${}^{\omega}2 \simeq \mathbb{R} \lneqq \mathcal{R} \simeq \mathcal{C}$.

Similarly, if ${}^{\omega}2 \lneqq \mathcal{C}$ holds, then take any $g : {}^{\omega}2 \xleftrightarrow{bij.} \mathbb{R}$ in order to obtain the partition $\mathcal{R} = \{f[p] \mid p \in \mathcal{C}\}$ that satisfies $\mathcal{C} \simeq \mathcal{R}$ which leads to $\mathbb{R} \lneqq \mathcal{R}$ since one has $\mathbb{R} \simeq {}^{\omega}2 \lneqq \mathcal{C} \simeq \mathcal{R}$.

So, in order to establish the result we simply prove that there exists some partition $\mathcal{C}$ of ${}^{\omega}2$ such that ${}^{\omega}2 \lneqq \mathcal{C}$. For this, we come back to Lemma 365 which stated that ${}^{\omega}2 \cup \omega_1 \overset{onto}{\lesssim} {}^{\omega}2$ holds and take any $f : {}^{\omega}2 \xrightarrow{onto} {}^{\omega}2 \cup \omega_1$ to form the partition

$$\begin{aligned}\mathcal{C} &= \Big\{ \{s \in {}^{\omega}2 \mid f(s) = x\} \mid x \in {}^{\omega}2 \cup \omega_1 \Big\} \\ &= \Big\{ f^{-1}\big[\{x\}\big] \mid x \in {}^{\omega}2 \cup \omega_1 \Big\}.\end{aligned}$$

We obtain

**${}^{\omega}2 \overset{1-1}{\lesssim} \mathcal{C}$:** The mapping $g : {}^{\omega}2 \longrightarrow \mathcal{C}$ defined by

$$\begin{aligned} g(x) &= \{s \in {}^{\omega}2 \mid f(s) = x\} \\ &= f^{-1}\big[\{x\}\big]\end{aligned}$$

is obviously 1-1, hence witnesses that ${}^{\omega}2 \overset{1-1}{\lesssim} \mathcal{C}$ holds.

**$\mathcal{C} \overset{1-1}{\not\lesssim} {}^{\omega}2$:** Towards a contradiction, we assume $\mathcal{C} \overset{1-1}{\lesssim} {}^{\omega}2$. We notice that $\omega_1 \overset{1-1}{\lesssim} \mathcal{C}$ holds for the following mapping is 1-1: $h : \omega_1 \longrightarrow \mathcal{C}$ defined by

$$\begin{aligned} h(x) &= \{s \in {}^{\omega}2 \mid f(s) = x\} \\ &= f^{-1}\big[\,\{x\}\,\big] \end{aligned}$$

Therefore, we have both

$$\omega_1 \overset{1-1}{\lesssim} \mathcal{C} \quad \text{and} \quad \mathcal{C} \overset{1-1}{\lesssim} {}^{\omega}2$$

which leads to $\omega_1 \overset{1-1}{\lesssim} {}^{\omega}2$, contradicting Proposition 366.

$\square$ 367

# Chapter 21

# Symmetric Submodels of Generic Extensions

## 21.1 Symmetry Groups and Hereditarily Symmetric $\mathbb{P}$-names

We recall Definition 342 which states that given $\mathbf{M}$ any *c.t.m.* of "**ZFC**" and $(\mathbb{P}, \leqslant, \mathbb{1})$ any partial order over $\mathbf{M}$, an *automorphism* of $\mathbb{P}$ is a mapping $\pi : \mathbb{P} \overset{bij.}{\longleftrightarrow} \mathbb{P}$ such that

$$\forall p \in \mathbb{P}\ \forall q \in \mathbb{P}\ \ \big(p \leqslant q \ \longleftrightarrow\ \pi(p) \leqslant \pi(q)\big).$$

We also recall that this definition implies $\pi(\mathbb{1}) = \mathbb{1}$ and any such automorphism $\pi$ induces an automorphism $\tilde{\pi}$ on the class of $\mathbb{P}$-names $\mathbf{M}^{\mathbb{P}}$ defined by transfinite recursion (see Definition 344):

$$\begin{aligned} \tilde{\pi} : \mathbf{M}^{\mathbb{P}} &\longrightarrow \mathbf{M}^{\mathbb{P}} \\ \tau &\longmapsto \{(\tilde{\pi}(\sigma), \pi(p)) \mid (\sigma, p) \in \tau\}\,. \end{aligned}$$

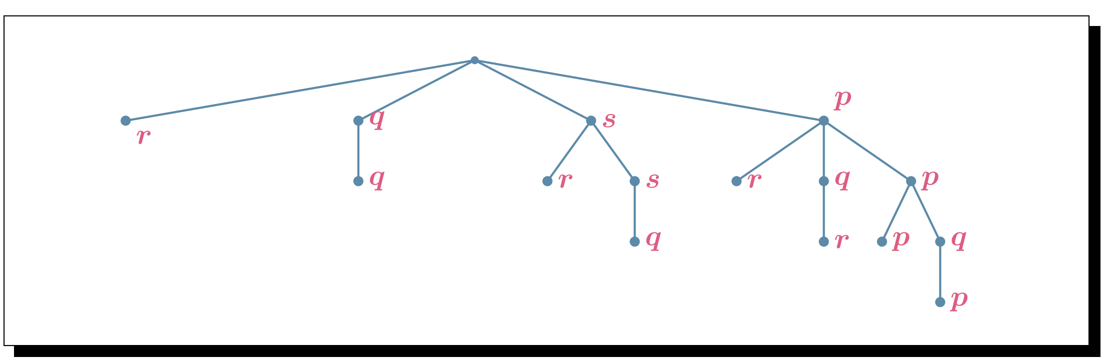



Figure 21.1: The $\mathbb{P}$-name $\tau$.

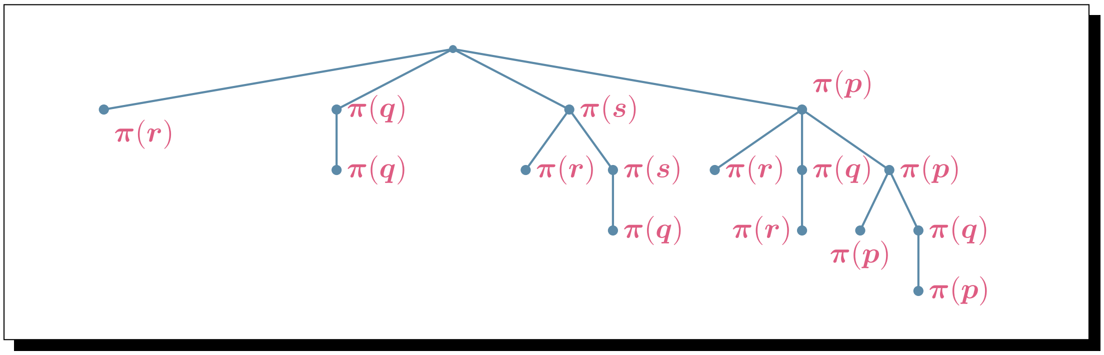



Figure 21.2: The $\mathbb{P}$-name $\tilde{\pi}(\tau)$.

Notice that we have $\tilde{\pi}(\varnothing) = \varnothing$. Also, for every canonical $\mathbb{P}$-name $\check{x}$, we have $\tilde{\pi}(\check{x}) = \check{x}$.

By Lemma 346, for all $G$ $\mathbb{P}$-generic over $\mathbf{M}$, and all automorphism $\pi$,

$$\mathbf{M}\big[\pi[G]\big] = \mathbf{M}[G].$$

Moreover, by Lemma 347, for all $\tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$, and $p \in \mathbb{P}$,

$$p \Vdash \varphi(\tau_1, \ldots, \tau_n) \iff \pi(p) \Vdash \varphi\big(\tilde{\pi}(\tau_1), \ldots, \tilde{\pi}(\tau_n)\big).$$

**Definition 368** (Symmetry Group)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*.*
*For each* $\mathbb{P}$*-name* $\tau$ *the symmetry group of* $\tau$ *is*

$$sym_{\mathcal{G}}(\tau) = \{\pi \in \mathcal{G} \mid \tilde{\pi}(\tau) = \tau\}.$$

**Lemma 369.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*. Let* $\tau$ *be any* $\mathbb{P}$*-name.*

$$sym_{\mathcal{G}}(\tau) \text{ is a subgroup of } \mathcal{G}.$$

*Proof of Lemma* 369: We check that $sym_{\mathcal{G}}(\tau)$ is closed under products and inverses.

**Closed under $(\pi, \rho) \mapsto \rho \circ \pi$:** Given any automorphisms $\pi, \rho \in sym_{\mathscr{G}}(\tau)$,

$$\begin{aligned}\tilde{\rho} \circ \tilde{\pi}(\tau) &= \tilde{\rho}\big(\tilde{\pi}(\tau)\big) \\ &= \tilde{\rho}\big(\tau\big) \\ &= \tau\end{aligned}$$

which shows that $\rho \circ \pi \in sym_{\mathscr{G}}(\tau)$.

**Closed under $\pi \mapsto \pi^{-1}$:** Given any automorphism $\pi \in \mathbb{P}$, the following holds:

$$\forall (\sigma, p) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \;\; \Big( (\sigma, p) \in \tau \longleftrightarrow \big(\tilde{\pi}(\sigma), \pi(p)\big) \in \tilde{\pi}(\tau) \Big).$$

Notice that we have $\pi \in sym_{\mathscr{G}}(\tau)$ if and only if $\tau = \tilde{\pi}(\tau)$. Or, to say it differently,

$$\tau = \tilde{\pi}(\tau) \iff \forall (\sigma, p) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \;\; \Big( (\sigma, p) \in \tau \longleftrightarrow \big(\tilde{\pi}(\sigma), \pi(p)\big) \in \tau \Big).$$

Notice also that $\forall (\sigma, p) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P}$

$$(\sigma, p) \in \tau \iff \big(\widetilde{\pi^{-1}}(\sigma), \pi^{-1}(p)\big) \in \widetilde{\pi^{-1}}(\tau)$$

and

$$\begin{aligned}\big(\tilde{\pi}(\sigma), \pi(p)\big) \in \tau &\iff \big(\widetilde{\pi^{-1}} \circ \tilde{\pi}(\sigma), \pi^{-1} \circ \pi(p)\big) \in \widetilde{\pi^{-1}}(\tau) \\ &\iff (\sigma, p) \in \widetilde{\pi^{-1}}(\tau).\end{aligned}$$

So, we end up with

$$\begin{aligned}\tau = \tilde{\pi}(\tau) &\iff \forall (\sigma, p) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \;\; \Big( (\sigma, p) \in \tau \longleftrightarrow \big(\tilde{\pi}(\sigma), \pi(p)\big) \in \tau \Big) \\ &\iff \forall (\sigma, p) \in \mathbf{M}^{\mathbb{P}} \times \mathbb{P} \;\; \Big( (\sigma, p) \in \tau \longleftrightarrow (\sigma, p) \in \widetilde{\pi^{-1}}(\tau) \Big) \\ &\iff \tau = \widetilde{\pi^{-1}}(\tau).\end{aligned}$$

which shows that $\pi \in sym_{\mathscr{G}}(\tau)$ if and onyl if $\pi^{-1} \in sym_{\mathscr{G}}(\tau)$.

□ 369

We notice:

(1) For any canonical $\mathbb{P}$-name $\check{x}$,

$$sym_{\mathscr{G}}(\check{x}) = \{\pi \in \mathcal{G} \mid \pi(\check{x}) = \check{x}\} = \mathcal{G}.$$

(2) For all $\mathbb{P}$-name $\tau$, and all automorphism $\pi \in \mathcal{G}$, one has

$$sym_{\mathscr{G}}\left(\tilde{\pi}(\tau)\right) = \pi \circ sym_{\mathscr{G}}(\tau) \circ \pi^{-1}$$

since, given any automorphism $\mu \in sym_{\mathscr{G}}(\tau)$, we have

$$(\tilde{\pi} \circ \tilde{\mu} \circ \tilde{\pi}^{-1})(\tilde{\pi}(\tau)) = \tilde{\pi} \circ \tilde{\mu} \circ (\tilde{\pi}^{-1} \circ \tilde{\pi})(\tau) = \tilde{\pi} \circ \tilde{\mu}(\tau) = \tilde{\pi}(\tau).$$

**Definition 370** (Normal Filter)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*", $(\mathbb{P}, \leqslant, \mathbb{1})$ any partial order over* $\mathbf{M}$*, and $\mathcal{G}$ any subgroup of the group of automorphisms of $\mathbb{P}$.*

$$\mathscr{F} \textit{ is a normal filter on } \mathcal{G} \textit{ if}$$

*$\mathscr{F}$ is a set of subgroups of $\mathcal{G}$ such that for all subgroups $\mathcal{H}, \mathcal{K} \subseteq \mathcal{G}$ and all $\pi \in \mathcal{G}$:*

*(1)* $\mathcal{G} \in \mathscr{F}$

*(2) if* $\mathcal{H} \in \mathscr{F}$ and $\mathcal{H} \subseteq \mathcal{K}$, *then* $\mathcal{K} \in \mathscr{F}$

*(3) if* $\mathcal{H} \in \mathscr{F}$ and $\mathcal{K} \in \mathscr{F}$, *then* $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$

*(4) if* $\mathcal{H} \in \mathscr{F}$, *then* $\pi \circ \mathcal{H} \circ \pi^{-1} \in \mathscr{F}$

Granted with some normal filter, we focus on $\mathbb{P}$-names whose symmetry group belongs to the filter, and even more, whose symmetry group hereditarily belongs to the filter:

**Definition 371** (Hereditarily Symmetric $\mathbb{P}$-names)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*", $(\mathbb{P}, \leqslant, \mathbb{1})$ any partial order over* $\mathbf{M}$*, and $\mathcal{G}$ any subgroup of the group of automorphisms of $\mathbb{P}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$.*
*The set of all hereditarily symmetric $\mathbb{P}$-names* $\mathbf{HS}_{\mathscr{F}} \subseteq \mathbf{M}^{\mathbb{P}}$ *is defined by transfinite recursion:*

$$\tau \in \mathbf{HS}_{\mathscr{F}} \iff sym_{\mathscr{G}}(\tau) \in \mathscr{F} \text{ and } \{\sigma \mid \exists p \in \mathbb{P}\ (\sigma, p) \in \tau\} \subseteq \mathbf{HS}_{\mathscr{F}}.$$

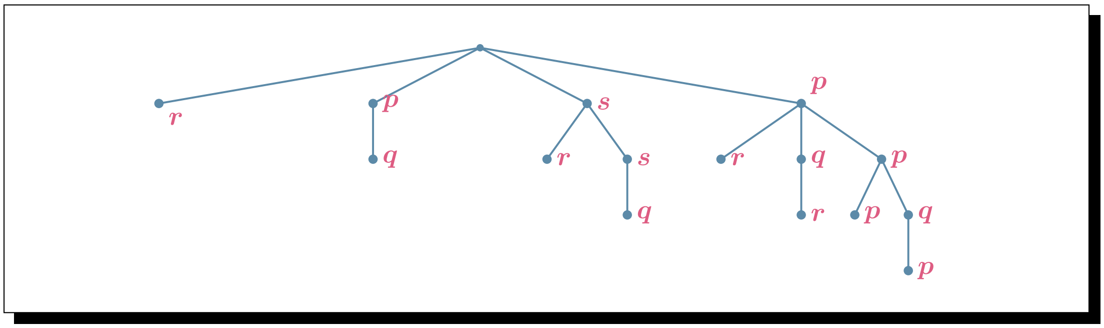



Figure 21.3: The $\mathbb{P}$-name $\tau$.

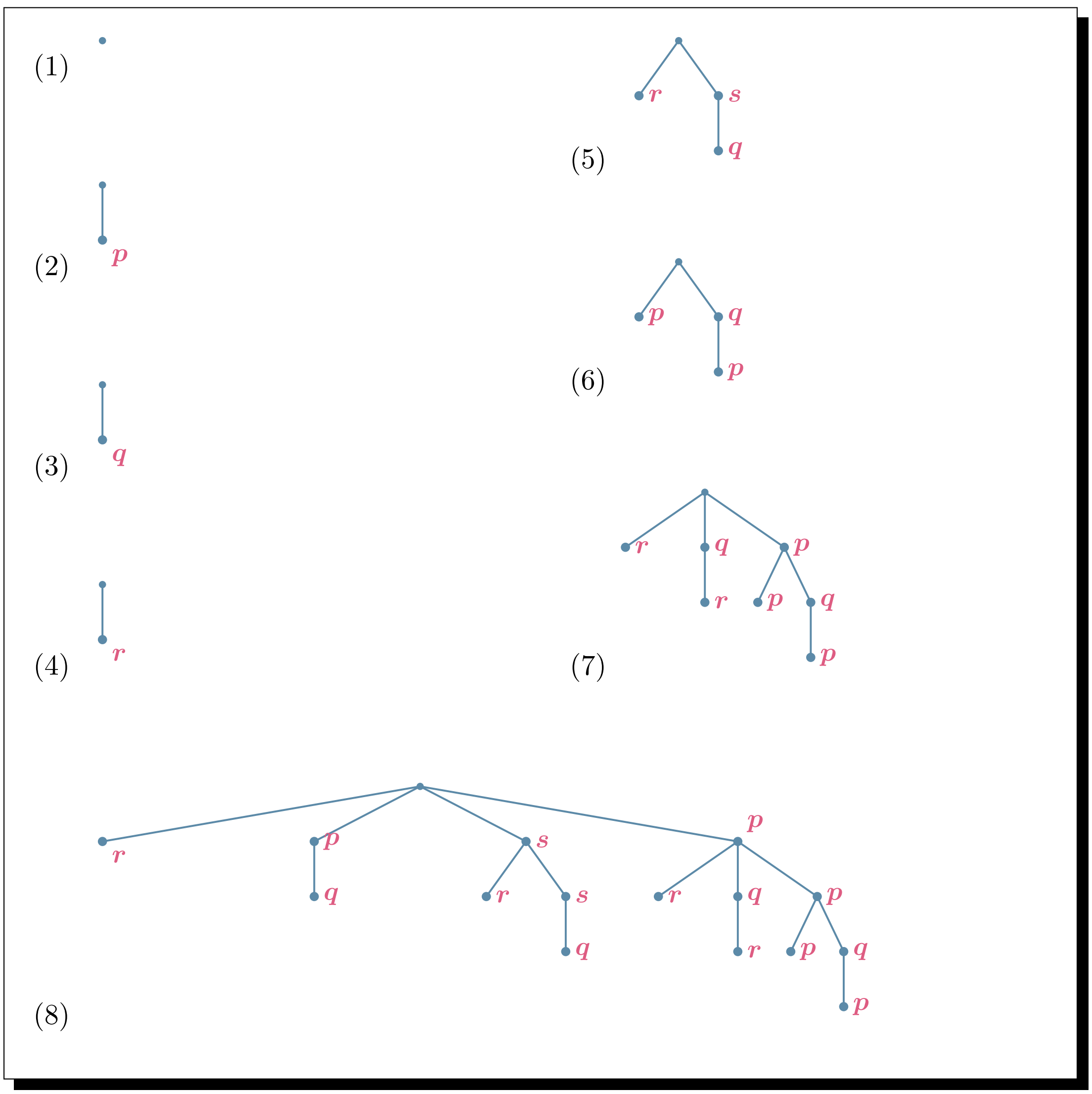



Figure 21.4: $\tau \in \mathbf{HS}_{\mathscr{F}}$ iff $sym_{\mathscr{G}}(\sigma) \in \mathscr{F}$ holds for every above $\mathbb{P}$-name $\sigma$.

So we have

$$\mathbf{HS}_{\mathscr{F}} = \left\{\tau \in \mathbf{M}^{\mathbb{P}} \mid sym_{\mathcal{G}}(\tau) \in \mathscr{F} \text{ and } \{\sigma \mid \exists p \in \mathbb{P}\ (\sigma, p) \in \tau\} \subseteq \mathbf{HS}_{\mathscr{F}}\right\}.$$

Notice that

- every canonical $\mathbb{P}$-name $\check{x} \in \mathbf{HS}_{\mathscr{F}}$ since $sym_{\mathcal{G}}(\check{x}) = \mathcal{G} \in \mathscr{F}$.
- If $\tau \in \mathbf{HS}_{\mathscr{F}}$, then $\tilde{\pi}(\tau) \in \mathbf{HS}_{\mathscr{F}}$ holds for any $\pi \in \mathcal{G}$, because we have

$$sym_{\mathcal{G}}\left(\tilde{\pi}(\tau)\right) = \pi \circ sym_{\mathcal{G}}(\tau) \circ \pi^{-1}$$

  Hence, for all $\mathbb{P}$-name $\tau \in \mathbf{M}^{\mathbb{P}}$ and all automorphism $\pi \in \mathcal{G}$, we have

$$\tau \in \mathbf{HS}_{\mathscr{F}} \iff \tilde{\pi}(\tau) \in \mathbf{HS}_{\mathscr{F}}.$$

## 21.2 The Symmetric Submodel

We now define the symmetric submodel of a generic extension as its restriction to the only elements that admit an hereditarily symmetric $\mathbb{P}$-name.

**Definition 372** (Symmetric Submodel)**.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*, and* $\mathscr{F}$ *any normal filter on* $\mathcal{G}$*. The symmetric submodel of the generic extension* $\mathbf{M}[G]$ *is*

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} = \left\{(\tau)_G \in \mathbf{M}[G] \mid \tau \in \mathbf{HS}_{\mathscr{F}}\right\}.$$

A symmetric submodel is a structure which lies in between $\mathbf{M}$ (since every canonical $\mathbb{P}$-name is hereditarily symmetric) and the generic extension $\mathbf{M}[G]$ (since hereditarily symmetric $\mathbb{P}$-names are particular $\mathbb{P}$-names). The properties that make the symmetric submodel very interesting is that it is transitive and satisfies all the axioms of **ZF**, but contrary to the generic extension, it does not necessarily satisfy the axiom of choice (**AC**).

**Lemma 373.** *Let* $\mathbf{M}$ *be any c.t.m. of "*$\mathbf{ZFC}$*",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*, and* $\mathscr{F}$ *any normal filter on* $\mathcal{G}$*.*

*(1)* $\mathbf{M} \subseteq \widehat{\mathbf{M}[G]}^{\mathscr{F}} \subseteq \mathbf{M}[G]$*;*

*(2)* $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ *is a transitive set;*

*(3)* $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ *satisfies "* $\mathbf{ZF}$ *".*

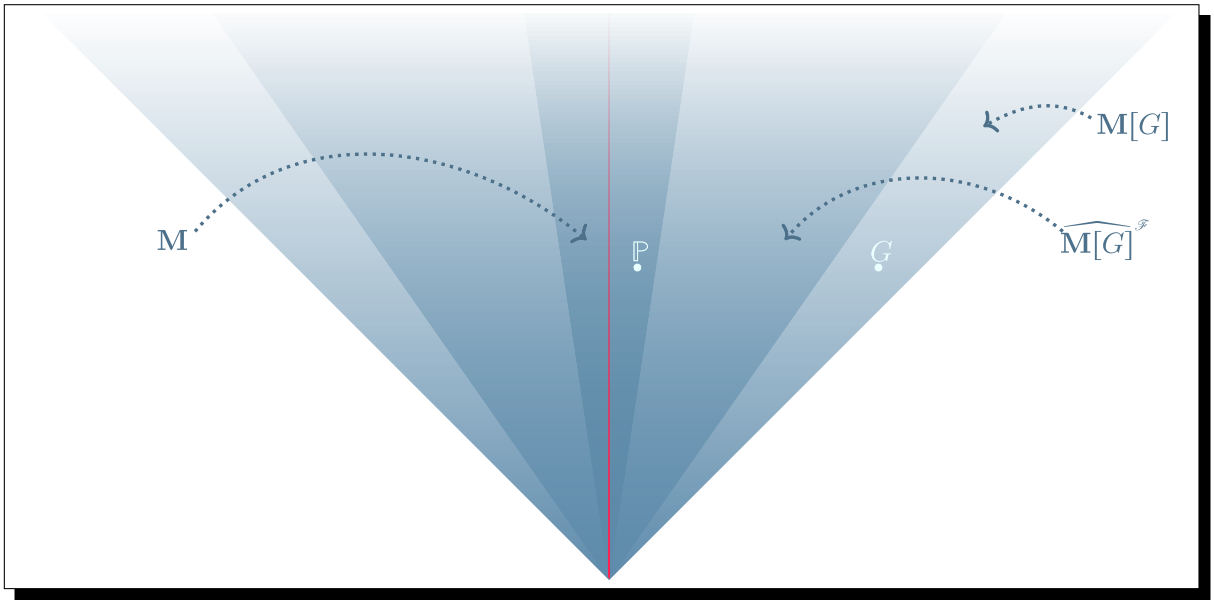



Figure 21.5: The symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ of $\mathbf{M}[G]$ the generic extension of $\mathbf{M}$.

Notice that if $G$ the generic filter over $\mathbf{M}$ is a member of $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ then the whole construction is not of much interest. Indeed, in Lemma 311, we showed that $\mathbf{M}[G]$ is the $\subseteq$-least transitive model $\mathbf{N}$ of "**ZFC**" with satisfies both $\mathbf{M} \subseteq \mathbf{N}$ and $G \in \mathbf{N}$. Therefore, if there exists some $\mathbb{P}$-name for $G$ that is hereditarily symmetric, then we end up with $\widehat{\mathbf{M}[G]}^{\mathscr{F}} = \mathbf{M}[G]$, rendering the entire construction useless.

*Proof of Lemma* 373:

(1) is immediate.

(2) Assume $x \in (\tau)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$ with $\tau \in \mathbf{HS}_{\mathscr{F}}$. Then, there exists $p \in \mathbf{G}$ and $\sigma \in \mathbf{M}^{\mathbb{P}}$ such that $(\sigma, p) \in \tau$ and $x = (\sigma)_G$. Since $\tau \in \mathbf{HS}_{\mathscr{F}}$, it follows that $\sigma \in \mathbf{HS}_{\mathscr{F}}$, hence $x = (\sigma)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

(3) $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ satisfies " **ZF** ":

**Extensionality** holds in $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ since $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ is transitive.

**Comprehension Schema** We want to show that for all $\sigma, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n \in \mathbf{HS}_{\mathscr{F}}$ and $\varphi(x, y_1, \ldots, y_n)$:

$$u = \left\{ z \in (\sigma)_G \mid \left(\varphi\big(z, (\underset{\sim}{z}_1)_G, \ldots, (\underset{\sim}{z}_n)_G\big)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}} \right\} \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}.$$

We must find some $\tau \in \mathbf{HS}_{\mathscr{F}}$ such that $u = (\tau)_G$. For this purpose, we first modify $\sigma$ and consider $\overline{\sigma}$ instead:

$$\overline{\sigma} = \{(\underset{\sim}{z}, p) \in \mathbf{HS}_{\mathscr{F}} \times \mathbb{P} \mid \exists q \in \mathbb{P}\ p \leqslant q \text{ and } (\underset{\sim}{z}, q) \in \sigma\}$$

Notice that $(\sigma)_G = (\overline{\sigma})_G$ holds because:

- $\sigma \subseteq \overline{\sigma}$ implies $(\sigma)_G \subseteq (\overline{\sigma})_G$, and
- for every $p \in G$ and $(\underset{\sim}{z}, p) \in \overline{\sigma}$, there exists some $(\underset{\sim}{z}, q) \in \sigma$ with $p \leqslant q$, hence $q \in G$, which shows that if $(\underset{\sim}{z})_G$ belongs to $(\overline{\sigma})_G$, it also belongs to $(\sigma)_G$ .

We now show that $\overline{\sigma} \in \mathbf{HS}_{\mathscr{F}}$.

**Claim 374.**

$$sym_{\mathscr{G}}(\sigma) \subseteq sym_{\mathscr{G}}(\overline{\sigma}).$$

*Proof of Claim 374:* Take any automorphism $\pi \in sym_{\mathscr{G}}(\sigma)$ and any $(\underset{\sim}{z}, p) \in \overline{\sigma}$ with $p \leqslant q$ and $(\underset{\sim}{z}, q) \in \sigma$. Since $\pi \in sym_{\mathscr{G}}(\sigma)$, $\big(\tilde{\pi}(\underset{\sim}{z}), \pi(q)\big) \in \sigma$ holds. Because $p \leqslant q$, we have $\pi(p) \leqslant \pi(q)$; thus, by the very definition of $\overline{\sigma}$ we have $\big(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\big) \in \overline{\sigma}$ which shows $\tilde{\pi}(\overline{\sigma}) \subseteq \overline{\sigma}$. To show the other inclusion, namely that $\tilde{\pi}(\overline{\sigma}) \supseteq \overline{\sigma}$ holds, it is enough to notice that

$$\pi \in sym_{\mathscr{G}}(\sigma) \text{ if and only if } \pi^{-1} \in sym_{\mathscr{G}}(\sigma)$$

and

$$\begin{aligned}
\widetilde{\pi^{-1}}(\overline{\sigma}) \subseteq \overline{\sigma} &\Longrightarrow \tilde{\pi}\left(\widetilde{\pi^{-1}}(\overline{\sigma})\right) \subseteq \tilde{\pi}(\overline{\sigma}) \\
&\Longrightarrow \tilde{\pi} \circ \widetilde{\pi^{-1}}(\overline{\sigma}) \subseteq \tilde{\pi}(\overline{\sigma}) \\
&\Longrightarrow \tilde{id}(\overline{\sigma}) \subseteq \tilde{\pi}(\overline{\sigma}) \\
&\Longrightarrow \overline{\sigma} \subseteq \tilde{\pi}(\overline{\sigma}).
\end{aligned}$$

□ 374

So, we obtain $sym_{\mathscr{G}}(\overline{\sigma}) \in \mathscr{F}$ since we have shown

$$\underbrace{sym_{\mathscr{G}}(\sigma)}_{\in \mathscr{F}} \subseteq sym_{\mathscr{G}}(\overline{\sigma}),$$

Moreover, since $dom(\overline{\sigma}) = dom(\sigma) \subseteq \mathbf{HS}_{\mathscr{F}}$, it follows $\overline{\sigma} \in \mathbf{HS}_{\mathscr{F}}$.

We set

$$\underset{\sim}{u} = \Big\{(\underset{\sim}{z}, p) \in \overline{\sigma} \mid p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{z}, \underset{\sim}{z_1}, \ldots, \underset{\sim}{z_n})\Big\}.$$

We show $\underset{\sim}{u} \in \mathbf{HS}_{\mathscr{F}}$. Since $\overline{\sigma} \in \mathbf{HS}_{\mathscr{F}}$, it only remains to show that $sym_{\mathscr{G}}(\underset{\sim}{u}) \in \mathscr{F}$. For this, we consider

$$\mathscr{G}' = sym_{\mathscr{G}}(\overline{\sigma}) \cap sym_{\mathscr{G}}(\underset{\sim}{z_1}) \cap \ldots \cap sym_{\mathscr{G}}(\underset{\sim}{z_n})$$

$\mathscr{G}' \in \mathscr{F}$ holds since $\mathscr{F}$ is a filter, and for each $\pi \in \mathscr{G}'$, one has

$$\tilde{\pi}(\overline{\sigma}) = \overline{\sigma}, \quad \tilde{\pi}(\underset{\sim}{z_1}) = \underset{\sim}{z_n}, \quad \ldots, \tilde{\pi}(\underset{\sim}{z_n}) = \underset{\sim}{z_n}.$$

and also

$$\begin{aligned}
\tilde{\pi}(\underset{\sim}{u}) &= \Big\{ \big(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\big) \mid (\underset{\sim}{z}, p) \in \underset{\sim}{u} \Big\} \\
&= \Big\{ \big(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\big) \mid (\underset{\sim}{z}, p) \in \overline{\sigma} \wedge p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{z}, \underset{\sim}{z_1}, \ldots, \underset{\sim}{z_n}) \Big\} \\
&= \Big\{ \big(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\big) \mid \big(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\big) \in \tilde{\pi}(\overline{\sigma}) \wedge \pi(p) \Vdash_{\mathbb{P},\mathbf{M}} \varphi\big(\tilde{\pi}(\underset{\sim}{z}), \tilde{\pi}(\underset{\sim}{z_1}), \ldots, \tilde{\pi}(\underset{\sim}{z_n})\big) \Big\} \\
&= \Big\{ \big(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\big) \mid \big(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\big) \in \overline{\sigma} \wedge \pi(p) \Vdash_{\mathbb{P},\mathbf{M}} \varphi\big(\tilde{\pi}(\underset{\sim}{z}), \underset{\sim}{z_1}, \ldots, \underset{\sim}{z_n}\big) \Big\} \\
&= \Big\{ \big(\tilde{\pi}(\underset{\sim}{z}), \pi(p)\big) \in \overline{\sigma} \mid \pi(p) \Vdash_{\mathbb{P},\mathbf{M}} \varphi\big(\tilde{\pi}(\underset{\sim}{z}), \underset{\sim}{z_1}, \ldots, \underset{\sim}{z_n}\big) \Big\} \\
&= \Big\{ (\underset{\sim}{z}', p') \in \overline{\sigma} \mid p' \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{z}'), \underset{\sim}{z_1}, \ldots, \underset{\sim}{z_n}) \Big\} \\
&= \underset{\sim}{u}
\end{aligned}$$

We have just shown $\mathscr{G}' \subseteq sym_{\mathscr{G}}(\underset{\sim}{u})$, which proves that $sym_{\mathscr{G}}(\underset{\sim}{u}) \in \mathscr{F}$ and completes the proof that $\underset{\sim}{u} \in \mathbf{HS}_{\mathscr{F}}$.

It remains to show that $(\underset{\sim}{u})_G = u$.

$(\underset{\sim}{\mathbf{u}})_{\mathbf{G}} \subseteq \mathbf{u}$**:** if $(\underset{\sim}{z})_G \in (\underset{\sim}{u})_G$, then there exists $p \in \mathbf{G}$ such that both the following hold:

$$(\underset{\sim}{z}, p) \in \overline{\sigma} \quad \text{and} \quad p \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{z}, \underset{\sim}{z_1}, \ldots, \underset{\sim}{z_n})$$

which yields

$$\Big(\varphi\big(\,(\underset{\sim}{z})_G\,, (\underset{\sim}{z_1})_G\,, \ldots, (\underset{\sim}{z_n})_G\,\big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}.$$

By construction, there exists $q \geqslant p$ such that $(\underset{\sim}{z}, q) \in \sigma$. Now, $q \geqslant p$ and $p \in G$ yields $q \in G$, which gives $(\underset{\sim}{z})_G \in (\sigma)_G$. Putting all this together, we obtain

$$(\underset{\sim}{z})_G \in (\sigma)_G \quad \text{and} \quad \Big(\varphi((\underset{\sim}{z})_G\,, (\underset{\sim}{z_1})_G\,, \ldots, (\underset{\sim}{z_n})_G)\Big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$$

which shows that $(\underset{\sim}{z})_G \in u$ holds.

$(\underset{\sim}{\mathbf{u}})_{\mathbf{G}} \supseteq \mathbf{u}$**:** if $z \in u$, then we have both

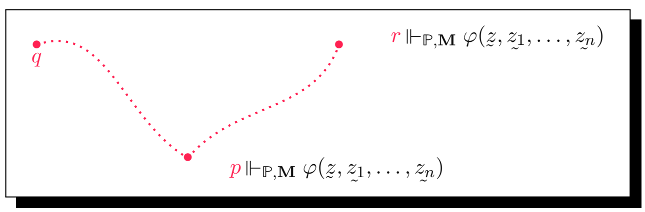
$$z \in (\sigma)_G \text{ and } \left(\varphi(z, (\underset{\sim}{z_1})_G, \dots, (\underset{\sim}{z_n})_G)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}.$$

So, there exists some $q \in G$ and some — necessarily hereditarily symmetric — $\mathbb{P}$-name $\underset{\sim}{z}$ such that $(\underset{\sim}{z}, q) \in \sigma$ and $(\underset{\sim}{z})_G = z$. By construction of $\overline{\sigma}$ we have $(\underset{\sim}{z}, q) \in \overline{\sigma}$. We have both

$$\left(\varphi((\underset{\sim}{z})_G, (\underset{\sim}{z_1})_G, \dots, (\underset{\sim}{z_n})_G)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}} \quad \text{and} \quad q \in G.$$

Therefore, by a direct application of the Truth Lemma, there exists some $r \in G$ such that

$$r \Vdash_{\mathbb{P},\mathbf{M}} \varphi(\underset{\sim}{z}, \underset{\sim}{z_1}, \dots, \underset{\sim}{z_n}).$$




Since $G$ is a filter and $q, r \in G$, there exists also some $p \in G$ which satisfies both $p \leqslant q$ and $p \leqslant r$, and also (since $p \leqslant q$):

$$p \Vdash_{\mathbb{P},\mathbf{M}} \left(\varphi(\underset{\sim}{z}, \underset{\sim}{z_1}, \dots, \underset{\sim}{z_n})\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}},$$

which shows that $(\underset{\sim}{z}, p) \in \underset{\sim}{u}$, and finally $z = (\underset{\sim}{z})_G \in (\underset{\sim}{u})_G$.

**Pairing** If $(\underset{\sim}{x})_G = x$, $(\underset{\sim}{y})_G = y \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$ — i.e., with $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbf{HS}_{\mathscr{F}}$ — then $\{(\underset{\sim}{x}, \mathbb{1}), (\underset{\sim}{y}, \mathbb{1})\} \in \mathbf{HS}_{\mathscr{F}}$ since $sym_{\mathscr{G}}\left(\{(\underset{\sim}{x}, \mathbb{1}), (\underset{\sim}{y}, \mathbb{1})\}\right) \supseteq \overbrace{\underbrace{sym_{\mathscr{G}}(\underset{\sim}{x})}_{\in\mathscr{F}} \cap \underbrace{sym_{\mathscr{G}}(\underset{\sim}{y})}_{\in\mathscr{F}}}^{\in\mathscr{F}}$. We obtain

$$\{(\underset{\sim}{x}, \mathbb{1}), (\underset{\sim}{y}, \mathbb{1})\}_G = \left\{(\underset{\sim}{x})_G, (\underset{\sim}{y})_G\right\} = \{x, y\} \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}.$$

**Union** Let $\sigma \in \mathbf{HS}_{\mathscr{F}}$, to prove that $\bigcup(\sigma)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$, it is enough to show that there exists $\tau \in \mathbf{HS}_{\mathscr{F}}$ such that $\bigcup(\sigma)_G \subseteq (\tau)_G$. We recall that

$$dom(\sigma) = \{\delta \in \mathbf{HS}_{\mathscr{F}} \mid \exists p \in \mathbb{P}\ (\delta, p) \in \sigma\}$$

We set

$$X = \bigcup \{dom(\delta) \mid \delta \in dom(\sigma)\}.$$

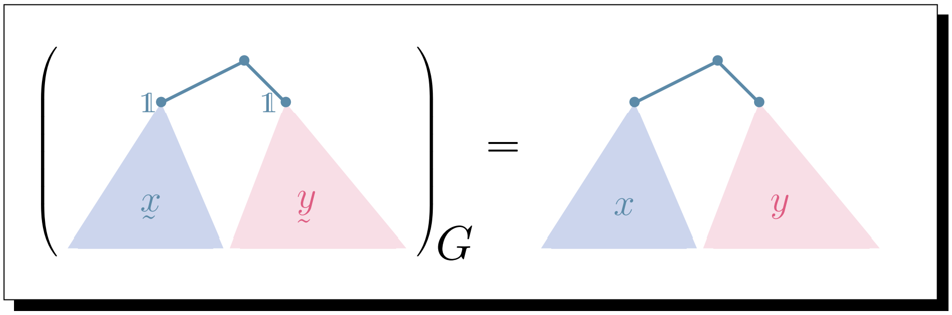



Figure 21.6: An hereditarily symmetric $\mathbb{P}$-name for the pair $\{x, y\}$.

Notice that, for every $\pi \in sym_{\mathscr{G}}(\sigma)$, $\tilde{\pi}(X) = X$ holds since $\sigma \in \mathbf{HS}_{\mathscr{F}}$. Thus, $X \times \{\mathbb{1}\}$ belongs to $\mathbf{HS}_{\mathscr{F}}$ and $(X \times \{\mathbb{1}\})_G \supseteq \bigcup \sigma$.

**Infinity** Since $\omega \in \mathbf{M}$, one has $\check{\omega} \in \mathbf{HS}_{\mathscr{F}}$, hence $\omega = (\check{\omega})_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

**Power Set** Let $\sigma \in \mathbf{HS}_{\mathscr{F}}$, it is enough to show there exists $\tau \in \mathbf{HS}_{\mathscr{F}}$ such that $\mathscr{P}\big((\sigma)_G\big) \cap \widehat{\mathbf{M}[G]}^{\mathscr{F}} \subseteq (\tau)_G$.

Notice first that for every subset $X \subseteq dom(\sigma)$, the $\mathbb{P}$-name $\sigma_X = \big\{(\underset{\sim}{x}, \mathbb{1}) \mid \underset{\sim}{x} \in X\big\}$ belongs to $\mathbf{HS}_{\mathscr{F}}$ since every $\underset{\sim}{x}$ belongs to $\mathbf{HS}_{\mathscr{F}}$ and $sym_{\mathscr{G}}(\sigma_X) \supseteq sym_{\mathscr{G}}(\sigma) \in \mathscr{F}$ holds for every $X \subseteq dom(\sigma)$.

We consider :

$$\tau = \big\{(\sigma_X, \mathbb{1}) \mid X \subseteq dom(\sigma)\big\}$$

Notice that $\tau \in \mathbf{HS}_{\mathscr{F}}$ since every $\sigma_X$ belongs to $\mathbf{HS}_{\mathscr{F}}$ and

$$sym_{\mathscr{G}}(\tau) \supseteq \underbrace{\bigcap_{X \subseteq dom(\sigma)} \underbrace{sym_{\mathscr{G}}(\sigma_X)}_{\supseteq sym_{\mathscr{G}}(\sigma)}}_{\supseteq sym_{\mathscr{G}}(\sigma) \in \mathscr{F}}.$$

Given any $Y \subseteq (\sigma)_G$, it appears that $Y \in (\tau)_G$, since

$$Y = \Big(\sigma_{\{\underset{\sim}{y} \mid (\underset{\sim}{y})_G \in Y\}}\Big)_G$$

and by construction $Y \in (\tau)_G$.

**Foundation** holds in $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ since $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ is transitive and **Foundation** holds in the ground model $\mathbf{M}$ as well as in the generic extension $\mathbf{M}[G]$. (Indeed, the $\in$-well-foundedness of each element of $\mathbf{M}^{\mathbb{P}}$ yields the $\in$-well-foundedness of each element of $\mathbf{HS}_{\mathscr{F}}$, hence each element of $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$.)

**Replacement Schema** for each formula $\varphi(x, y, z_1, \ldots, z_n)$, we want to prove that:

$\forall z_1 \in \mathbf{M}[G] \ldots \forall z_n \in \mathbf{M}[G]$

$$\left(\begin{array}{c} \forall x \in \widehat{\mathbf{M}[G]}^{\mathscr{F}} \ \exists! y \in \widehat{\mathbf{M}[G]}^{\mathscr{F}} \ \left(\varphi(x, y, z_1, \ldots, z_n)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}} \\ \longrightarrow \\ \forall u \in \widehat{\mathbf{M}[G]}^{\mathscr{F}} \ \exists v \in \widehat{\mathbf{M}[G]}^{\mathscr{F}} \ \forall x \in u \ \exists y \in v \ \left(\varphi(x, y, z_1, \ldots, z_n)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}} \end{array}\right).$$

We fix $z_1 = (\underset{\sim}{z}_1)_G, \ldots, z_n = (\underset{\sim}{z}_n)_G$, and $u = (\underset{\sim}{u})_G$.

Inside **M** we define:

$$\begin{array}{llll} \mathbf{F} : \mathit{dom}\,(\underset{\sim}{u}) \times \mathbb{P} \to \mathbf{On} & & \\ (\underset{\sim}{x}, p) & \to & \begin{cases} \text{least } \alpha \text{ s.t. } \exists \underset{\sim}{y} \in \mathbf{HS}_{\mathscr{F}} \cap \mathbf{V}_\alpha \ \ p \Vdash \varphi(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n)^{\mathbf{HS}_{\mathscr{F}}} \\ 0 \text{ otherwise.} \end{cases} \end{array}$$

Since **M** satisfies the instances of the various replacement schema that we need to complete the proof, there exists $\beta \in (\mathbf{On})^{\mathbf{M}}$ such that $\mathbf{F}\left[\mathit{dom}\,(\underset{\sim}{u}) \times \mathbb{P}\right] \subseteq \beta$. We claim that $v = \mathbf{V}_\beta$ works:

Let $(\underset{\sim}{x})_G \in (\underset{\sim}{u})_G$, by hypothesis there exists — a unique — $(\underset{\sim}{y})_G$ such that

$$\left(\varphi\left( (\underset{\sim}{x})_G, (\underset{\sim}{y})_G, (\underset{\sim}{z}_1)_G, \ldots, (\underset{\sim}{z}_n)_G \right)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}.$$

Therefore there exists $p \in G$ such that

$$p \Vdash \varphi(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n)^{\mathbf{HS}_{\mathscr{F}}}$$

It follows that there exists $\underset{\sim}{\hat{y}} \in \mathbf{V}_\beta$ such that

$$p \Vdash \varphi(\underset{\sim}{x}, \underset{\sim}{\hat{y}}, \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_n).$$

The Truth Lemma yields

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \varphi\left( (\underset{\sim}{x})_G, (\underset{\sim}{\hat{y}})_G, (\underset{\sim}{z}_1)_G, \ldots, (\underset{\sim}{z}_n)_G \right)^{\mathbf{HS}_{\mathscr{F}}}$$

so, by unicity, we obtain $(\underset{\sim}{y})_G = (\underset{\sim}{\hat{y}})_G$. Therefore $\mathbf{V}_\beta$ is (one of) the set we were looking for, since it satisfies

$$\left\{ (\underset{\sim}{y})_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}} \mid \exists\, (\underset{\sim}{x})_G \in (\underset{\sim}{u})_G \ \left(\varphi\left( (\underset{\sim}{x})_G, (\underset{\sim}{y})_G, (\underset{\sim}{z}_1)_G, \ldots, (\underset{\sim}{z}_n)_G \right)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}} \right\} \subseteq \mathbf{V}_\beta = \left(\check{\mathbf{V}}_\beta\right)_G.$$

□ 373

**Theorem 375.** *Let* $\mathbf{M}$ *be any c.t.m. of "***ZFC***",* $(\mathbb{P}, \leqslant, \mathbb{1})$ *any partial order over* $\mathbf{M}$*, and* $\mathcal{G}$ *any subgroup of the group of automorphisms of* $\mathbb{P}$*, and* $\mathscr{F}$ *any normal filter on* $\mathcal{G}$*.*

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \text{ is a countable transitive model of "}\mathbf{ZF}\text{"}$$

*Proof of Theorem* 375: Immediate from Lemma 373. □ 375

# Chapter 22

# Some Applications of the Symmetric Submodel Technique

## 22.1 Forcing $\mathbb{R}$ as a Countable Union of Countable Sets

The following blatantly contradicts the countable axiom of choice (**CC**) which is precisely the statement that we used in order to prove Lemma 104 which established that any countable union of countable sets is countable. Keep in mind that, with the help of Cantor's Theorem (see page 61), we proved that the set of reals is uncountable. So, we will have an uncountable set (**R**) which is a countable union of countable sets.

**Theorem 376** (Feferman & Lévy)**.**

$$cons\big(\mathbf{ZF}\big) \Longrightarrow cons\big(\mathbf{ZF} + \text{“}\mathbb{R} \textit{ is a countable union of countable sets”}\big).$$

*Proof of Theorem 376:* Instead of showing that there exists some model that satisfies

“ $\mathbb{R}$ is a countable union of countable sets”

we will show that there exists some model that satisfies the equivalent[1] statement

“$\mathcal{P}(\omega)$ is a countable union of countable sets”.

We start with **M** any *c.t.m.* of “ $\mathbf{ZFC} + \forall n < \omega \;\; 2^{\aleph_n} = \aleph_{n+1}$ ” and force with the poset $(\mathbb{P}_{\text{Levy}}, \leqslant, \mathbb{1})$ defined inside **M** by

$$\mathbb{P}_{\text{Levy}} = \Big\{ f \;:\; \omega \times \omega \to \aleph_\omega \mid dom(f) \text{ is finite} \wedge \forall n \in \omega \; \forall m \in \omega \;\; f(n,m) \in \aleph_n \Big\}$$

[1]See Lemma 362 on page 313 for the equivalence between the two statements.

and $f \leqslant g \iff f \supseteq g$, so that $\mathbb{1} = \varnothing$.
We let $G$ be $\mathbb{P}_{\text{Levy}}$-generic over $\mathbf{M}$ and construct a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ of $\mathbf{M}[G]$. For this, we consider the group $\mathscr{G}_{per.}$ of the permutations of $\omega \times \omega$ which do not move the first coordinate. i.e.,

$$\mathscr{G}_{per.} = \Big\{\pi : \omega \times \omega \overset{bij.}{\longleftrightarrow} \omega \times \omega \mid \forall i, j, k, n < \omega \ \big(\pi(k,i) = (n,j) \longrightarrow k = n\big)\Big\}.$$

For each integer $n$, any such permutation $\pi$ induces a permutation $\pi_n$ of $\omega$ defined by

$$\forall i \quad < \omega \ \pi(n,i) = \big(n, \pi_n(i)\big).$$

Every permutation $\pi \in \mathscr{G}_{per.}$ induces an automorphism $\pi : \mathbb{P} \xrightarrow{aut.} \mathbb{P}$ by having for each forcing condition $p \in \mathbb{P}$:

$$dom\,(\pi(p)) = \pi\,[dom\,(\pi)] = \{(n, \pi_n(i)) \mid (n,i) \in dom\,(p)\,\}$$

and

$$\pi(p)\big(n, \pi_n(i)\big) = p(n,i).$$

Or, to say it directly,

$$\pi(p) = \Big\{\big(n, \pi_n(i), \alpha\big) \mid (n,i,\alpha) \in p\Big\}.$$

We let $\mathscr{G}_{aut.}$ be the group of automorphisms of $\mathbb{P}$ induced by the group of permutations $\mathscr{G}_{per.}$:

$$\mathscr{G}_{aut.} = \big\{\pi \mid \pi \in \mathscr{G}_{per.}\big\}.$$

We check that $\mathscr{G}_{aut.}$ is closed under products and inverses.

- Closed under $(\rho, \pi) \mapsto \rho \circ \pi$: Given any $\pi, \rho \in \mathscr{G}_{aut.}$, we have

$$\begin{aligned}\rho \circ \pi(p) &= \rho\Big(\Big\{\big(n, \pi_n(i), \alpha\big) \mid (n,i,\alpha) \in p\Big\}\Big)\\ &= \Big\{\Big(n, \rho_n\big(\pi_n(i)\big), \alpha\Big) \mid (n,i,\alpha) \in p\Big\}\\ &= \Big\{\big(n, \rho_n{\circ}\pi_n(i), \alpha\big) \mid (n,i,\alpha) \in p\Big\}.\end{aligned}$$

  Since both $\rho$ and $\pi$ belong to $\mathscr{G}_{per.}$, it is immediate to see that

$$\forall i \quad < \omega \ \rho{\circ}\pi(n,i) = \big(n, \rho_n{\circ}\pi_n(i)\big),$$

  hence $\rho{\circ}\pi$ belongs to $\mathscr{G}_{per.}$; so that $\rho \circ \pi \in \mathscr{G}_{aut.}$.

- Closed under $\pi \mapsto \pi^{-1}$: it is immediate to see that if $\pi$ belongs to $\mathscr{G}_{per.}$, so does $\pi^{-1}$ (indeed, $\mathscr{G}_{per.}$ is a subgroup of the group of permutations on $\omega \times \omega$) and

$$\pi^{-1}(p) = \Big\{\big(n, \pi_n^{-1}(i), \alpha\big) \mid (n,i,\alpha) \in p\Big\}.$$

We let $\mathscr{H}_k$ be the subgroup of $\mathscr{G}_{aut.}$ formed of all automorphisms $\pi$ such that for every integer $n < k$, the permutation $\pi_n$ is the identity. i.e.,

$$\mathscr{H}_k = \Big\{\pi \in \mathscr{G}_{aut.} \mid \forall n < k \;\; \forall i < \omega \;\; \pi(n,i) = (n,i)\Big\}.$$

We also let $\mathscr{F}$ be the filter:

$$\mathscr{F} = \Big\{\mathscr{S} \subseteq \mathscr{G}_{aut.} \mid \text{“ } \mathscr{S} \text{ is a subgroup ”} \;\wedge\; \exists n < \omega \;\; \mathscr{H}_n \subseteq \mathscr{S}\Big\}.$$

$\mathscr{F}$ is a normal filter on $\mathscr{G}_{aut.}$ since $\mathscr{F}$ is a set of subgroups of $\mathscr{G}_{aut.}$ such that for all subgroups $\mathscr{S}$, $\mathscr{T}$ of $\mathscr{G}_{aut.}$:

(1) $\mathscr{G}_{aut.} \in \mathscr{F}$ holds since $\mathscr{G}_{aut.} = \mathscr{H}_0$;

(2) if $\mathscr{S} \in \mathscr{F}$ and $\mathscr{S} \subseteq \mathscr{T}$ for $\mathscr{T}$ some subgroup of $\mathscr{G}_{aut.}$, then there exists $n < \omega$ such that $\mathscr{H}_n \subseteq \mathscr{S} \subseteq \mathscr{T}$, hence $\mathscr{T} \in \mathscr{F}$;

(3) if $\mathscr{S} \in \mathscr{F}$ and $\mathscr{T} \in \mathscr{F}$, then there exist $n, k < \omega$ $\mathscr{H}_n \subseteq \mathscr{S}$ and $\mathscr{H}_k \subseteq \mathscr{T}$, hence $\mathscr{H}_{\sup\{n,k\}} \subseteq \mathscr{S} \cap \mathscr{T}$ which gives $\mathscr{S} \cap \mathscr{T} \in \mathscr{F}$;

(4) if $\mathscr{S} \in \mathscr{F}$, then there exists $n < \omega$ such that $\mathscr{H}_n \subseteq \mathscr{S}$ and for all $\pi \in \mathscr{G}_{aut.}$, one has $\pi \circ \mathscr{H}_n \circ \pi^{-1} = \mathscr{H}_n$, so that $\mathscr{H}_n = \pi \circ \mathscr{H}_n \circ \pi^{-1} \subseteq \pi \circ \mathscr{S} \circ \pi^{-1}$ which shows $\pi \circ \mathscr{S} \circ \pi^{-1} \in \mathscr{F}$.

We then construct a canonical symmetric $\mathbb{P}$-name for each real[2] (subset of the integers).

**Claim 377.** *Let* $\big(x \in \mathcal{P}(\omega)\big)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$ *and* $\underset{\sim}{x}$ *be any* $\mathbb{P}$*-name in* $\mathbf{HS}_{\mathscr{F}}$ *for* $x$*. Then,*

$$\underset{\sim}{x} = \Big\{(\underset{\sim}{k}, p) \in \mathbf{HS}_{\mathscr{F}} \times \mathbb{P} \mid \exists (\underset{\sim}{k}, r) \in \underset{\sim}{x} \;\; \big(p \leqslant r \;\wedge\; p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \check{\omega}\big)\Big\}$$

*is another* $\mathbb{P}$*-name in* $\mathbf{HS}_{\mathscr{F}}$ *which satisfies*

$$(\underset{\sim}{x})_G = (\underset{\sim}{x})_G = x.$$

*Proof of Claim 377:* We show $\underset{\sim}{x} \in \mathbf{HS}_{\mathscr{F}}$. It is enough to show that $sym_{\mathscr{G}}(\underset{\sim}{x}) \subseteq sym_{\mathscr{G}}(\underset{\sim}{x})$. For this, take any $\pi \in sym_{\mathscr{G}}(\underset{\sim}{x})$. We have

$$\pi \in sym_{\mathscr{G}}(\underset{\sim}{x}) \iff \forall (\underset{\sim}{k}, r) \in \mathbf{HS}_{\mathscr{F}} \times \mathbb{P} \;\; \Big((\underset{\sim}{k}, r) \in \underset{\sim}{x} \longleftrightarrow \big(\tilde{\pi}(\underset{\sim}{k}), \pi(r)\big) \in \underset{\sim}{x}\Big)$$

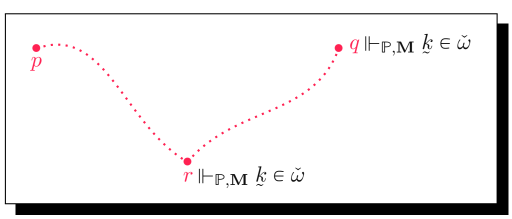
[2]As is common among set theorists, we freely use the word “real” to designate any subset of $\omega$.

Now, for all $(\underset{\sim}{k}, p) \in \mathbf{HS}_{\mathscr{F}} \times \mathbb{P}$ we have

$$\begin{aligned}
(\underset{\sim}{k}, p) \in \underset{\sim}{x} &\iff \exists (\underset{\sim}{k}, r) \in \underset{\sim}{x} \ \big(r \geqslant p \ \wedge \ p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \check{\omega}\big) \\
&\iff \exists \big(\tilde{\pi}\,(\underset{\sim}{k})\,, \pi(r)\big) \in \tilde{\pi}\,(\underset{\sim}{x}) \ \ \big(\pi(r) \geqslant \pi(p) \ \wedge \ \pi(p) \Vdash_{\mathbb{P},\mathbf{M}} \tilde{\pi}\,(\underset{\sim}{k}) \in \tilde{\pi}\,(\check{\omega})\,\big) \\
&\iff \exists \big(\tilde{\pi}\,(\underset{\sim}{k})\,, \pi(r)\big) \in \underset{\sim}{x} \ \ \big(\pi(r) \geqslant \pi(p) \ \wedge \ \pi(p) \Vdash_{\mathbb{P},\mathbf{M}} \tilde{\pi}\,(\underset{\sim}{k}) \in \check{\omega}\big) \\
&\iff \big(\tilde{\pi}\,(\underset{\sim}{k})\,, \pi(p)\big) \in \underset{\sim}{x},
\end{aligned}$$

which shows that $\pi \in sym_{\mathscr{G}}\,(\underset{\sim}{x})$.

We show $(\underset{\sim}{x})_G = (\underset{\sim}{x})_G$

- $(\underset{\sim}{x})_G \supseteq (\underset{\sim}{x})_G$: if $(\underset{\sim}{k})_G \in (\underset{\sim}{x})_G$, then there exist $p \in G$ and $(\underset{\sim}{k}, p) \in \underset{\sim}{x}$. By construction, there exists $r \geqslant p$ such that $(\underset{\sim}{k}, r) \in \underset{\sim}{x}$. Since $p \in G$ and $p \leqslant r$, it follows that $r \in G$, hence $(\underset{\sim}{k})_G \in (\underset{\sim}{x})_G$.
- $(\underset{\sim}{x})_G \subseteq (\underset{\sim}{x})_G$: if $(\underset{\sim}{k})_G \in (\underset{\sim}{x})_G$, then there exists $(\underset{\sim}{k}, p) \in \underset{\sim}{x}$ with $p \in G$. Now, since the following holds:
$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models (\underset{\sim}{k})_G \in x \subseteq \omega$$
also holds
$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models (\underset{\sim}{k})_G \in \omega$$
or to say it differently,
$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models (\underset{\sim}{k})_G \in (\check{\omega})_G\,,$$
by the Truth Lemma, there exists some $q \in G$ which satisfies
$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \check{\omega}.$$
Since both $p, q \in G$, there also exists some $r \in G$, $r \leqslant p, q$ which necessarily satisfies
$$r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \check{\omega}.$$


By construction we have $(\underset{\sim}{k}, r) \in \underset{\sim}{x}$ and since $r \in G$, we obtain $(\underset{\sim}{k})_G \in (\underset{\sim}{x})_G$.

$\square$ 377

So, Without loss of generality we may ask that

$$\underset{\sim}{x} \subseteq \Big\{(\underset{\sim}{k}, p) \in \mathbf{HS}_{\mathscr{F}} \times \mathbb{P} \mid p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \check{\omega}\Big\}.$$

Notice also that because for any filter $J$ which is $\mathbb{P}$-generic over $\mathbf{M}$ and contains $p$, we have

$$\mathbf{M}[J] \models (\underset{\sim}{k})_J \in (\underset{\sim}{x})_J\,,$$

we also have for any $(\underset{\sim}{k}, p) \in \underset{\sim}{x}$,

$$p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \underset{\sim}{x}.$$

So, by combining both results mentioned above, we have

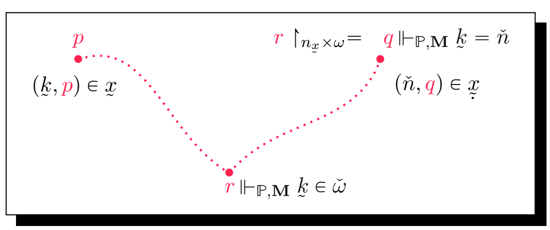
$$\forall(\underset{\sim}{k}, p) \in \underset{\sim}{x} \quad p \Vdash_{\mathbb{P},\mathbf{M}} \big(\underset{\sim}{k} \in \check{\omega} \wedge \underset{\sim}{k} \in \underset{\sim}{x}\big).$$

Now, since $\underset{\sim}{x} \in \mathbf{HS}_{\mathscr{F}}$, there exists some integer $n_{\underset{\sim}{x}}$ such that $\mathscr{H}_{n_{\underset{\sim}{x}}} \subseteq sym_{\mathscr{G}_{aut.}}(\underset{\sim}{x})$. So, given any automorphism $\pi \in \mathscr{H}_{n_{\underset{\sim}{x}}}$, since $\pi \in sym_{\mathscr{G}_{aut.}}(\underset{\sim}{x})$ we have:

$$\begin{aligned}\tilde{\pi}(\underset{\sim}{x}) &= \Big\{(\tilde{\pi}(\underset{\sim}{k}), \pi(p)) \mid (\underset{\sim}{k}, p) \in \underset{\sim}{x}\Big\}\\ &= \underset{\sim}{x}\end{aligned}$$

We define another *canonical* $\mathbb{P}$-name $\underset{\dot{\sim}}{x}$ for $x$ by:

$$\underset{\dot{\sim}}{x} = \Big\{(\check{n}, q) \mid \exists(\underset{\sim}{k}, p) \in \underset{\sim}{x}\ \ \exists r \in \mathbb{P}\ \ \big(r \leqslant p \wedge q = r \restriction_{n_{\underset{\sim}{x}} \times \omega} \wedge r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n}\big)\Big\}.$$




We first check that $\underset{\dot{\sim}}{x} \in \mathbf{HS}_{\mathscr{F}}$. Notice first that given any $\pi \in \mathscr{H}_{n_{\underset{\sim}{x}}}$ and $(\check{n}, q) \in \underset{\dot{\sim}}{x}$, we have

$$\tilde{\pi}(\check{n}, q) = \big(\check{n}, \pi(q)\big).$$

Since $dom(q) \subseteq n_{\underset{\sim}{x}} \times \omega$ and $\pi \in \mathscr{H}_{n_{\underset{\sim}{x}}}$, we also have $\pi(q) = q$. So that we obtain

$$\tilde{\pi}(\check{n}, q) = (\check{n}, q),$$

which yields

$$\tilde{\pi}(\underset{\dot{\sim}}{x}) = \underset{\dot{\sim}}{x}.$$

We have shown that $\underset{\dot\sim}{x}$ is symmetric. Since every $\mathbb{P}$-name $\sigma \in dom(\underset{\dot\sim}{x})$ is of the form $\sigma = \check{n}$, it follows that $\sigma \in \mathbf{HS}_{\mathscr{F}}$, which shows that $\underset{\dot\sim}{x}$ belongs to $\mathbf{HS}_{\mathscr{F}}$.

Now that we know that both $\underset{\sim}{x}$ and $\underset{\dot\sim}{x}$ belong to $\mathbf{HS}_{\mathscr{F}}$, we need to show that they both give birth to the same set $x$. Namely,

**Claim 378.**

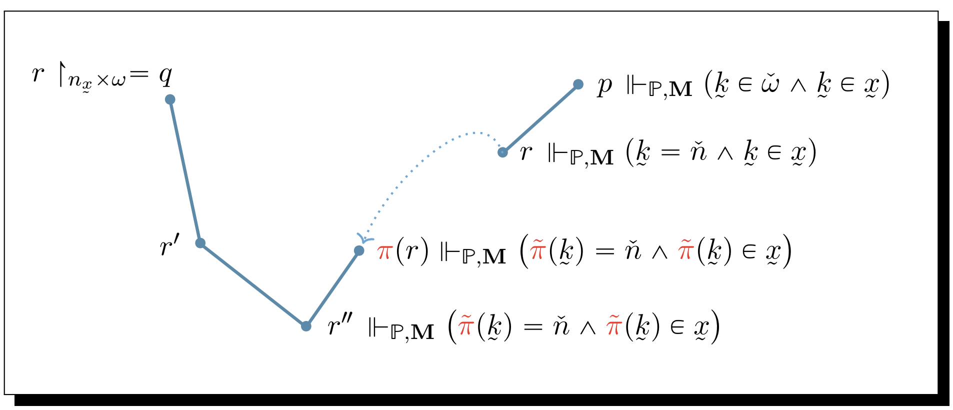
$$(\underset{\dot\sim}{x})_G = (\underset{\sim}{x})_G .$$

*Proof of Claim 378:*

$\boxed{(\underset{\dot\sim}{x})_G \subseteq (\underset{\sim}{x})_G}$ : Let $n \in (\underset{\dot\sim}{x})_G$, and consider any $q \in G$ such that $(\check{n}, q) \in \underset{\dot\sim}{x}$. So, there exists $(\underset{\sim}{k}, p) \in \underset{\sim}{x}$ and $r \leqslant p$ such that

$$r \leqslant p \text{ and } r \restriction_{n_{\underset{\sim}{x}} \times \omega} = q \text{ and } r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n}.$$

Now, for every condition $r' \leqslant q$, we may find a condition $r''$ and an automorphism $\pi \in \mathscr{H}_{n_{\underset{\sim}{x}}}$ such that the picture below holds.




i.e., both $r'' \leqslant r'$ and $r'' \leqslant \pi(r)$ hold. Notice first the following:

- $p \Vdash_{\mathbb{P},\mathbf{M}} (\underset{\sim}{k} \in \check{\omega} \wedge \underset{\sim}{k} \in \underset{\sim}{x})$ holds because $(\underset{\sim}{k}, p) \in \underset{\sim}{x}$ (this was shown on page 339).
- $r \Vdash_{\mathbb{P},\mathbf{M}} (\underset{\sim}{k} = \check{n} \wedge \underset{\sim}{k} \in \underset{\sim}{x})$ holds since we have both
  - $r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n}$, by assumption on $\underset{\dot\sim}{x}$, and
  - $r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \underset{\sim}{x}$ since $r \leqslant p$ and $p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \underset{\sim}{x}$ holds.
- $\pi(r) \Vdash_{\mathbb{P},\mathbf{M}} \left(\tilde{\pi}(\underset{\sim}{k}) = \check{n} \wedge \tilde{\pi}(\underset{\sim}{k}) \in \underset{\sim}{x}\right)$ since $\pi(r) \Vdash_{\mathbb{P},\mathbf{M}} \left(\tilde{\pi}(\underset{\sim}{k}) = \check{n} \wedge \tilde{\pi}(\underset{\sim}{k}) \in \tilde{\pi}(\underset{\sim}{x})\right)$ and $\tilde{\pi}(\underset{\sim}{x}) = \underset{\sim}{x}$.
- $q = r \restriction_{n_{\underset{\sim}{x}} \times \omega} = \pi(r) \restriction_{n_{\underset{\sim}{x}} \times \omega}$

So, in particular since $q = \pi(r) \upharpoonright_{n_{\underset{\sim}{x}} \times \omega}$ and $r' \leqslant q$ holds, we necessarily have $\pi(r)$, $r$ and $r'$ agree on $dom(q)$, which is a finite subset of $n_{\underset{\sim}{x}} \times \omega$. i.e.,

$$\pi(r) \upharpoonright_{n_{\underset{\sim}{x}} \times \omega} = r \upharpoonright_{n_{\underset{\sim}{x}} \times \omega} = q = r' \upharpoonright_{dom(q)} .$$

So, for each $n \geqslant n_{\underset{\sim}{x}}$ we consider the following sets:

- $U_n = \big\{ i \in \omega \big( \mid (n,i) \in dom(r) \cap dom(r') \quad \wedge \quad r(n,i) \neq r'(n,i) \big) \big\}$
- $V_n \subseteq \big\{ i \in \omega \mid (n,i) \notin dom(r) \cup dom(r') \big\}$ is any set of the same cardinality as $U_n$.
- $f_n : U_n \overset{bij.}{\longleftrightarrow} V_n$, any permutation between $U_n$ and $V_n$.

Let $\pi \in \mathscr{G}_{per.}$ be such that

(1) for every integer $n < n_{\underset{\sim}{x}}$, the permutation $\pi_n$ is the identity.

(2) for every integer $n \geqslant n_{\underset{\sim}{x}}$, the permutation $\pi_n$ satisfies:

  (a) If $(n,i) \notin U_n \cup V_n$, then $\pi_n(i) = i$;

  (b) If $(n,i) \in U_n$, then $\pi_n(i) = f_n(i)$;

  (c) If $(n,i) \in V_n$, then $\pi_n(i) = f_n^{-1}(i)$.

The family $(\pi_n)_{n<\omega}$ induces a permutation $\pi \in \mathscr{G}_{per.}$, which itself induces an automorphism $\pi \in \mathscr{G}_{aut.}$.

We notice that $\pi$ belongs to $\mathscr{H}_{n_{\underset{\sim}{x}}}$ and also $\pi(r)$ agrees with $r'$ on their common domain. Therefore, there exists some $r'' \in \mathbb{P}_{\text{Levy}}$ such that both $r'' \leqslant r'$ and $r'' \leqslant \pi(r)$ hold. Such a forcing condition $r''$ necessarily satisfies

$$r'' \Vdash_{\mathbb{P},\mathbf{M}} \big( \tilde{\pi}(\underset{\sim}{k}) = \check{n} \wedge \tilde{\pi}(\underset{\sim}{k}) \in \underset{\sim}{x} \big),$$

since we have $r'' \leqslant \pi(r)$ and this statement is already forced by $\pi(r)$. So, given any $r' \leqslant q$, we have found $r'' \leqslant r'$ such that $r'' \Vdash_{\mathbb{P},\mathbf{M}} \check{n} \in \underset{\sim}{x}$ holds. This shows that the set $\mathscr{D} = \big\{ r'' \in \mathbb{P} \mid r'' \Vdash_{\mathbb{P},\mathbf{M}} \check{n} \in \underset{\sim}{x} \big\}$ is *dense below* $q$. So, there exists some $r'' \in \mathscr{D} \cap G$, and by the Truth Lemma, we finally obtain $\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models n \in (\underset{\sim}{x})_G$.

$\boxed{(\underset{\sim}{x})_G \subseteq (\underset{\sim}{\dot{x}})_G}$ : If $k \in (\underset{\sim}{x})_G$, then there exists $(\underset{\sim}{k}, p) \in \underset{\sim}{x}$ such that both $p \in G$ and $p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} \in \check{\omega}$.

Since there exists some integer $n$ such that

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \big( k \in \omega \wedge k \in x \wedge n = k \big),$$

by the Truth Lemma, there exists some $q \in G$ such that

$$q \Vdash_{\mathbb{P},\mathbf{M}} \big( \underset{\sim}{k} \in \check{\omega} \wedge \underset{\sim}{k} \in \underset{\sim}{x} \wedge \check{n} = \underset{\sim}{k} \big);$$

hence, there exists some $r \leqslant p$ and $r \leqslant q$ with $r \in G$ which satisfies

$$r \Vdash_{\mathbb{P},\mathbf{M}} \left(\underset{\sim}{k} \in \check{\omega} \wedge \underset{\sim}{k} \in \underset{\sim}{x} \wedge \check{n} = \underset{\sim}{k}\right).$$

This yields $(\check{n}, r \restriction_{n_{\underset{\sim}{x}} \times \omega}) \in \underset{\sim}{x}$ because

$$\underset{\sim}{x} = \left\{(\check{n}, q) \mid \exists(\underset{\sim}{k}, p) \in \underset{\sim}{x} \ \exists r \in \mathbb{P} \ \left(r \leqslant p \wedge q = r \restriction_{n_{\underset{\sim}{x}} \times \omega} \wedge r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n}\right)\right\}.$$

Finally, from $r \leqslant r \restriction_{n_{\underset{\sim}{x}}}$ and $r \in G$, we obtain $r \restriction_{n_{\underset{\sim}{x}}} \in G$, which gives both $k = (\underset{\sim}{k})_G$ and $k \in \left(\underset{\sim}{x}\right)_G$.

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Now that we have these "*canonical*" $\mathbb{P}$-names $\underset{\sim}{x}$ for the reals that belong to $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, we shift our attention to some $\mathbb{P}$-names for which we will show that

(1) they give countable sets of reals and (2) their union gives the entire set of reals.

For each integer $n$, we set

$$\underset{\sim}{R_n} = D_n \times \{\mathbb{1}\} = \left\{(\underset{\sim}{x}, \mathbb{1}) \mid \text{“}\underset{\sim}{x} \text{ is a canonical } \mathbb{P}\text{-name for a real”} \ \wedge \ \underset{\sim}{x} \in D_n\right\}$$

where

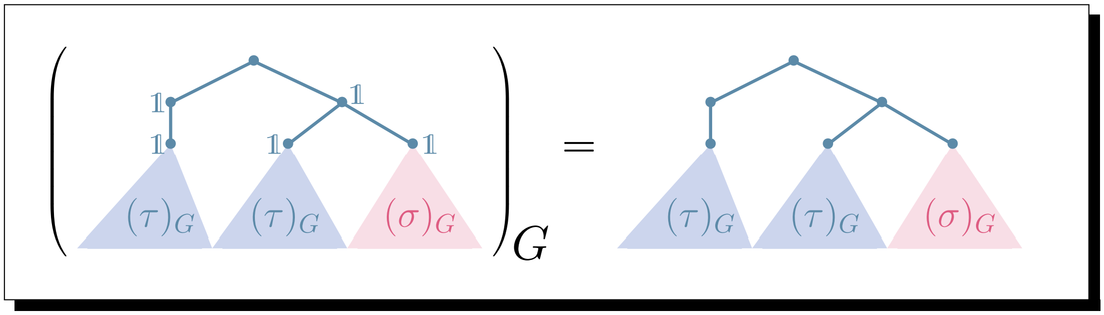
$$D_n = \left\{\underset{\sim}{x} \in \mathbb{P} \mid \forall m \in \omega \ \forall p \in \mathbb{P} \ \forall (i,j) \in dom(p) \ \left((\check{m}, p) \in \underset{\sim}{x} \longrightarrow i < n\right)\right\}.$$

Now, for every $\pi \in \mathscr{H}_n$, we have $\tilde{\pi}(\underset{\sim}{R_n}) = \underset{\sim}{R_n}$, hence $\mathscr{H}_n \subseteq sym_{\mathscr{G}_{aut.}}\left(\underset{\sim}{R_n}\right) \in \mathscr{F}$. Since every $\underset{\sim}{x} \in dom\left(\underset{\sim}{R_n}\right)$ belongs to $\mathbf{HS}_{\mathscr{F}}$, it follows that $\underset{\sim}{R_n} \in \mathbf{HS}_{\mathscr{F}}$, hence

$$\left(\underset{\sim}{R_n}\right)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}.$$

We recall that $couple : \mathbf{M}^{\mathbb{P}} \times \mathbf{M}^{\mathbb{P}} \rightarrow \mathbf{M}^{\mathbb{P}}$ was introduced in Example 309 so that given any $\tau, \sigma \in \mathbf{M}^{\mathbb{P}}$, and any $G$ $\mathbb{P}$-generic over $\mathbf{M}$ one has $\left(couple(\tau,\sigma)\right)_G = \left((\tau)_G, (\sigma)_G\right)$. This is

$$couple(\tau,\sigma) = \left\{\left(\left\{(\tau,\mathbb{1})\right\},\mathbb{1}\right), \left(\left\{(\tau,\mathbb{1}),(\sigma,\mathbb{1})\right\},\mathbb{1}\right)\right\}.$$




We set $R_n = \left(\underset{\sim}{R}_n\right)_G = \left\{ (\underset{\approx}{x})_G \mid \underset{\approx}{x} \in D_n \right\}$ and define some $\mathbb{P}$-name for the function that maps $n$ to $R_n$:

$$\underset{\sim}{F} = \left\{ \left( couple(\check{n}, \underset{\sim}{R}_n), \mathbb{1} \right) \mid n \in \omega \right\}$$

By construction, $\underset{\sim}{F} \in \mathbf{HS}_{\mathscr{F}}$, therefore, the function

$$\begin{aligned} (\underset{\sim}{F})_G : \omega &\to \widehat{\mathbf{M}[G]}^{\mathscr{F}} \\ n \mapsto &\quad R_n \end{aligned}$$

belongs to $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$. Notice that, for any real $x \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$, by the very construction of $\underset{\approx}{x}$, we have

$$\begin{aligned} \underset{\approx}{x} &= \left\{ (\check{n}, q) \mid \exists (\underset{\sim}{k}, p) \in \underset{\approx}{x} \ \exists r \in \mathbb{P} \ \left( r \leqslant p \wedge q = r \restriction_{n_{\underset{\approx}{x}} \times \omega} \wedge r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n} \right) \right\} \\ &\subseteq \left\{ (\check{n}, q) \mid dom\,(q) \subseteq n_{\underset{\approx}{x}} \times \omega \right\} \end{aligned}$$

so that $(\underset{\approx}{x}, \mathbb{1}) \in \underset{\sim}{R}_{n_{\underset{\approx}{x}}}$, and finally $x \in R_{n_{\underset{\approx}{x}}}$. This shows that every real that belongs to $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ belongs to some $R_n$, so that

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \mathcal{P}(\omega) = \bigcup \{ R_n \mid n \in \omega \},$$

or equivalently

$$\left( \mathcal{P}(\omega) = \bigcup \{ R_n \mid n \in \omega \} \right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}.$$

So, it just remains to prove that, for each $n$,

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \text{“} R_n \text{ is countable”}.$$

(1) We first compute the size of $\underset{\sim}{R}_n$ inside the ground model. Since

$$\underset{\sim}{R}_n = \left\{ (\underset{\approx}{x}, \mathbb{1}) \mid \text{“}\underset{\approx}{x} \text{ is a canonical } \mathbb{P}\text{-name for a real”} \ \wedge \ \underset{\approx}{x} \in D_n \right\}$$

it is enough to count how many canonical $\mathbb{P}$-names of the form $\underset{\approx}{x}$ there are inside $D_n$.

$$\underset{\sim}{x} = \Big\{(\check{n}, q) \mid \exists(\underset{\sim}{k}, p) \in \underset{\sim}{x}\, \exists r \in \mathbb{P}\, \big(r \leqslant p \wedge q = r \restriction_{n_{\underset{\sim}{x}} \times \omega} \wedge r \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{k} = \check{n}\big)\Big\}.$$

By construction, if $(\check{n}, q) \in \underset{\sim}{x}$ and $(\underset{\sim}{x}, \mathbb{1}) \in \underset{\sim}{R_n}$, then $dom(q)$ is finite and $q : n \times \omega \to \aleph_n$. So, inside $\mathbf{M}$ there are at most $\aleph_n$ many such forcing conditions $q$ and $\aleph_0$ many canonical $\mathbb{P}_{\text{Levy}}$-names of the form $\check{n}$. So, there are $\aleph_n \cdot \aleph_0 = \aleph_n$ many $(\check{n}, q)$, which yields $2^{\aleph_n}$ many $\underset{\sim}{x}$. Since $\mathbf{M}$ satisfies $\forall k < \omega \;\; 2^{\aleph_k} = \aleph_{k+1}$, we obtain

$$\Big(|\underset{\sim}{R_n}| \leqslant 2^{\aleph_n} = \aleph_{n+1}\Big)^{\mathbf{M}}.$$

(2) We define, for each integer $n$, a $\mathbb{P}$-name $\underset{\sim}{f_n}$ that we will show gives rise to some mapping $f_n : \omega \xrightarrow{onto} R_n$:

$$\underset{\sim}{f_n} = \Big\{\Big(couple(\check{k}, \check{\alpha}), p\Big) \mid p \in \mathbb{P} \wedge dom(p) \subseteq (n+1) \times \omega \wedge p(n,k) = \alpha\Big\}.$$

(a) By construction, $sym_{\mathscr{G}_{aut.}}\left(\underset{\sim}{f_n}\right) \supseteq \mathscr{H}_{n+1}$ and $couple(\check{k}, \check{\alpha}) \in \mathbf{HS}_{\mathscr{F}}$. Therefore, $\underset{\sim}{f_n} \in \mathbf{HS}_{\mathscr{F}}$, hence $f_n = \big(\underset{\sim}{f_n}\big)_G \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

(b) We now show that, inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, we have $\big(\underset{\sim}{f_n}\big)_G = f_n : \omega \xrightarrow{onto} (\aleph_n)^{\mathbf{M}}$:

**$f_n$ is a function from $\omega$ to $(\aleph_n)^{\mathbf{M}}$:**

- $f_n \subseteq \omega \times (\aleph_n)^{\mathbf{M}}$ holds by construction.
- If both $(k, \alpha)$ and $(k, \beta)$ belong to $f_n$, then there exist $p_\alpha, p_\beta \in G$ with $p_\alpha(n,k) = \alpha$ and $p_\beta(n,k) = \beta$. Since both $p_\alpha$ and $p_\beta$ belong to $G$, they agree on their common domain, hence $p_\alpha(n,k) = p_\beta(n,k)$. i.e., $\alpha = \beta$.

**$f_n$ is onto:** Given any $\alpha \in (\aleph_n)^{\mathbf{M}}$, the set

$$\Big\{p \in \mathbb{P} \mid dom(p) \subseteq (n+1) \times \omega \wedge \exists k \in \omega\, \big((n,k) \in dom(p) \wedge p(n,k) = \alpha\big)\Big\}.$$

is dense which shows that there exists some integer $k$ such that $f_n(k) = \alpha$.

Inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, since $\omega \subseteq (\aleph_n)^{\mathbf{M}}$, we have $\omega \overset{1-1}{\lesssim} (\aleph_n)^{\mathbf{M}}$. Now, define $g_n : (\aleph_n)^{\mathbf{M}} \xrightarrow{1-1} \omega$ by

$$g_n(\alpha) = \bigcap \{k \in \omega \mid f_n(k) = \alpha\},$$

or, in other words, $g_n(\alpha)$ is the least integer $k$ such that $f_n(k) = \alpha$ (notice that such an integer $k$ always exists because $f_n$ is onto). So, we have shown

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models (\aleph_n)^{\mathbf{M}} \overset{1-1}{\lesssim} \omega.$$

Notice that by Cantor-Schröder-Bernstein Theorem (see page 57), we obtain $\omega \simeq (\aleph_n)^{\mathbf{M}}$.

It remains to show that $\left(R_n \overset{1-1}{\lesssim} (\aleph_n)^{\mathbf{M}}\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$. For this purpose, we notice that since

$$\left(|\underset{\sim}{R_n}| \leqslant 2^{\aleph_n} = \aleph_{n+1}\right)^{\mathbf{M}},$$

so there exists <u>inside **M**</u> some mapping

$$g_n : dom\left(\underset{\sim}{R_n}\right) \xrightarrow{1-1} (\aleph_{n+1})^{\mathbf{M}}$$

which maps *injectively* each $\underset{\sim}{x} \in dom\left(\underset{\sim}{R_n}\right)$ to some ordinal $\alpha \in (\aleph_{n+1})^{\mathbf{M}}$. We consider

$$\underset{\sim}{B} = \left\{\left(couple(\underset{\sim}{x}, \check{\alpha}), \mathbb{1}\right) \mid \alpha = g_n(\underset{\sim}{x}).\right\}$$

We set $(\underset{\sim}{B})_G = B$. Since every $\underset{\sim}{x}$ belongs to $\mathbf{HS}_{\mathscr{F}}$, we notice $\underset{\sim}{B}$ also belongs to $\mathbf{HS}_{\mathscr{F}}$, hence $B \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$ and

$$\begin{aligned} B &= \left\{\left(couple(\underset{\sim}{x}, \check{\alpha})\right)_G \mid \left(\left(\underset{\sim}{x}, \mathbb{1}\right) \in \underset{\sim}{R_n} \wedge g_n(\underset{\sim}{x}) = \alpha\right)\right\} \\ &= \left\{\left(\left(\underset{\sim}{x}\right)_G, \alpha\right) \mid \left(\left(\underset{\sim}{x}, \mathbb{1}\right) \in \underset{\sim}{R_n} \wedge g_n(\underset{\sim}{x}) = \alpha\right)\right\}. \end{aligned}$$

<u>Inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$</u>, we define $h_n : R_n \xrightarrow{1-1} (\aleph_{n+1})^{\mathbf{M}}$ by

$$h_n(x) = \bigcap\left\{\alpha \in (\aleph_{n+1})^{\mathbf{M}} \mid (x, \alpha) \in B\right\}.$$

(Notice this is well defined since $g_n$ is $1-1$, and also an injection for the same reason.) Since inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ we already have $(\aleph_{n+1})^{\mathbf{M}} \overset{1-1}{\lesssim} \omega$, we have proved

$$R_n \overset{1-1}{\lesssim} (\aleph_{n+1})^{\mathbf{M}} \overset{1-1}{\lesssim} \omega$$

hence $R_n \overset{1-1}{\lesssim} \omega$, which means

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \text{“ } R_n \text{ is countable ”}.$$

All in all, we have shown

$$\left(\mathcal{P}(\omega) = \bigcup\left\{R_n \mid n \in \omega\right\} \wedge \forall n \in \omega \left(R_n \overset{1-1}{\lesssim} (\aleph_n)^{\mathbf{M}} \wedge (\aleph_n)^{\mathbf{M}} \overset{1-1}{\lesssim} \aleph_0\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$$

i.e.,

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \mathcal{P}(\omega) = \bigcup\left\{R_n \mid n \in \omega\right\} \wedge \forall n \in \omega \text{ “ } R_n \text{ is countable ”}.$$

which proves

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \text{“}\mathcal{P}(\omega) \text{ is a countable union of countable sets”}.$$

Moreover, since $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ satisfies “ **ZF** ” and “ **ZF** ” $\vdash_c \mathbb{R} \simeq \mathcal{P}(\omega)$, we have

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \mathbb{R} \simeq \mathcal{P}(\omega);$$

from which we easily obtain

$$\widehat{\mathbf{M}[G]}^{\mathscr{F}} \models \text{ “}\mathbb{R} \text{ is a countable union of countable sets”}.$$

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Among the major consequences of this result is Proposition 366 which states that if $\mathbb{R}$ is a countable union of countable sets, then

$$\omega_1 \overset{1-1}{\not\lesssim} \mathbb{R}.$$

Notice that this result holds in a model which satisfies “ **ZF** ”, hence in which there is no bijection between $\omega$ and $\mathbb{R}$. In other words, the real numbers are uncountable and the model knows it, but there is no injection from the least uncountable ordinal to the set of real numbers.

Another disturbing result, which is a consequence of the real numbers being a countable union of countable sets, is Corollary 367 which states that if $\mathbb{R}$ is a countable union of countable sets, then there exists some partition $\mathcal{R}$ of $\mathbb{R}$ together with an injection from $\mathbb{R}$ to $\mathcal{R}$ (showing that this partition is extremely fine) but somehow, no injection from the partition to the real numbers:

$$\mathbb{R} \lnsim \mathcal{R}.$$

Such a result of course, highly contradicts the axiom of choice since

(1) inside a world where **AC** holds, one could precisely make use of this axiom to pick from every element of the partition which is non-empty, some element to then form a 1-1 mapping from $\mathcal{R}$ to $\mathbb{R}$.

(2) Also, getting used of working with the axiom of choice at hand, our initial reaction is to understand $\mathbb{R} \lnsim \mathcal{R}$ as saying that the set of all the real numbers is strictly smaller than some partition of it, which seems extremely bizarre.

## 22.2 Forcing the Well-Orderings of the Reals Out

In this section we show that it is consistent with **ZF** that there exists no well-ordering of the reals. Notice that this implies that there is no bijection between any ordinal and the set of all reals, and even that there is no injection of the reals into the class of the ordinals.

**Theorem 379** (Cohen)**.**

$$cons(\mathbf{ZF}) \Longrightarrow cons(\mathbf{ZF} + \text{“}\textit{there is no well-ordering of } \mathbb{R}\text{”}).$$

*Proof of Theorem 379*: To do so, we prove that given $\mathbf{M}$ any *c.t.m.* of “**ZFC**” with $\mathbb{P}_{\aleph_0} \in \mathbf{M}$, if $G$ is $\mathbb{P}_{\aleph_0}$-generic over $\mathbf{M}$, then there exists some countable set of reals[3] $A \in \mathbf{M}[G]$ and a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, such that $\underline{\text{inside } \widehat{\mathbf{M}[G]}^{\mathscr{F}}}$: $A$ still exists, remains infinite, but contains no proper subset $B \subsetneq A$ such that $B \simeq A$.
This will clearly give the result because if there would exist a well-ordering of the reals, then every subset of the reals would also be well-ordered. So, in particular there would exist a well ordering $(A, <_A)$ whose order-type would be some infinite ordinal. From there, designing a proper subset $B \subsetneq A$ such that $B \simeq A$ would be an easy exercise.

We force with $\mathbb{P} = \left(\mathbb{P}_{\aleph_0}, \leqslant, \mathbb{1}\right)$ where

$$\mathbb{P}_{\aleph_0} = \Big\{f : \omega \times \omega \longrightarrow \{0,1\} \mid dom\,(f) \ \textit{ is finite}\Big\}; \quad f \leqslant g \iff f \supseteq g; \quad \mathbb{1} = \varnothing.$$

Given any $G$ $\mathbb{P}$-generic over $\mathbf{M}$, we have $\bigcup G = \mathcal{F} \in \mathbf{M}[G]$ satisfies

$$\mathcal{F} : \omega \times \omega \to \{0,1\}.$$

For each integer $k$, we set

$$\underset{\sim}{a_k} = \big\{(\check{n}, p) \in dom\,(\check{\omega}) \times \mathbb{P} \mid p(k,n) = 1\big\} \qquad \text{and} \qquad \underset{\sim}{A} = \big\{(\underset{\sim}{a_k}, \mathbb{1}) \mid k \in \omega\big\}.$$

We let $\big(\underset{\sim}{a_k}\big)_G = a_k$ and $(\underset{\sim}{A})_G = A$, so that we have

$$a_k = \big\{n < \omega \mid \mathcal{F}(k,n) = 1\big\} \qquad \text{and} \qquad A = \big\{a_k \mid k \in \omega\big\}.$$

Since for all integers $l, m, n$ the following sets $D_{n,l}$ and $E_{n,m}$ are dense in $\mathbb{P}$:

$$D_{n,l} = \big\{p \in \mathbb{P} \mid \exists k > l \ \ p(n,k) = 1\big\}$$

and

$$E_{n,m} = \big\{p \in \mathbb{P} \mid \exists k \leqslant \omega \ \ p(n,k) \neq p(m,k)\big\}.$$

Using the notation $[\omega]^\omega$ for the set of infinite subsets of $\omega$, it follows that

- $\left(a_n \in [\omega]^\omega\right)^{\mathbf{M}[G]}$,
- $A \in \mathbf{M}[G] \smallsetminus \mathbf{M}$, and

[3] Here, reals stand for subsets of integers.

∘ $\Big(\forall n \in \omega \left(a_n \in A \wedge \forall m \in \omega \; (n \neq m \longleftrightarrow a_n \neq a_m)\right)\Big)^{\mathbf{M}[G]}$.

Therefore, $A$ is infinite.

We then construct a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ which *still contains the infinite set $A$ but no injection* from $\omega$ to $A$.

Every permutation of the integers $\rho : \omega \overset{bij.}{\longleftrightarrow} \omega$ induces an automorphism $\pi_\rho : \mathbb{P} \overset{aut.}{\longleftrightarrow} \mathbb{P}$ defined by

$$\pi_\rho(p) = \Big\{ \big( (\rho(n), m), i \big) \subseteq (\omega \times \omega) \times \omega \mid \big( (n, m), i \big) \in p \Big\}.$$

We consider the group of such automorphisms

$$\mathscr{G} = \left\{ \pi_\rho \mid \rho : \omega \overset{bij.}{\longleftrightarrow} \omega \right\}$$

and $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ the filter generated by

$$\left\{ \mathit{fix}_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{\mathit{fin}}(\omega) \right\}$$

where

$$\mathit{fix}_{\mathscr{G}}(F) = \left\{ \pi_\rho \in \mathscr{G} \mid \forall n \in F \;\; \rho(n) = n \right\}.$$

$\mathscr{F}$ is a normal filter on $\mathcal{G}$ since $\mathscr{F}$ is a set of subgroups of $\mathcal{G}$ such that for all subgroups $\mathcal{H}$, $\mathcal{K}$ of $\mathcal{G}$ and all $\pi \in \mathcal{G}$:

(1) $\mathscr{G} \in \mathscr{F}$ because $\mathscr{G} = \mathit{fix}_{\mathscr{G}}(\varnothing)$.

(2) If $\mathcal{H} \in \mathscr{F}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathcal{H} \subseteq \mathcal{K}$ holds for some finite $F \subseteq \omega$, which shows $\mathcal{K} \in \mathscr{F}$.

(3) If $\mathcal{H} \in \mathscr{F}$ and $\mathcal{K} \in \mathscr{F}$, then both $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathcal{H}$ and $\mathit{fix}_{\mathscr{G}}(E) \subseteq \mathcal{K}$ hold for finite $E, F \subseteq \omega$. Thus, $\mathit{fix}_{\mathscr{G}}(E \cup F) \subseteq \mathcal{H} \cap \mathcal{K} \in \mathscr{F}$ holds which shows that $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$.

(4) If $\mathcal{H} \in \mathscr{F}$, then given any finite $F \subseteq \omega$ such that $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathcal{H}$, one has $\pi_\rho \circ \mathit{fix}_{\mathscr{G}}(F) \circ \pi_\rho^{-1} = \mathit{fix}_{\mathscr{G}}(\rho[F])$; so that $\mathit{fix}_{\mathscr{G}}(\rho[F]) \subseteq \pi_\rho \circ \mathcal{H} \circ \pi_\rho^{-1}$. Thus, $\pi_\rho \circ \mathcal{H} \circ \pi_\rho^{-1} \in \mathscr{F}$.

So, we can define $\mathbf{HS}_{\mathscr{F}}$ as the class of all hereditarily symmetric $\mathbb{P}$-names, and $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ as the symmetric submodel of the generic extension $\mathbf{M}[G]$ induced by $\mathbf{HS}_{\mathscr{F}}$.

Notice that for each integer $k$ and each $\pi_\rho \in \mathscr{G}$, we have

$$\begin{aligned}
\tilde{\pi}_\rho(\underset{\sim}{a}_k) &= \Big\{ \big(\tilde{\pi}_\rho(\check{n}), \pi_\rho(p)\big) \in \mathit{dom}\,(\check{\omega}) \times \mathbb{P} \mid p(k, n) = 1 \Big\} \\
&= \Big\{ \big(\check{n}, \pi_\rho(p)\big) \in \mathit{dom}\,(\check{\omega}) \times \mathbb{P} \mid \pi_\rho(p)\big(\rho(k), n\big) = 1 \Big\} \\
&= \Big\{ \big(\check{n}, q\big) \in \mathit{dom}\,(\check{\omega}) \times \mathbb{P} \mid q\big(\rho(k), n\big) = 1 \Big\} \\
&= \underset{\sim}{a}_{\rho(k)}.
\end{aligned}$$

We see that for every permutation $\rho$ such that $\rho(k) = k$, we have $\tilde{\pi}_\rho(\underset{\sim}{a_k}) = \underset{\sim}{a}_{\rho(k)} = \underset{\sim}{a}_k$. Therefore, $\mathit{fix}_{\mathscr{G}}(\{k\}) \subseteq \mathit{sym}_{\mathscr{G}}(\underset{\sim}{a}_k)$. And since each element of $\mathit{dom}(\underset{\sim}{a}_k)$ which is of the form $\check{n}$ is in $\mathbf{HS}_{\mathscr{F}}$, we have $\underset{\sim}{a}_k \in \mathbf{HS}_{\mathscr{F}}$, hence $a_k \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$. Moreover, for each $\pi_\rho \in \mathscr{G}$, we have

$$\begin{aligned}\tilde{\pi}_\rho(\underset{\sim}{A}) &= \left\{\left(\tilde{\pi}_\rho(\underset{\sim}{a_k}), \pi_\rho(\mathbb{1})\right) \mid k \in \omega\right\}\\ &= \left\{\left(\underset{\sim}{a}_{\rho(k)}, \mathbb{1}\right) \mid k \in \omega\right\}\\ &= \underset{\sim}{A}.\end{aligned}$$

Which shows that $\mathscr{G} \subseteq \mathit{sym}_{\mathscr{G}}(\underset{\sim}{A})$, hence $\underset{\sim}{A} \in \mathbf{HS}_{\mathscr{F}}$ and $A \in \widehat{\mathbf{M}[G]}^{\mathscr{F}}$.

We already have $\left(a_n \in [\omega]^\omega\right)^{\mathbf{M}[G]}$ and

$$\Big(\forall n \in \omega \left(a_n \in A \wedge \forall m \in \omega \ (n \neq m \longleftrightarrow a_n \neq a_m)\right)\Big)^{\mathbf{M}[G]}.$$

The set $[\omega]^\omega$ belongs to $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ since it belongs to $M$. So, we have for each integer $n$, $\left(a_n \in [\omega]^\omega\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$ and $\left(\text{“}A \text{ is infinite”}\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$. We show that $\left(\text{“}A \text{ is Dedekind-finite”}\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}$ — see Definition 353.

**Claim 380.**

$$\left(\textit{“There is no 1-1 mapping from } \omega \textit{ to } A\textit{”}\right)^{\widehat{\mathbf{M}[G]}^{\mathscr{F}}}.$$

*Proof of Claim 380:* Towards a contradiction, we assume that there exists in $\mathbf{M}$ an hereditarily symmetric name $\underset{\sim}{f} \in \mathbf{HS}_{\mathscr{F}}$ for some mapping that exists inside $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, namely,

$$f = \left(\underset{\sim}{f}\right)_G : \omega \xrightarrow{1-1} A.$$

So, there exists $p \in G$ such that

$$p \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f} : \check{\omega} \xrightarrow{1-1} \underset{\sim}{A}.$$

Since $\underset{\sim}{f} \in \mathbf{HS}_{\mathscr{F}}$, we have $\mathit{sym}_{\mathscr{G}}\left(\underset{\sim}{f}\right) \in \mathscr{F}$, hence there exists some finite set $F_{\underset{\sim}{f}} \subsetneq \omega$ such that $\mathit{fix}_{\mathscr{G}}(F_{\underset{\sim}{f}}) \subseteq \mathit{sym}_{\mathscr{G}}\left(\underset{\sim}{f}\right)$. Then, $f$ being injective, there exist $n_f \in \omega \smallsetminus F_f$ and $k \in \omega$ such that $f(k) = a_{n_f}$. So, by the Truth Lemma, there also exists $p_f \in G$ with $p_f \leqslant p$ and

$$p_f \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{k}) = \underset{\sim}{a}_{n_f}.$$

We consider any permutation $\rho : \omega \overset{bij.}{\longleftrightarrow} \omega$ such that $\pi_\rho \in \mathit{fix}_{\mathscr{G}}(F_{\underset{\sim}{f}})$, $\rho(n_f) \neq n_f$ and there exists $q \leqslant \pi_\rho(p_f), p_f$ — i.e., $\pi_\rho(p_f)$ and $p_f$ are compatible.

From $\pi_\rho \in fix_{\mathscr{G}}(F_{\underset{\sim}{f}})$, we obtain $\pi_\rho \in sym_{\mathscr{G}}\big(\underset{\sim}{f}\big)$, hence $\tilde{\pi}_\rho(\underset{\sim}{f}) = \underset{\sim}{f}$. So, we have

$$\pi_\rho(p_f) \Vdash_{\mathbb{P},\mathbf{M}} \tilde{\pi}_\rho(\underset{\sim}{f})\big(\tilde{\pi}_\rho(\check{k})\big) = \tilde{\pi}_\rho(\underset{\sim}{a}_{n_f})$$

i.e.,

$$\pi_\rho(p_f) \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}\big(\check{k}\big) = \underset{\sim}{a}_{\rho(n_f)}.$$

Any $q \in \mathbb{P}$ which satisfies both $q \leqslant \pi_\rho(p_f)$ and $q \leqslant p_f$ yields both

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}(\check{k}) = \underset{\sim}{a}_{n_f}$$

and

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{f}\big(\check{k}\big) = \underset{\sim}{a}_{\rho(n_f)},$$

hence

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a}_{n_f} = \underset{\sim}{a}_{\rho(n_f)}.$$

Now, for every filter $H$ which is $\mathbb{P}$-generic over $\boldsymbol{\mathcal{M}}$ and contains $q$ we have

$$\big(a_{n_f} = a_{\rho(n_f)}\big)^{\mathbf{M}[H]}$$

but since $n_f \neq \rho(n_f)$, this contradicts

$$\Big(\forall n \in \omega\, \forall m \in \omega\ (n \neq m \longrightarrow a_n \neq a_m)\big)\Big)^{\mathbf{M}[H]}.$$

This proves that there is no hereditarily symmetric $\mathbb{P}$-name for an injection from $\omega$ to $A$.

□ 380

So, we have shown that there is no hereditarily symmetric name for an injection from $\omega$ to the infinite set $A$. This result implies that there is no well-ordering of $\mathbb{R}$, since any well-ordering of the reals would yield some bijection $f : \alpha \xleftrightarrow{\text{bij.}} \mathbb{R}$ which would yield an injection $g : \omega \xrightarrow{1-1} A$ defined by recursion on the integers by

$$\begin{aligned} g(n) = f(\beta) \text{ where } \beta &= \min\big\{\xi \in \alpha \mid f(\xi) \in A \smallsetminus \{g(i) \mid i < n\}\,\big\} \\ &= \min\Big\{\xi \in \alpha \mid \big(f(\xi) \in A \,\wedge\, \forall i < n\ f(\xi) \neq g(i)\big)\Big\}. \end{aligned}$$

□ 379

## 22.3 Forcing Every Ultrafilter on $\omega$ is Principal

**Definition 381.** *Let $X$ be any non-empty set.*

- *An ultrafilter $\mathscr{U}$ on $X$ is any non-empty set $\mathscr{U} \subseteq \mathscr{P}(X)$ which satisfies*

$$\left.\begin{array}{l}(1)\ \varnothing \notin \mathscr{U}\\ (2)\ \text{if } A, B \in \mathscr{U},\ \text{then } A \cap B \in \mathscr{U}\\ (3)\ \text{if } A \in \mathscr{U} \text{ and } A \subseteq B,\ \text{then } B \in \mathscr{U}\end{array}\right\}\ \text{Filter}$$
$$(4)\ \text{for all } A \subseteq X,\ A \in \mathscr{U} \text{ or } A^{\complement} \in \mathscr{U}\quad \}\ \text{Ultra}$$

- *An ultrafilter $\mathscr{U}$ on $X$ is principal (or trivial) if there exists some $A \subseteq X$ such that*

$$\mathscr{U} = \{B \subseteq X \mid A \subseteq B\}.$$

- *An ultrafilter $\mathscr{U}$ on $X$ is free if it is non-principal*

An ultrafilter is trivial if and only if it contains some $\subseteq$-least element. Every filter $\mathscr{F} \subseteq \mathscr{P}(X)$ which contains some $\subseteq$-least element $A$ can trivially be extended into an ultrafilter, namely

$$\begin{aligned}\mathscr{U} &= \{B \subseteq X \mid \exists C \in \mathscr{F}\ C \subseteq B\}\\ &= \{B \subseteq X \mid A \subseteq B\}\end{aligned}$$

This question is far more involved with non-trivial filters. With the axiom of choice, of course, every filter can be extended into an ultrafilter. But the converse is not necessarily true. Even for the Fréchet Filter — $\mathscr{F}_{réchet} = \{A \subseteq \omega \mid \omega \setminus A \text{ is finite}\}$ — as shown by the next result, it is consistent with **ZF** that it cannot be extended by any ultrafilter.

**Theorem 382** (Feferman)**.**

$$cons(\mathbf{ZF}) \Longrightarrow cons(\mathbf{ZF} + \text{"every ultrafilter on } \omega \text{ is trivial"}).$$

*Proof of Theorem 382*: As with the proof of Theorem 379, we start with **M** any *c.t.m.* of "**ZFC**" and force with $\mathbb{P} = (\mathbb{P}_{\aleph_0}, \leqslant, \mathbb{1})$ where

$$\mathbb{P}_{\aleph_0} = \{f : \omega \times \omega \longrightarrow \{0,1\} \mid dom(f) \text{ is finite}\};\quad f \leqslant g \iff f \supseteq g;\quad \mathbb{1} = \varnothing.$$

Given any $G$ which is $\mathbb{P}$-generic over $\mathbf{M}$, we have $\bigcup G = \mathcal{F} \in \mathbf{M}[G]$ satisfies

$$\mathcal{F} : \omega \times \omega \to \{0, 1\}.$$

For each integer $k$, we set

$$\underset{\sim}{a_k} = \{(\check{n}, p) \in dom(\check{\omega}) \times \mathbb{P} \mid p(k, n) = 1\}.$$

We let $\left(\underset{\sim}{a_k}\right)_G = a_n$, so that we have

$$a_k = \{n < \omega \mid \mathcal{F}(k, n) = 1\}.$$

Since for all integers $l, m, n$ the sets $D_{n,l}$ and $E_{n,m}$ below are dense in $\mathbb{P}$:

$$D_{n,l} = \{p \in \mathbb{P} \mid \exists k > l \;\; p(n, k) = 1\}$$

and

$$E_{n,m} = \{p \in \mathbb{P} \mid \exists k \leqslant \omega \;\; p(n, k) \neq p(m, k)\}$$

it follows that

$$\Big(a_n \in [\omega]^{\omega} \;\wedge\; \forall n \in \omega \, \forall m \in \omega \; (n \neq m \longleftrightarrow a_n \neq a_m)\Big)^{\mathbf{M}[G]}.$$

We construct a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ by considering, for each $S \subseteq \omega \times \omega$, an automorphism $\pi_S : \mathbb{P} \xrightarrow{aut.} \mathbb{P}$ defined for each $p \in \mathbb{P}$ by:

$$\begin{aligned} \pi_S(p) : dom(p) &\longrightarrow 2 \\ (n, m) &\mapsto \begin{cases} 1 - p(n, m) & \text{if } \;(n, m) \in S \\ p(n, m) & \text{if } \;(n, m) \notin S. \end{cases} \end{aligned}$$

We let $\mathscr{G}$ be the group of all such automorphisms and given any $F \in \mathcal{P}_{fin}(\omega)$,

$$fix_{\mathscr{G}}(F \times \omega) = \{\pi_S \in \mathscr{G} \mid S \cap (F \times \omega) = \varnothing\},$$

and $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ be the filter generated by

$$\{fix_{\mathscr{G}}(F \times \omega) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\omega)\}.$$

We verify that $\mathscr{F}$ is a normal filter on $\mathcal{G}$.
$\mathscr{F}$ is a set of subgroups of $\mathcal{G}$ such that for all subgroups $\mathcal{H}$, $\mathcal{K}$ of $\mathcal{G}$ and all $\pi \in \mathcal{G}$:

(1) $\mathscr{G} \in \mathscr{F}$ because $\mathscr{G} = fix_{\mathscr{G}}(\varnothing) = fix_{\mathscr{G}}(\varnothing \times \omega)$

(2) if $\mathcal{H} \in \mathscr{F}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $fix_{\mathscr{G}}(F \times \omega) \subseteq \mathcal{H} \subseteq \mathcal{K}$ holds for some finite $F \subseteq \omega$, which shows $\mathcal{K} \in \mathscr{F}$

(3) if $\mathcal{H} \in \mathscr{F}$ and $\mathcal{K} \in \mathscr{F}$, then both $\mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \mathcal{H}$ and $\mathit{fix}_{\mathscr{G}}(E \times \omega) \subseteq \mathcal{K}$ hold for finite $E, F \subseteq \omega$. Thus, $\mathit{fix}_{\mathscr{G}}\big((E \cup F) \times \omega\big) \subseteq \mathcal{H} \cap \mathcal{K} \in \mathscr{F}$ holds which shows that $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$.

(4) if $\mathcal{H} \in \mathscr{F}$, then given any finite $F \subseteq \omega$ such that $\mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \mathcal{H}$, one has

$$\mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \pi_S \circ \mathit{fix}_{\mathscr{G}}(F \times \omega) \circ \pi_S^{-1},$$

thus, $\pi_S \circ \mathcal{H} \circ \pi_S^{-1} \in \mathscr{F}$.

So, we can define $\mathbf{HS}_{\mathscr{F}}$ as the class of all hereditarily symmetric $\mathbb{P}$-names, and $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ as the symmetric submodel of the generic extension $\mathbf{M}[G]$ induced by $\mathbf{HS}_{\mathscr{F}}$.

We let $\mathscr{U}$ be any ultrafilter in $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$, together with $\underset{\sim}{\mathscr{U}} \in \mathbf{HS}_{\mathscr{F}}$ any $\mathbb{P}$-name for $\mathscr{U}$, and any $p \in G$ with

$$p \Vdash_{\mathbb{P},\mathbf{M}} \text{“}\underset{\sim}{\mathscr{U}} \text{ is an ultrafilter over } \check{\omega}\text{”}.$$

We take any finite $F \in \mathcal{P}_{\mathit{fin}}(\omega)$ such that $\mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \mathit{sym}_{\mathscr{G}}(\underset{\sim}{\mathscr{U}})$ as well as any integer $k \notin F$. We distinguish between $a_k \in \mathscr{U}$ and $a_k \notin \mathscr{U}$.

If $a_k \in \mathscr{U}$: we pick any $q \in G$ such that $q \leqslant p$ and

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a_k} \in \underset{\sim}{\mathscr{U}}.$$

we consider any $k' \in \omega$ large enough such that

$$\big\{(k,n) \in \omega \times \omega \mid n \geqslant k'\big\} \cap \mathit{dom}(q) = \varnothing$$

We notice that $S = \big\{(k,n) \in \omega \times \omega \mid n \geqslant k'\big\}$ satisfies $S \cap (F \times \omega) = \varnothing$ and form $\pi_S$ and consider $\underset{\sim}{b_k} = \tilde{\pi}_S(\underset{\sim}{a_k})$ and write $b_k$ for $(\underset{\sim}{b_k})_G$. By construction, we see that for each integer $n \geqslant k'$ we have

$$n \in a_k \iff n \notin b_k$$

which yields

$$a_k \cap b_k \subseteq k'$$

which shows that this set is finite. Building on $q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a_k} \in \underset{\sim}{\mathscr{U}}$, we reach

$$\pi_S(q) \Vdash_{\mathbb{P},\mathbf{M}} \tilde{\pi}_S(\underset{\sim}{a_k}) \in \tilde{\pi}_S(\underset{\sim}{\mathscr{U}}).$$

Since $\mathit{fix}_{\mathscr{G}}(F \times \omega) = \big\{\pi_{S'} \in \mathscr{G} \mid S' \cap (F \times \omega) = \varnothing\big\}$ and $S \cap (F \times \omega) = \varnothing$, we have

$$\pi_S \in \mathit{fix}_{\mathscr{G}}(F \times \omega) \subseteq \mathit{sym}_{\mathscr{G}}(\underset{\sim}{\mathscr{U}}),$$

which gives

$$\pi_S(q) \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{b_k} \in \underset{\sim}{\mathscr{U}};$$

and since $S \cap dom(q) = \varnothing$, we have $\pi_S(q) = q$, so that we finally obtain

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{b}_k \in \underset{\sim}{\mathscr{U}}.$$

So, we end up with both $a_k \in \mathscr{U}$ and $b_k \in \mathscr{U}$, hence $a_k \cap b_k \in \mathscr{U}$. Since $a_k \cap b_k \subseteq k'$, we obtain

$$\mathscr{U} = \{X \subseteq \omega \mid c \subseteq X\}$$

where $c$ is the finite set defined by

$$c = \bigcap \{Y \in \mathscr{U} \mid Y \subseteq a_k \cap b_k\}.$$

Thus, $\mathscr{U}$ is principal.

If $a_k \notin \mathscr{U}$: we pick any $q \in G$ such that $q \leqslant p$ and

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a}_k \notin \underset{\sim}{\mathscr{U}}.$$

we consider any $k' \in \omega$ large enough such that

$$\{(k,n) \in \omega \times \omega \mid n \geqslant k'\} \cap dom(q) = \varnothing$$

We notice that $S = \{(k,n) \in \omega \times \omega \mid n \geqslant k'\}$ satisfies $S \cap (F \times \omega) = \varnothing$ and form $\pi_S$ and consider $\underset{\sim}{b}_k = \tilde{\pi}_S(\underset{\sim}{a}_k)$ and write $b_k$ for $(\underset{\sim}{b}_k)_G$. By construction, we see that for each integer $n \geqslant k'$ we have

$$n \in a_k \iff n \notin b_k$$

which yields

$$(\omega \smallsetminus a_k) \cap (\omega \smallsetminus b_k) = \{n \in \omega \mid n \notin a_k \wedge n \notin b_k\} \subseteq k',$$

hence this set is finite. From $q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{a}_k \notin \underset{\sim}{\mathscr{U}}$, we get

$$\pi_S(q) \Vdash_{\mathbb{P},\mathbf{M}} \tilde{\pi}_S(\underset{\sim}{a}_k) \notin \tilde{\pi}_S(\underset{\sim}{\mathscr{U}}).$$

Since $fix_{\mathscr{G}}(F \times \omega) = \{\pi_{S'} \notin \mathscr{G} \mid S' \cap (F \times \omega) = \varnothing\}$ and $S \cap (F \times \omega) = \varnothing$, we have

$$\pi_S \in fix_{\mathscr{G}}(F \times \omega) \subseteq sym_{\mathscr{G}}(\underset{\sim}{\mathscr{U}}),$$

which gives

$$\pi_S(q) \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{b}_k \notin \underset{\sim}{\mathscr{U}};$$

and since $S \cap dom(q) = \varnothing$, we have $\pi_S(q) = q$, so that we finally obtain

$$q \Vdash_{\mathbb{P},\mathbf{M}} \underset{\sim}{b}_k \notin \underset{\sim}{\mathscr{U}}.$$

So, we end up with both $a_k \notin \mathscr{U}$ and $b_k \notin \mathscr{U}$, which gives $(\omega \smallsetminus a_k) \in \mathscr{U}$ and $(\omega \smallsetminus b_k) \in \mathscr{U}$ and finally $(\omega \smallsetminus a_k) \cap (\omega \smallsetminus b_k) \in \mathscr{U}$. Now, since $(\omega \smallsetminus a_k) \cap (\omega \smallsetminus b_k) \subseteq k'$, we obtain

$$\mathscr{U} = \{X \subseteq \omega \mid c \subseteq X\}$$

where $c$ is the finite set defined by

$$c = \bigcap \{Y \in \mathscr{U} \mid Y \subseteq (\omega \smallsetminus a_k) \cap (\omega \smallsetminus b_k)\}.$$

Thus, $\mathscr{U}$ is principal.

□ 382

We have constructed a symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ in which there is no free ultrafilter on $\omega$ because every ultrafilter on $\omega$ is principal. So, in particular, the Fréchet filter — which belongs to $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$ because it belongs to $\mathbf{M}$ and $\mathbf{M} \subseteq \widehat{\mathbf{M}[G]}^{\mathscr{F}}$ — cannot be extended into any ultrafilter.