Part VII

Set Theory with Atoms

Chapter 23

Atoms and Permutation Models

Set theory with atoms — denoted **ZFA** — is slightly different from **ZF** in that the empty set is not the only set which does not contain any other set. There are also some other sets that do not contain anything but are still different from each other. This is a blatant contradiction to the axiom of **Extensionality**.These other basic elements are known as *atoms* or as *Urelements* (usually written *urelements*).

23.1 Zermelo-Fraenkel with Atoms (ZFA)

The language of **ZFA** is the same as the language of **ZF** augmented with a constant symbol $\mathcal{A}$ whose interpretation is a "set of atoms" denoted by $\mathbb{A}$. So, **ZFA** is still a theory of first order logic with equality, but the signature of the language becomes now $\{\mathcal{A}, \in\}$. Essentially, the axioms remain the same except that one must take into account the special features of the atoms. So, **Extensionality** and **Comprehension Schema** are modified, and an **Empty Set Existence for ZFA** as well as an **axiom of Atoms** are added to the theory. Since atoms do not contain any element, the axiom of extensionality is modified and a few other axioms are added to **ZF**.

Empty Set Existence for ZFA

$$\exists x(\forall y\ y \notin x \wedge x \notin \mathcal{A}).$$

This axiom claims that the empty set exists and is different from any atom.

Extensionality for ZFA

$$\forall x\,\forall y\Big((x \notin \mathcal{A} \wedge y \notin \mathcal{A}) \longrightarrow \big(\forall z\ (z \in x \longleftrightarrow z \in y) \longrightarrow x = y\big)\Big).$$

This axiom claims that all sets that are not atoms are the same if and only if they contain the same elements. Notice that the Axiom of **Extensionality for ZFA** implies that the empty set is unique, which guarantees the use of the usual constant symbol $\varnothing$.

axiom of Atoms

$$\forall x\ \big(x \in \mathcal{A} \longleftrightarrow (x \neq \varnothing \wedge \forall z\ z \notin x)\big).$$

This axiom claims that, apart from the empty set — which is not an atom — the atoms are the only sets which do not contain any other set.

Comprehension Schema for ZFA

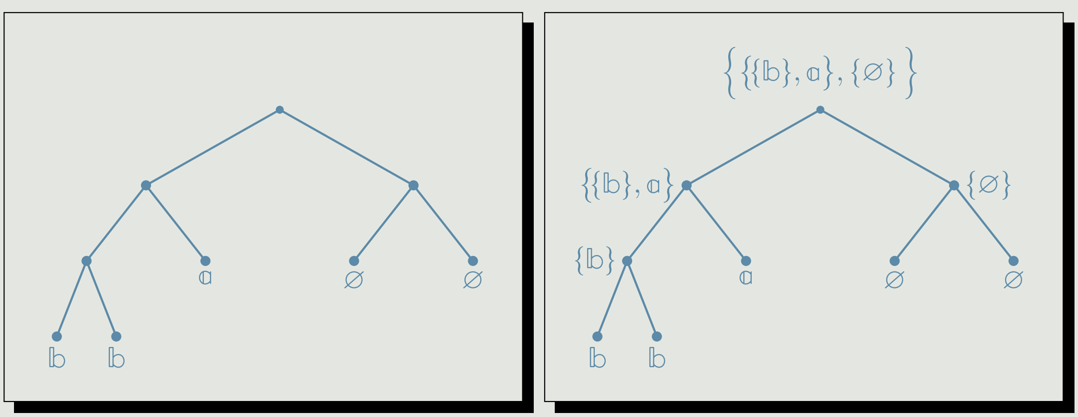
By convention, we assume also that the **Comprehension Schema** is strengthened in order to have the empty subset of any set to be the empty set as opposed to some atom. This gives rise to the **Comprehension Schema for ZFA**:

For each formula $\varphi_{(x,z,w_1\ldots w_n)}$ whose free variables — if any — are among $x, z, w_1, \ldots, w_n$ and y is not free in $\varphi_{(x,z,w_1\ldots w_n)}$, the following formula is an axiom:

$$\forall z \forall w_1 \ldots \forall w_n \exists y \forall x \Big(\big(x \in y \longleftrightarrow (x \in z \wedge \varphi_{(x,z,w_1,\ldots,w_n)})\big) \wedge \big((x \in z \longrightarrow \neg\varphi_{(x,z,w_1,\ldots,w_n)}) \longrightarrow y = \varnothing\big)\Big).$$

The whole theory of **ZFA** is developed the same way the theory of **ZF** is — in particular ordinals are constructed from the empty set and not from atoms. The tree representation of a set — as a well-founded tree, since we work with **Foundation** — still holds. The only difference is that with **ZF** each leaf corresponds to the empty set, whereas with **ZFA**, the leaves can also represent atoms and not just the empty set.

Example 383. *We assume* $\mathbb{a}, \mathbb{b} \in \mathbb{A}$*. Below is a tree that represents the set* $\Big\{\{\{\mathbb{b}\}, \mathbb{a}\}, \{\varnothing\}\Big\}$.



Definition 384 (ZFA). *Given any set* S*, we define* $\boldsymbol{\mathcal{P}}^{\infty}(S)$ *by transfinite recursion.*

- $\boldsymbol{\mathcal{P}}^{0}(S) = S$

- $\boldsymbol{\mathcal{P}}^{\alpha+1}(S) = \boldsymbol{\mathcal{P}}^{\alpha}(S) \cup \mathscr{P}\big(\boldsymbol{\mathcal{P}}^{\alpha}(S)\big)$
- $\boldsymbol{\mathcal{P}}^{\alpha}(S) = \bigcup_{\xi<\alpha} \boldsymbol{\mathcal{P}}^{\xi}(S)$ *(when α is limit).*

$$\boldsymbol{\mathcal{P}}^{\infty}(S) = \bigcup_{\alpha\in\mathbf{On}} \boldsymbol{\mathcal{P}}^{\alpha}(S)\,.$$

If $\mathbb{A}$ is any set of atoms, and $\boldsymbol{\mathcal{M}}$ is any model of **ZFA**, then $\boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A}) = \bigcup_{\alpha\in\mathbf{On}} \boldsymbol{\mathcal{P}}^{\alpha}(\mathbb{A})$ is a subdivision of the elements of $\boldsymbol{\mathcal{M}}$, into some hierarchy, the same way the von Neumann hierarchy $\mathbf{V} = \bigcup_{\alpha\in\mathbf{On}} \mathbf{V}(\alpha)$ is a subdivision of the sets from any model $\mathbf{M}$ of **ZF** or **ZFC**.

$$\boldsymbol{\mathcal{M}} \models \forall x \;\; x \in \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})\,.$$

Definition 385 (**ZFA**)**.** *If $\mathbb{A}$ is any set of atoms, $\boldsymbol{\mathcal{M}}$ any model of* **ZFA***, and $x \in \boldsymbol{\mathcal{M}}$, then*

$$rk_{\boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})}(x) = \textit{least } \alpha \in \mathbf{On} \;\; \textit{such that} \;\; x \in \boldsymbol{\mathcal{P}}^{\alpha+1}(\mathbb{A})\,.$$

Remark 386. If $\mathbb{A}$ is any set of atoms, and $\boldsymbol{\mathcal{M}}$ is any model of **ZFA**, then the **kernel** :

$$\boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \cap \boldsymbol{\mathcal{M}}$$

is the domain of a model of **ZF**.
Moreover, for each $x \in \boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \cap \boldsymbol{\mathcal{M}}$ we have

$$rk_{\boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})}(x) = rk_{\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)}(x) = rk(x)\,.$$

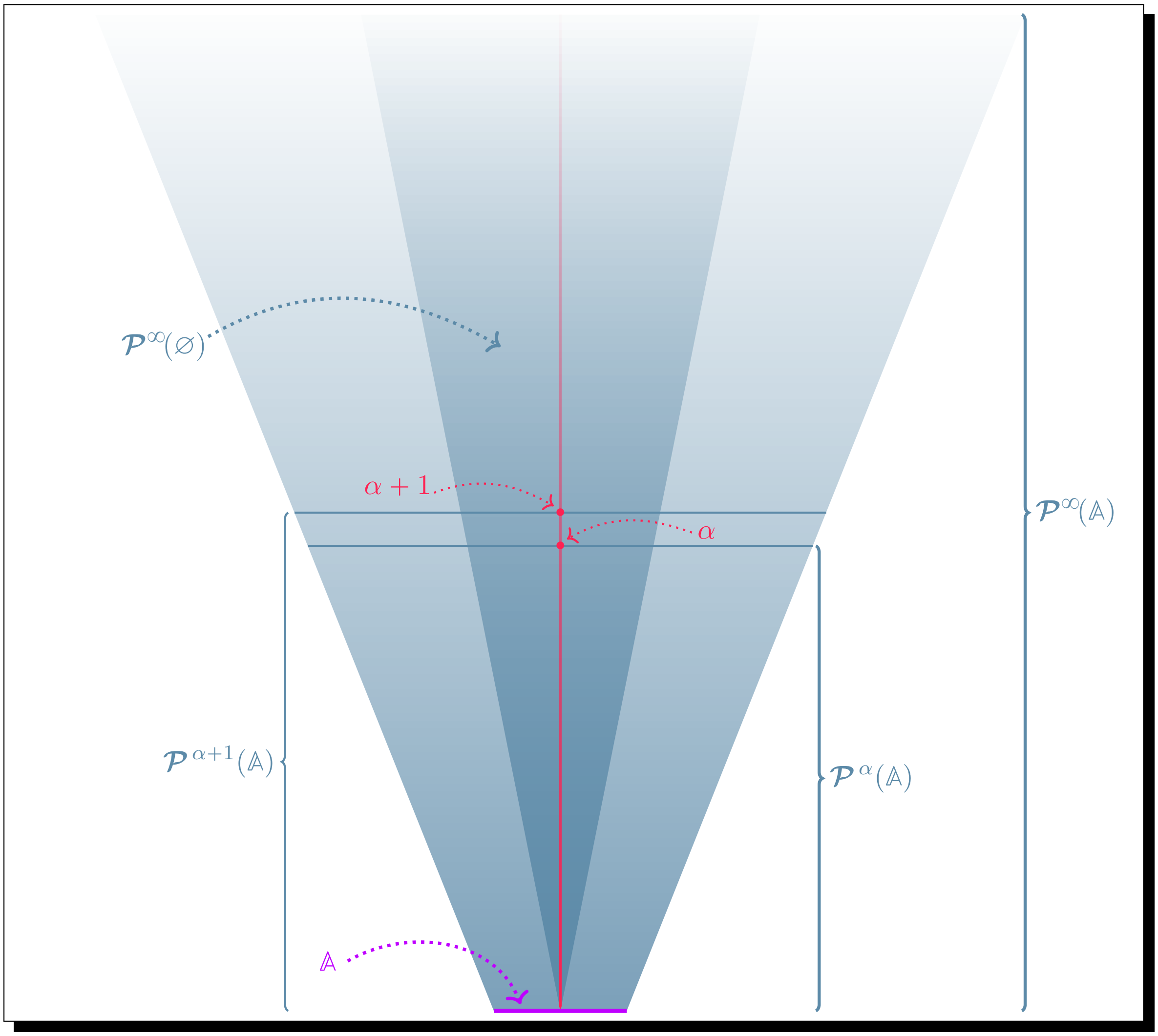


Figure 23.1: The Classes $\mathcal{P}^\infty(\mathbb{A})$ and $\mathcal{P}^\infty(\varnothing)$.

23.2 Permutation Models

The key idea that gave rise to the notion of permutation model is not far removed from the one that brought the concept of symmetric submodel of a generic extension. A permutation model of set theory is obtained from a model of **ZFA** by mean of a group of permutations of the atoms, whereas a symmetric submodel is constructed using a group of automorphism of a forcing poset. But unlike in the cas of symmetric submodel where the refinement of the model obtained came from working on $\mathbb{P}$-names, here we sort sets directly without reference to another structure (except for the permutation normal filter).

Definition 387 (ZFA). *Let* $\pi : \mathbb{A} \overset{bij.}{\longleftrightarrow} \mathbb{A}$ *be any permutation.*
The class-function $\breve{\pi} : \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A}) \to \boldsymbol{\mathcal{P}}^{\infty}(\mathbb{A})$ *is defined for every set* x *by*

- *if* $x = \varnothing$, *then* $\breve{\pi}(\varnothing) = \varnothing$
- *if* $x \in \mathbb{A}$, *then* $\breve{\pi}(x) = \pi(x)$
- *if* $x \notin \mathbb{A} \cup \{\varnothing\}$, *then* $\breve{\pi}(x) = \big\{\breve{\pi}(y) \mid y \in x\big\}$.

Example 388. *We assume the set of atoms is* $\mathbb{A} = \{\mathbb{a}, \mathbb{b}, \mathbb{c}\}$.
From the tree that represents the set in the left picture:

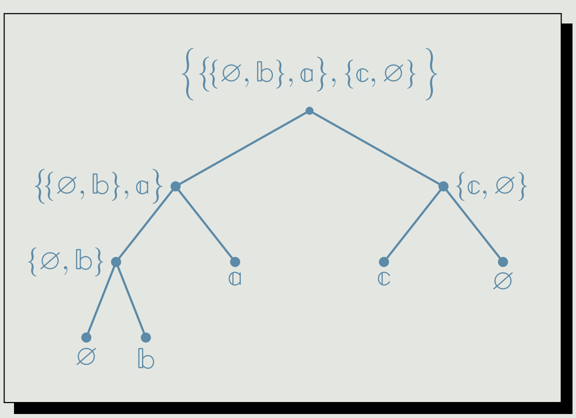
$$x = \Big\{\big\{\{\varnothing, \mathbb{b}\}, \mathbb{a}\big\}, \{\mathbb{c}, \varnothing\}\Big\}$$

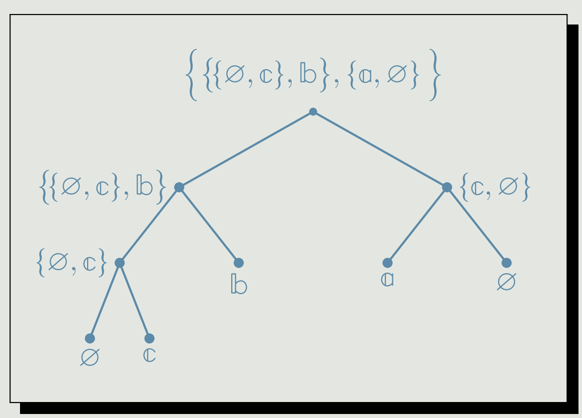
we obtain the set in the right picture:

$$\breve{\pi}(x) = \Big\{\big\{\{\varnothing, \mathbb{c}\}, \mathbb{b}\big\}, \{\mathbb{a}, \varnothing\}\Big\}$$

by applying the following permutation:

$$\begin{array}{cccc} \pi : & \mathbb{A} & \overset{bij.}{\longleftrightarrow} & \mathbb{A} \\ & \mathbb{a} & \longmapsto & \mathbb{b} \\ & \mathbb{b} & \longmapsto & \mathbb{c} \\ & \mathbb{c} & \longmapsto & \mathbb{a} \end{array}$$





Definition 389. *Given any model* $\mathcal{M}$*, a class-function* $f : \mathcal{M} \xrightarrow{bij.} \mathcal{M}$ *is an* $\in$*-automorphism if it satisfies for all* $x, y \in \mathcal{M}$

$$x \in y \iff f(x) \in f(y).$$

Lemma 390. *Given any set of atoms* $\mathbb{A}$*, any permutation* $\pi : \mathbb{A} \xleftrightarrow{bij.} \mathbb{A}$ *yields a class-function* $\breve{\pi} : \mathcal{P}^{\infty}(\mathbb{A}) \to \mathcal{P}^{\infty}(\mathbb{A})$ *which satisfies*

(1) $\forall x \in \mathcal{P}^{\infty}(\mathbb{A})\ \forall y \in \mathcal{P}^{\infty}(\mathbb{A})\ \Big(x \in y \longleftrightarrow \breve{\pi}(x) \in \breve{\pi}(y)\Big)$*;*

(2) $\breve{\pi}$ *is 1-1 and onto, and* $\breve{\pi}^{-1} = \breve{\pi^{-1}}$*.*

Proof of Remark 390:

(1) is immediate.

(2) (a) $\breve{\pi}$ is 1-1: if $x \neq y$, then by symmetry, there exists $z \in x \setminus y$, thence

$$\left.\begin{array}{c} z \in x \Longrightarrow \breve{\pi}(z) \in \breve{\pi}(x) \\ \text{and} \\ z \notin y \Longrightarrow \breve{\pi}(z) \notin \breve{\pi}(y) \end{array}\right\} \Longrightarrow \breve{\pi}(z) \in \breve{\pi}(x) \setminus \breve{\pi}(y)$$

(b) $\breve{\pi}$ is onto: towards a contradiction assume that for some minimal ordinal α there exists some $y \in \mathcal{P}^{\alpha+1}(\mathbb{A})$ such that $\breve{\pi}(x) \neq y$ holds for all $x \in \mathcal{M}$. By minimality of α, every element of y is in the range or $\breve{\pi}$, hence there exists $S \in \mathcal{M}$ such that $y = \big\{\breve{\pi}(x) \mid x \in S\big\}$, which yields $y = \breve{\pi}(S)$, a contradiction.

(c) is immediate by induction on $rk_{\mathcal{P}^{\infty}(\mathbb{A})}(x) =$ least ordinal α such that $x \in \mathcal{P}^{\alpha+1}(\mathbb{A})$.

□ 390

So, any permutation $\pi : \mathbb{A} \xleftrightarrow{bij.} \mathbb{A}$ yields an $\in$-automorphism $\breve{\pi} : \mathcal{P}^{\infty}(\mathbb{A}) \to \mathcal{P}^{\infty}(\mathbb{A})$. We recall that a group $\mathscr{G}$ of permutations of $\mathbb{A}$ is some subgroup of $\big\{\pi : \mathbb{A} \to \mathbb{A} \mid \pi \text{ is 1-1 and onto}\big\}$. equipped with $\circ : (g, f) \mapsto g \circ f$.

Definition 391 (Permutation Normal Filter)**.** *Let* $\boldsymbol{\mathcal{M}}$ *be any model of* **ZFA** *with* $\mathbb{A}$ *as set of atoms, and* $\mathscr{G}$ *be a group of permutations of* $\mathbb{A}$*.* $\mathscr{F}$ *is a* ***normal filter*** *on* $\mathcal{G}$ *if* $\mathscr{F}$ *is a set of subgroups of* $\mathcal{G}$ *such that for all subgroups* $\mathcal{H}$*,* $\mathcal{K}$ *of* $\mathcal{G}$ *and all* $\pi \in \mathcal{G}$*:*

(1) $\mathcal{G} \in \mathscr{F}$*,*

(2) if $\mathcal{H} \in \mathscr{F}$ *and* $\mathcal{H} \subseteq \mathcal{K}$*, then* $\mathcal{K} \in \mathscr{F}$*,*

(3) if $\mathcal{H} \in \mathscr{F}$ *and* $\mathcal{K} \in \mathscr{F}$*, then* $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$*,*

(4) if $\mathcal{H} \in \mathscr{F}$*, then* $\pi \circ \mathcal{H} \circ \pi^{-1} \in \mathscr{F}$*,*

(5) for each atom $a \in \mathbb{A}$*,* $\left\{\pi \in \mathcal{G} \mid \pi(a) = a\right\} \in \mathscr{F}$*.*

Definition 392 (Symmetry Group, Symmetric and Hereditarily Symmetric Set)**.** *Let* $\boldsymbol{\mathcal{M}}$ *be any model of* **ZFA** *with* $\mathbb{A}$ *as set of atoms,* $\mathcal{G}$ *any subgroup of the group of permutations of* $\mathbb{A}$*, and* $\mathscr{F}$ *any normal filter on* $\mathcal{G}$*. For each set* $x \in \boldsymbol{\mathcal{M}}$*,*

- *the* ***symmetry group*** *of* x *is*

$$sym_{\mathscr{G}}(x) = \{\pi \in \mathcal{G} \mid \breve{\pi}(x) = x\};$$

- *we write "*x *is* ***symmetric****" for*

$$sym_{\mathscr{G}}(x) \in \mathscr{F};$$

- *we write "*x *is* ***hereditarily symmetric****" for*

$$x \in \mathbf{HS}_{\mathscr{F}} \iff \begin{cases} sym_{\mathscr{G}}(x) \in \mathscr{F} \\ \text{and} \\ x \subseteq \mathbf{HS}_{\mathscr{F}}. \end{cases}$$

Remark 393. Let $\boldsymbol{\mathcal{M}}$ be any model of **ZFA** with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$. For all sets $x, y \in \boldsymbol{\mathcal{M}}$ and all permutations $\pi \in \mathcal{G}$,

(1) $x = y \iff \breve{\pi}(x) = \breve{\pi}(y)$;

(2) $x \in \mathbf{HS}_{\mathscr{F}} \iff \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}$.

Proof of Remark 393:

(1) $x \in y \iff \breve{\pi}(x) \in \breve{\pi}(y)$ yields
$$x \subseteq y \iff \forall z \in x \;\; z \in y \iff \forall z \in x \;\; \breve{\pi}(z) \in \breve{\pi}(y) \iff \forall \breve{\pi}(z) \in \breve{\pi}(x) \;\; \breve{\pi}(z) \in \breve{\pi}(y) \iff \breve{\pi}(x) \subseteq \breve{\pi}(y).$$

Thus,
$$\begin{aligned} x = y &\iff x \subseteq y \text{ and } y \subseteq x \\ &\iff \breve{\pi}(x) \subseteq \breve{\pi}(y) \text{ and } \breve{\pi}(y) \subseteq \breve{\pi}(x) \\ &\iff \breve{\pi}(x) = \breve{\pi}(y). \end{aligned}$$

(2) (a) We show that $sym_{\mathscr{G}}(x) \in \mathscr{F} \implies sym_{\mathscr{G}}\big(\breve{\pi}(x)\big) \in \mathscr{F}$.

For every permutation ρ we have
$$\begin{aligned} \breve{\pi}^{-1} \circ \breve{\rho} \circ \breve{\pi}(x) = x &\implies \breve{\pi} \circ \breve{\pi}^{-1} \circ \breve{\rho} \circ \breve{\pi}(x) = \breve{\pi}(x) \\ &\implies \breve{\rho}\big(\breve{\pi}(x)\big) = \breve{\pi}(x) \\ &\implies \rho \in sym_{\mathscr{G}}\big(\breve{\pi}(x)\big); \end{aligned}$$

which shows
$$\breve{\pi}^{-1} \circ \breve{\rho} \circ \breve{\pi} \in sym_{\mathscr{G}}(x) \implies \rho \in sym_{\mathscr{G}}\big(\breve{\pi}(x)\big)$$

or, equivalenty,
$$\breve{\pi} \circ sym_{\mathscr{G}}(x) \circ \breve{\pi}^{-1} \subseteq sym_{\mathscr{G}}\big(\breve{\pi}(x)\big).$$

Since $\mathscr{F}$ is a normal filter, it follows that
$$\underbrace{sym_{\mathscr{G}}(x)}_{\in \mathscr{F}} \implies \underbrace{\breve{\pi} \circ sym_{\mathscr{G}}(x) \circ \breve{\pi}^{-1}}_{\in \mathscr{F}} \quad \text{and} \quad \underbrace{\breve{\pi} \circ sym_{\mathscr{G}}(x) \circ \breve{\pi}^{-1}}_{\in \mathscr{F}} \subseteq \underbrace{sym_{\mathscr{G}}\big(\breve{\pi}(x)\big)}_{\in \mathscr{F}}.$$

(b) We show that $x \in \mathbf{HS}_{\mathscr{F}} \implies \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}$ by induction on $rk_{\mathcal{P}^{\infty}(\mathbb{A})}(x) =$ least ordinal α such that $x \in \mathcal{P}^{\alpha+1}(\mathbb{A})$.

- If $rk_{\mathcal{P}^{\infty}(\mathbb{A})}(x) = 0$, then x is a set — possibly empty — of atoms.

$$x \in \mathbf{HS}_{\mathscr{F}} \implies \left\{ \begin{array}{ccc} sym_{\mathscr{G}}(x) \in \mathscr{F} & \overset{by\ (a)}{\implies} & sym_{\mathscr{G}}(\breve{\pi}(x)) \in \mathscr{F} \\ \text{and} & & \text{and} \\ \underbrace{\forall a \in x \;\; a \in \mathbf{HS}_{\mathscr{F}}}_{\forall a \in \mathbb{A} \;\; sym_{\mathscr{G}}(a) \in \mathscr{F}} & \implies & \underbrace{\forall a \in x \;\; \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}}_{\forall a \in \mathbb{A} \;\; sym_{\mathscr{G}}(a) \in \mathscr{F}} \end{array} \right\} \implies \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}.$$

- If $rk_{\mathcal{P}^{\alpha}(\mathbb{A})}(x) > 0$

$$x \in \mathbf{HS}_{\mathscr{F}} \Longrightarrow \left\{ \begin{array}{ccc} sym_{\mathscr{G}}(x) \in \mathscr{F} & \overset{by\ (a)}{\Longrightarrow} & sym_{\mathscr{G}}(\breve{\pi}(x)) \in \mathscr{F} \\ \text{and} & & \text{and} \\ \forall y \in x \ \ y \in \mathbf{HS}_{\mathscr{F}} & \Longrightarrow & \underbrace{\forall y \in x \ \ \breve{\pi}(y) \in \mathbf{HS}_{\mathscr{F}}}_{by\ induction\ hypothesis} \end{array} \right\} \Longrightarrow \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}.$$

We have shown $x \in \mathbf{HS}_{\mathscr{F}} \Longrightarrow \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}}$ holds for all x and all $\breve{\pi}$. So, in particular for $x := \breve{\pi}(x)$ and $\breve{\pi} := \breve{\pi}^{-1}$ we have

$$\begin{aligned} \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}} & \Longrightarrow \breve{\pi}^{-1} \circ \breve{\pi}(x) \in \mathbf{HS}_{\mathscr{F}} \\ & \Longrightarrow x \in \mathbf{HS}_{\mathscr{F}}. \end{aligned}$$

□ 393

Lemma 394. *Let* $\mathcal{M}$ *be any model of* **ZFA** *with* $\mathbb{A}$ *as set of atoms,* $\mathcal{G}$ *any subgroup of the group of permutations of* $\mathbb{A}$*. Let* $\varphi(z_1, \ldots, z_n)$ *be any* $\mathcal{L}_{\mathrm{ST}}$*-formula whose free variables are among* $x_1, \ldots, x_n$*. If* $\pi \in \mathcal{G}$*, then for all* $b_1, \ldots, b_n \in \mathcal{M}$*,*

$$\mathcal{M} \models \varphi(b_1, \ldots, b_n) \Longleftrightarrow \mathcal{M} \models \varphi\big(\breve{\pi}(b_1), \ldots, \breve{\pi}(b_n)\big).$$

Proof of Lemma 394: As always, the proof is by induction on the height of φ. Without loss of generality, we may assume that φ only contains $\neg$ and $\wedge$ as connectors and $\exists$ as sole quantifier.

(1) If φ is an atomic formula, then we already saw that

- $\mathcal{M} \models b_1 = b_2 \iff \mathcal{M} \models \breve{\pi}(b_1) = \breve{\pi}(b_2)$
- $\mathcal{M} \models b_1 \in b_2 \iff \mathcal{M} \models \breve{\pi}(b_1) \in \breve{\pi}(b_2)$.

(2) If $\varphi = \neg\psi$, then

$$\begin{aligned} \mathcal{M} \models \varphi(b_1, \ldots, b_n) & \Longleftrightarrow \mathcal{M} \not\models \psi(b_1, \ldots, b_n) \\ & \Longleftrightarrow \mathcal{M} \not\models \psi\big(\breve{\pi}(b_1), \ldots, \breve{\pi}(b_n)\big) \\ & \Longleftrightarrow \mathcal{M} \models \varphi\big(\breve{\pi}(b_1), \ldots, \breve{\pi}(b_n)\big). \end{aligned}$$

(3) If $\varphi = (\psi \wedge \theta)$, then

$$\begin{aligned} \mathcal{M} \models \varphi(b_1, \ldots, b_n) & \Longleftrightarrow \mathcal{M} \models \psi(b_1, \ldots, b_n) \text{ and } \mathcal{M} \models \theta(b_1, \ldots, b_n) \\ & \Longleftrightarrow \mathcal{M} \models \psi\big(\breve{\pi}(b_1), \ldots, \breve{\pi}(b_n)\big) \text{ and } \mathcal{M} \models \theta\big(\breve{\pi}(b_1), \ldots, \breve{\pi}(b_n)\big) \\ & \Longleftrightarrow \mathcal{M} \models \varphi\big(\breve{\pi}(b_1), \ldots, \breve{\pi}(b_n)\big). \end{aligned}$$

(4) If $\varphi = \exists x\, \psi(x, x_1, \ldots, x_n)$, then

$$\begin{aligned}
\mathcal{M} \models \varphi(b_1, \ldots, b_n) &\iff \text{there exists } y \in \mathcal{M},\quad \mathcal{M} \models \psi(y, b_1, \ldots, b_n)\\
&\iff \text{there exists } \breve{\pi}(y) \in \mathcal{M},\quad \mathcal{M} \models \psi\big(\breve{\pi}(y), \breve{\pi}(b_1), \ldots, \breve{\pi}(b_n)\big)\\
&\iff \mathcal{M} \models \exists x\, \psi\big(x, \breve{\pi}(b_1), \ldots, \breve{\pi}(b_n)\big)\\
&\iff \mathcal{M} \models \varphi\big(\breve{\pi}(b_1), \ldots, \breve{\pi}(b_n)\big).
\end{aligned}$$

□ 394

We now define the symmetric submodel of $\mathcal{M}$ — denoted by $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ — as the restriction of $\mathcal{M}$ to $\mathbf{HS}_{\mathscr{F}}$.

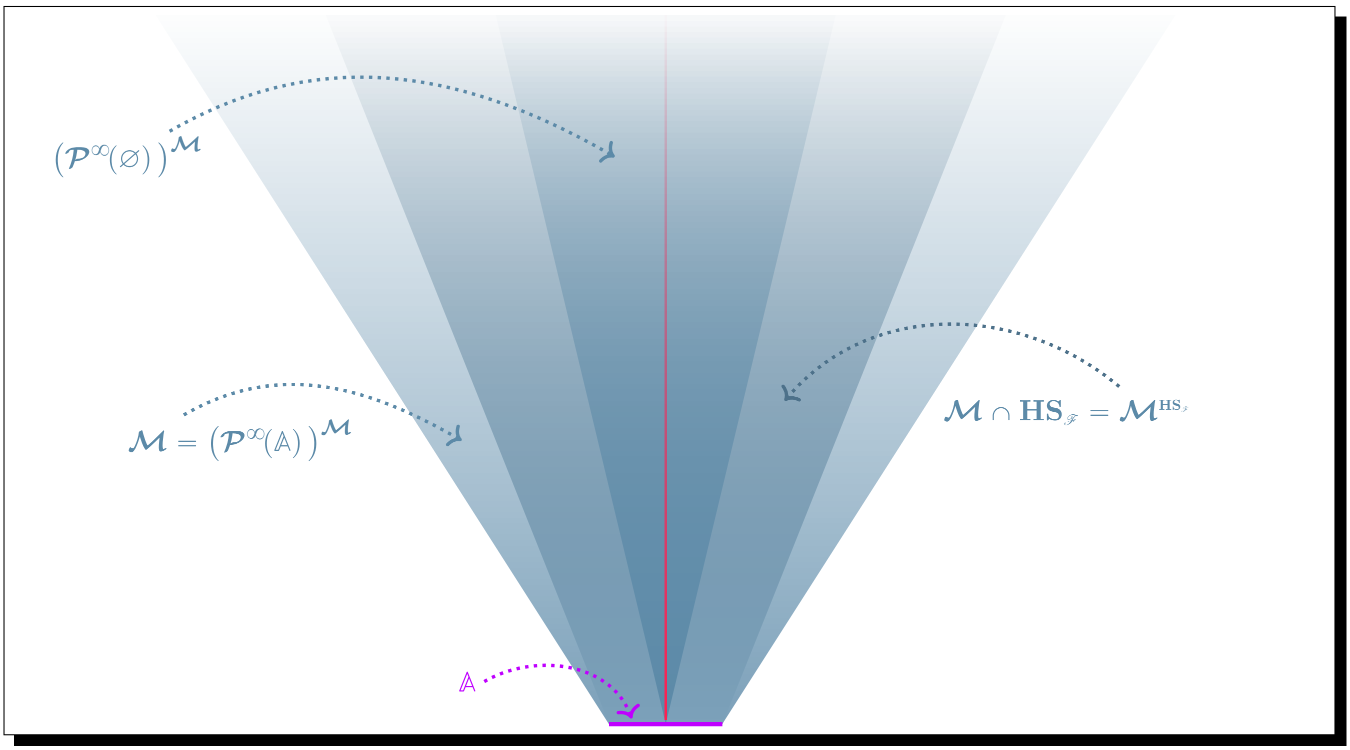


Figure 23.2: The class $\mathcal{M}$, its core model $\big(\mathcal{P}^{\infty}(\varnothing)\big)^{\mathcal{M}}$, and the permutation model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$.

Definition 395 (Permutation Model). *Let $\mathcal{M}$ be any model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$. The submodel of $\mathcal{M}$ formed of all the symmetric sets of $\mathcal{M}$ is called the permutation model and denoted by:*

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} = \mathcal{M} \cap \mathbf{HS}_{\mathscr{F}}.$$

We show that every permutation model satisfies **ZFA**.

Proposition 396. *Let $\mathcal{M}$ be any transitive model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathscr{F}$ any normal filter on $\mathcal{G}$.*

(1) $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ *is transitive;*

(2) $\mathcal{P}^{\infty}(\varnothing) \subseteq \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$*;*

(3) $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ *satisfies* **ZFA***.*

Proof of Proposition 396:

(1) If $x \in y \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$, then $x \in \mathcal{M}$ since $\mathcal{M}$ is transitive and $x \in \mathbf{HS}_{\mathscr{F}}$ since $y \subseteq \mathbf{HS}_{\mathscr{F}}$, hence $x \in \mathcal{M} \cap \mathbf{HS}_{\mathscr{F}} = \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$

(2) For all $x \in \mathcal{P}^{\infty}(\varnothing)$, and for all $\pi \in \mathcal{G}$,

$$\left.\begin{array}{l} \circ\ \breve{\pi}(x) = x, \text{ hence } sym_{\mathcal{G}}(x) = \mathcal{G} \in \mathscr{F}; \\ \circ\ tc(x) \subseteq \mathcal{P}^{\infty}(\varnothing), \text{ hence } x \subseteq \mathbf{HS}_{\mathscr{F}} \end{array}\right\} x \in \mathbf{HS}_{\mathscr{F}}$$

(3) $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ satisfies **ZFA**:

Empty Set Existence for ZFA comes from $\mathcal{P}^{\infty}(\varnothing) \subseteq \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$.

Extensionality for ZFA is from $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ being transitive.

Comprehension Schema We want to show that for all $w, w_1, \ldots, w_n \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ and formula $\varphi(x, y, y_1, \ldots, y_n)$:

$$u = \left\{ v \in w \mid \Big(\varphi\big(v/x, w/y, w_1/y_1, \ldots, w_n/y_n\big)\Big)^{\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}} \right\} \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}.$$

For this it is enough to consider the following subgroup of $\mathcal{S} \in \mathscr{F}$:

$$\mathcal{S} = sym_{\mathcal{G}}(w) \cap sym_{\mathcal{G}}(w_1) \cap sym_{\mathcal{G}}(w_2) \cap \ldots \cap sym_{\mathcal{G}}(w_n).$$

Notice that, for any $\pi \in \mathcal{S}$ and $v \in w$, we have

$$\begin{aligned} \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\big(v/x, w/y, w_1/y_1, \ldots, w_n/y_n\big) &\iff \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\Big(\breve{\pi}(v)/x, \breve{\pi}(w)/y, \breve{\pi}(w_1)/y_1, \ldots, \breve{\pi}(w_n)/y_n\Big) \\ &\iff \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\big(\breve{\pi}(v)/x, w/y, w_1/y_1, \ldots, w_n/y_n\big) \end{aligned}$$

So, for every $v \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ and every $\pi \in \mathcal{S}$, we have $v \in u \iff \breve{\pi}(v) \in u$. Since $\breve{\pi}(u) = \{\breve{\pi}(v) \mid v \in u\}$, we have shown that shows that $\breve{\pi}(u) = u$ holds for every

$v \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$. Hence,

$$\mathcal{S} = \underbrace{sym_{\mathscr{G}}(w) \cap sym_{\mathscr{G}}(w_1) \cap sym_{\mathscr{G}}(w_2) \cap \ldots \cap sym_{\mathscr{G}}(w_n)}_{\in \mathscr{F}} \subseteq \underbrace{sym_{\mathscr{G}}(u)}_{\in \mathscr{F}}.$$

which shows that $u \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$.

Pairing If $x, y \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$, then $\{x, y\} \in \mathbf{HS}_{\mathscr{F}}$ since $sym_{\mathscr{G}}\big(\{x, y\}\big) \supseteq sym_{\mathscr{G}}(x) \cap sym_{\mathscr{G}}(y)$ and $x, y \in \mathbf{HS}_{\mathscr{F}}$. We obtain

Union Let $x \in \mathbf{HS}_{\mathscr{F}}$, to prove that $\bigcup x \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$, it is enough to show that there exists $u \in \mathbf{HS}_{\mathscr{F}}$ such that $\bigcup x \subseteq u$.

$$u = \Big\{\breve{\pi}(z) \in \breve{\pi}\Big[\underbrace{\bigcup x}_{\in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}}\Big] \mid \exists y \in x \quad (z \in y \ \wedge \ \pi \in \mathcal{G})\Big\}.$$

As described u belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ and $\bigcup x \subseteq u$ holds. To show that u belongs to $\mathbf{HS}_{\mathscr{F}}$, it is enough to notice that $sym_{\mathscr{G}}(u) = \mathcal{G}$ and every $\breve{\pi}(z) \in u$ satisfies $\breve{\pi}(z) \in \mathbf{HS}_{\mathscr{F}}$ since $z \in \mathbf{HS}_{\mathscr{F}}$ holds.

Infinity Since ω belongs to the kernel, it belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$.

Power Set Let $x \in \mathbf{HS}_{\mathscr{F}}$, it is enough to show there exists $u \in \mathbf{HS}_{\mathscr{F}}$ such that $\mathscr{P}\big((x)\big) \cap \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \subseteq u$.

$$\begin{aligned} u &= \Big\{\breve{\pi}(y) \mid (y \in \mathcal{P}(x) \ \wedge \ \pi \in \mathcal{G})\Big\} \\ &= \bigcup \Big\{\breve{\pi}\Big[\mathcal{P}(x)\Big] \mid \pi \in \mathcal{G}\Big\}. \end{aligned}$$

As described u belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ and $\mathcal{P}(x) \subseteq u$ holds. To show that u belongs to $\mathbf{HS}_{\mathscr{F}}$, it is enough to notice that $sym_{\mathscr{G}}(u) = \mathcal{G}$ since given any $\rho \in \mathcal{G}$, we have

$$\begin{aligned} \breve{\rho}(u) &= \Big\{\breve{\rho}\big(\breve{\pi}(y)\big) \mid (y \in \mathcal{P}(x) \ \wedge \ \pi \in \mathcal{G})\Big\} \\ &= \Big\{\breve{\rho} \circ \breve{\pi}(y) \mid (y \in \mathcal{P}(x) \ \wedge \ \pi \in \mathcal{G})\Big\} \\ &= \Big\{\breve{\rho} \circ \breve{\rho}^{-1} \circ \breve{\pi}'(y) \mid \big(y \in \mathcal{P}(x) \ \wedge \ \breve{\rho}^{-1} \circ \breve{\pi}' \in \mathcal{G}\big)\Big\} \\ &= \Big\{\breve{\pi}'(y) \mid (y \in \mathcal{P}(x) \ \wedge \ \breve{\pi}' \in \mathcal{G})\Big\} \\ &= u. \end{aligned}$$

Moreover, every $\breve{\pi}(y) \in u$ satisfies $\breve{\pi}(y) \in \mathbf{HS}_{\mathscr{F}}$ since $y \in \mathbf{HS}_{\mathscr{F}}$ holds.

Foundation holds in $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ since $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ is transitive and **Foundation** holds in $\mathcal{M}$.

Replacement Schema for each formula $\varphi(x, y, z_1, \ldots, z_n)$, we want to prove that given any $w_1 \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}, \ldots, w_n \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$:

$$\left(\begin{array}{c} \forall x \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}\ \exists! y \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}\ \ \big(\varphi(x, y, w_1/z_1, \ldots, w_n/z_n)\big)^{\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}} \\ \longrightarrow \\ \forall u \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}\ \exists v \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}\ \forall x \in u\ \exists y \in v\ \ \big(\varphi(x, y, w_1/z_1, \ldots, w_n/z_n)\big)^{\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}} \end{array} \right).$$

We fix $w_1 \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}, \ldots, w_n \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ and $u \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$ and consider (inside $\mathcal{M}$ which satisfies the **Replacement Schema** since it satisfies **ZFA**) the following set

$$v = \left\{ y \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \mid \big(\exists x \in u\ \ \varphi(x, y, w_1/z_1, \ldots, w_n/z_n)\big)^{\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}} \right\}$$

We consider the subgroup

$$\mathcal{S} = \underbrace{sym_{\mathscr{G}}(u) \cap sym_{\mathscr{G}}(w_1) \cap sym_{\mathscr{G}}(w_2) \cap \ldots \cap sym_{\mathscr{G}}(w_n)}_{\in \mathscr{F}}.$$

Notice that, for any $\pi \in \mathcal{S}$, any $x \in u$ and any $y \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$, we have $\breve{\pi}(x) \in \breve{\pi}(u) = u$ and

$$\begin{aligned} \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi(x, y, w_1, \ldots, w_n) &\Longleftrightarrow \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\Big(\breve{\pi}(x), \breve{\pi}(y), \breve{\pi}(w_1), \ldots, \breve{\pi}(w_n)\Big) \\ &\Longleftrightarrow \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \varphi\big(\breve{\pi}(x), \breve{\pi}(y), w_1, \ldots, w_n\big). \end{aligned}$$

Since we have $x \in u \iff \breve{\pi}(x) \in u$, we have

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \exists x \in u\ \ \varphi\big(x, y, w_1, \ldots, w_n\big) \iff \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}} \models \exists \breve{\pi}(x) \in u\ \ \varphi\big(\breve{\pi}(x), \breve{\pi}(y), w_1, \ldots, w_n\big).$$

Therefore, we have $y \in v \iff \breve{\pi}(y) \in v$, which shows that $\breve{\pi}(v) = v$, hence

$$\mathcal{S} = \underbrace{sym_{\mathscr{G}}(u) \cap sym_{\mathscr{G}}(w_1) \cap sym_{\mathscr{G}}(w_2) \cap \ldots \cap sym_{\mathscr{G}}(w_n)}_{\in \mathscr{F}} \subseteq \underbrace{sym_{\mathscr{G}}(v)}_{\in \mathscr{F}},$$

which shows that $v \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}$.

□ 396

23.3 The Basic Fraenkel Model

Definition 397 (Basic Fraenkel Model)**.** *Let $\mathcal{M}$ be any transitive model of* **ZFA** *with any countable infinite set of atoms $\mathbb{A}$, $\mathscr{G}$ be the group of permutations of $\mathbb{A}$, and $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ be the normal filter generated by*

$$\{fix_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\mathbb{A})\}$$

where

$$fix_{\mathscr{G}}(F) = \{\pi \in \mathscr{G} \mid \forall x \in F \ \ \breve{\pi}(x) = x\}.$$

The submodel of $\mathcal{M}$ formed of all its symmetric sets is the permutation model known as the basic Fraenkel Model:

$$\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathscr{F}}} = \mathcal{M} \cap \mathbf{HS}_{\mathscr{F}}.$$

To say that the filter $\mathscr{F}$ is generated by $\{fix_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\mathbb{A})\}$ means that

$$\mathscr{F} = \Big\{\mathcal{H} \subseteq \mathcal{G} \mid \mathcal{H} \text{ is a subgroup of } \mathcal{G} \text{ and } fix_{\mathscr{G}}(F) \subseteq \mathcal{H} \text{ holds for some finite } F \subseteq \mathbb{A}\Big\}.$$

This set $\mathscr{F}$ of subgroups of $\mathcal{G}$ is a normal filter on $\mathcal{G}$ because for all subgroups $\mathcal{H}$, $\mathcal{K}$ of $\mathcal{G}$ and all permutations $\pi \in \mathcal{G}$ we have:

(1) $\mathscr{G} \in \mathscr{F}$ because $\mathscr{G} = fix_{\mathscr{G}}(\varnothing)$.

(2) If $\mathcal{H} \in \mathscr{F}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $fix_{\mathscr{G}}(F) \subseteq \mathcal{H} \subseteq \mathcal{K}$ holds for some finite $F \subseteq \mathbb{A}$, which shows $\mathcal{K} \in \mathscr{F}$.

(3) If $\mathcal{H} \in \mathscr{F}$ and $\mathcal{K} \in \mathscr{F}$, then $fix_{\mathscr{G}}(E) \subseteq \mathcal{H}$ and $fix_{\mathscr{G}}(F) \subseteq \mathcal{K}$ holds for some finite $E, F \subseteq \mathbb{A}$. Thus, $fix_{\mathscr{G}}(E \cup F) \subseteq \mathcal{H} \cap \mathcal{K}$ also holds, which shows that $\mathcal{H} \cap \mathcal{K} \in \mathscr{F}$.

(4) If $\mathcal{H} \in \mathscr{F}$, then given any finite $F \subseteq \mathbb{A}$ such that $fix_{\mathscr{G}}(F) \subseteq \mathcal{H}$, one has, for any permutation π, $\pi \circ fix_{\mathscr{G}}(F) \circ \pi^{-1} = fix_{\mathscr{G}}(\pi[F])$; so that $fix_{\mathscr{G}}(\pi[F]) \subseteq \pi \circ \mathcal{H} \circ \pi^{-1}$, which shows that $\pi \circ \mathcal{H} \circ \pi^{-1} \in \mathscr{F}$.

(5) For each atom $a \in \mathbb{A}$, $\{\pi \in \mathcal{G} \mid \pi(a) = a\} = fix_{\mathscr{G}}(\{a\}) \in \mathscr{F}$.

For any set x, we call **support** of x any $F_x \in \mathcal{P}_{fin}(\mathbb{A})$ which satisfies $fix_{\mathscr{G}}(F_x) \subseteq sym_{\mathscr{G}}(x)$. Notice that if F_x is a support of x and $F_x \subseteq F \in \mathcal{P}_{fin}(\mathbb{A})$ holds, then $fix_{\mathscr{G}}(F) \subseteq fix_{\mathscr{G}}(F_x) \subseteq sym_{\mathscr{G}}(x)$ holds as well, so that F is also a support of x.

Lemma 398. *We use the same assumptions as in Definition 397 (the definition of $\mathcal{M}_{\mathcal{F}_0}^{\mathbf{HS}_{\mathscr{F}}}$). If $F \in \mathcal{P}_{fin}(\mathbb{A})$, $S \subseteq \mathbb{A}$ and $fix_{\mathscr{G}}(F) \subseteq sym_{\mathscr{G}}(S) \in \mathscr{F}$, then*

- *S is either finite or co-finite (i.e., $\mathbb{A} \setminus S$ is finite);*
- *if S is finite, then $S \subseteq F$;*
- *if S is co-finite, then $(\mathbb{A} \setminus S) \subseteq F$.*

Proof of Lemma 398: We distinguish between $S \cap (\mathbb{A} \setminus F) = \varnothing$ and $S \cap (\mathbb{A} \setminus F) \neq \varnothing$.

- If $S \cap (\mathbb{A} \setminus F) = \varnothing$, then $S \subseteq F$, and in particular S is finite.
- If $S \cap (\mathbb{A} \setminus F) \neq \varnothing$, we show that $S \supseteq (\mathbb{A} \setminus F)$, and in particular S is co-finite. We fix some $a \in S \cap (\mathbb{A} \setminus F)$ and consider any $b \in (\mathbb{A} \setminus F)$ such that $b \neq a$ — since $\mathbb{A} \setminus F$ is infinite, such b exists. The permutation $\pi_{a \leftrightarrow b}$ which exchanges a and b, and is the identity everywhere else, belongs to $fix_{\mathscr{G}}(F)$. Now we have

$$\begin{aligned} a \in S \quad &\Longleftrightarrow \quad \breve{\pi}_{a\leftrightarrow b}(a) \in \breve{\pi}_{a\leftrightarrow b}(S) \\ &\Longleftrightarrow \quad \breve{\pi}_{a\leftrightarrow b}(a) \in S \\ &\Longleftrightarrow \quad b \in S \end{aligned}$$

  which shows that $(\mathbb{A} \setminus F) \subseteq S$.

So, we have shown that we have either $S \subseteq F$ or $(\mathbb{A} \setminus S) \subseteq F$, which also shows that S is either finite or co-finite.

□ 398

We now show that inside the basic Fraenkel model, there exists some set which is both infinite and Dedekind-finite.

Proposition 399. *Let $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ be the basic Fraenkel model with $\mathbb{A}$ as set of atoms.*

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \aleph_0 \overset{1-1}{\not\lesssim} \mathbb{A}.$$

So, although the basic Fraenkel model is built from a set of atoms which is infinite and countable, the model itself cannot recognize this fact, for there is no injection from the integers to the set of atoms.

Proof of Proposition 399: Towards a contradiction, we assume that inside $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ there exists $f : \aleph_0 \xrightarrow{1-1} \mathbb{A}$. Then the set

$$S = \{f(2n) \in \mathbb{A} \mid n \in \omega\}$$

belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ since $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ is a model of **ZFA**. So, we have $sym_{\mathscr{G}}(S) \in \mathscr{F}$, hence there exists some finite $F \subseteq \mathbb{A}$ such that $fix_{\mathscr{G}}(F) \subseteq sym_{\mathscr{G}}(S)$. By Lemma 398, either S finite or $\mathbb{A} \setminus S$ is finite; a contradiction.

□ 399

Proposition 400. *Let $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ be the basic Fraenkel model with $\mathbb{A}$ as set of atoms.*

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \aleph_0 \overset{1-1}{\not\lesssim} \mathcal{P}(\mathbb{A}).$$

Proof of Proposition 400: Towards a contradiction, we assume that inside $\mathcal{M}^{\mathbf{HS}_{\mathscr{G}}}_{\mathcal{F}_0}$ there exists $f : \aleph_0 \xrightarrow{1-1} \mathcal{P}(\mathbb{A})$. Since f belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{G}}}_{\mathcal{F}_0}$, there exists some finite $F_f \subseteq \mathbb{A}$ such that

$$\mathit{fix}_{\mathscr{G}}(F_f) \subseteq \mathit{sym}_{\mathscr{G}}(f).$$

By Lemma 398, any $S \subseteq \mathbb{A}$ that satisfies $\mathit{fix}_{\mathscr{G}}(F_f) \subseteq \mathit{sym}_{\mathscr{G}}(S)$ satisfies also either $S \subseteq F_f$ or $(\mathbb{A} \smallsetminus S) \subseteq F_f$. Therefore, there exist only finitely many such sets S. So, take any $n \in \omega$ such that $f(n) \subseteq \mathbb{A}$ satisfies

$$\mathit{fix}_{\mathscr{G}}(F_f) \nsubseteq \mathit{sym}_{\mathscr{G}}\big(f(n)\big).$$

Take any $\pi \in \mathit{fix}_{\mathscr{G}}(F_f) \smallsetminus \mathit{sym}_{\mathscr{G}}\big(f(n)\big)$ in order to have both

$$\breve{\pi}(f) = f \quad \text{and} \quad \breve{\pi}\big(f(n)\big) \neq f(n).$$

Since n belongs to the kernel, $\breve{\pi}(n) = n$ holds, which leads to $f\big(\breve{\pi}(n)\big) = f(n)$.
By construction,

$$\begin{aligned}\breve{\pi}(f) &= \breve{\pi}\Big(\big\{\big(k, f(k)\big) \mid k \in \omega\big\}\Big)\\ &= \Big\{\Big(\breve{\pi}(k), \breve{\pi}\big(f(k)\big)\Big) \mid k \in \omega\Big\}\\ &= \Big\{\Big(k, \breve{\pi}\big(f(k)\big)\Big) \mid k \in \omega\Big\}.\end{aligned}$$

So that, in particular, we have

$$\breve{\pi}(f)(n) = \breve{\pi}\big(f(n)\big).$$

But, since $\pi \in \mathit{fix}_{\mathscr{G}}(F_f)$, we also have $\breve{\pi}(f) = f$, hence $\breve{\pi}(f)(n) = f(n)$ which contradicts $\breve{\pi}\big(f(n)\big) \neq f(n)$.

□ 400

23.4 The Second Fraenkel Model

Definition 401 (Second Fraenkel Model)**.** *Let* $\mathcal{M}$ *be any transitive model of* **ZFA** *whose set of atoms is*

$$\mathbb{A} = \bigcup_{n\in\omega} P_n, \text{ where } (P_n)_{n\in\omega} \text{ is a family of disjoint pairs}^1;$$

and the subgroup of permutations $\mathscr{G}$ *is*

$$\mathscr{G} = \big\{\pi : \mathbb{A} \xrightarrow{bij.} \mathbb{A} \mid \forall n \in \omega \;\; \breve{\pi}(P_n) = P_n\big\};$$

(i.e., $\mathscr{G}$ *is the group of permutations of* $\mathbb{A}$ *which preserves the pairs).*

Let also $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ be the normal filter generated by

$$\left\{ \mathit{fix}_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{\mathit{fin}}(\mathbb{A}) \right\}$$

where

$$\mathit{fix}_{\mathscr{G}}(F) = \left\{ \pi \in \mathscr{G} \mid \forall x \in F \;\; \breve{\pi}(x) = x \right\}.$$

The submodel of $\boldsymbol{\mathcal{M}}$ formed of all its symmetric sets is the permutation model known as the second Fraenkel Model:

$$\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} = \boldsymbol{\mathcal{M}} \cap \mathbf{HS}_{\mathscr{F}}.$$

Notice that the set of atoms of the second Fraenkel model is made of the elements of countably many disjoints pairs.

Lemma 402. *Let $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ be the second Fraenkel model as described in Definition 401. i.e., its set of atoms is $\mathbb{A} = \bigcup_{n\in\omega} P_n$ with each P_n being a pair, and for all $\neq m$, $P_n \cap P_m = \varnothing$. We then have*

(1) for each integer n, the set P_n belongs to $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$,

(2) the mapping $f = \left\{ (n, P_n) \mid n \in \omega \right\}$ belongs to $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$.

Proof of Lemma 402:

(1) By construction, every $\pi \in \mathcal{G}$ satisfies $\breve{\pi}(P_n) = P_n$, hence $\mathit{sym}_{\mathscr{G}}\big(P_n\big) = \mathcal{G} \in \mathscr{F}$. So, each set P_n is symmetric, hence hereditarily symmetric, so it belongs to $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$.

(2) For each $\pi \in \mathcal{G}$, and each $n \in \omega$, we have $\breve{\pi}(n) = n$ because n belongs to the kernel. Therefore,

$$\breve{\pi}(f) = \breve{\pi}\Big(\left\{(n, P_n) \mid n \in \omega\right\}\Big) = \Big\{ \big(\breve{\pi}(n), \breve{\pi}(P_n)\big) \mid n \in \omega \Big\} = \left\{(n, P_n) \mid n \in \omega\right\} = f.$$

So, $\mathit{sym}_{\mathscr{G}}(f) = \mathcal{G} \in \mathscr{F}$ which shows that f is symmetric. Since all elements of f are hereditarily symmetric, f is hereditarily symmetric as well, hence f belongs to $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$.

[1]It would not be wise here to present the pair P_n by saying $P_n = \{a_n, b_n\}$, because this would mean that we already have some ordering on the elements of P_n, from which we could easily get a choice function $\mathsf{c} : \omega \longrightarrow \mathbb{A}$ which chooses one item in each pair. For instance the following

$$\begin{aligned} \mathsf{c} : \omega &\longrightarrow \mathbb{A} \\ n &\mapsto a_n \end{aligned}$$

which is precisely what we want to prevent from happening as Theorem 403 will show on page 376.

□ 402

Theorem 403. *Let $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ be the second Fraenkel model — described in Definition 401 — whose set of atoms is $\mathbb{A} = \bigcup_{n\in\omega} P_n$, and each P_n is a pair.*

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \text{“}\{P_n \mid n \in \omega\} \textit{ does not admit any choice function”}.$$

i.e.,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \not\models \text{“}\textit{there is some mapping } \mathsf{c} : \omega \longrightarrow \mathbb{A} \textit{ that satisfies } \ \forall n \in \omega \ \ \mathsf{c}(n) \in P_n\text{”}.$$

This theorem says that there is no function $\mathsf{c} : \omega \longrightarrow \mathbb{A}$ which satisfies that for each integer n, $\mathsf{c}(n)$ belongs to P_n. Since each set P_n contains exactly two elements, one may think of these elements as socks which the model considers so indistinguishable that it cannot make up its mind when it comes to picking exactly one of them in each pair.

Proof of Theorem 403: Towards a contradiction, we assume that inside $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ there exists such a choice function $\mathsf{c} : \omega \xrightarrow{1-1} \bigcup\{P_n \mid n \in \omega\}$. Since c belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$, there exists some finite $F_\mathsf{c} \subseteq \mathbb{A}$ such that

$$\mathit{fix}_{\mathscr{G}}(F_\mathsf{c}) \subseteq \mathit{sym}_{\mathscr{G}}(\mathsf{c}).$$

Pick k large enough such that $F_\mathsf{c} \cap P_k = \varnothing$ as well as $\pi \in \mathit{fix}_{\mathscr{G}}(F_\mathsf{c})$ that satisfies $\breve{\pi}\big(\mathsf{c}(k)\big) \neq \mathsf{c}(k)$. i.e., if $P_k = \{a_k, b_k\}$, this means that we both have $\breve{\pi}(a_k) = b_k$ and $\breve{\pi}(b_k) = a_k$ (we should not mention a_k nor b_k, but rather go with the more convoluted $\breve{\pi}\big(\mathsf{c}(k)\big) \neq \mathsf{c}(k)$).

We then have the following contradiction:

- $\breve{\pi}(\mathsf{c}) = \mathsf{c}$ (because $\pi \in \mathit{fix}_{\mathscr{G}}(F_\mathsf{c})$),
- $\breve{\pi}\big(\mathsf{c}(k)\big) \neq \mathsf{c}(k)$ (by construction), and
- $$\begin{aligned}\breve{\pi}(\mathsf{c}) &= \Big\{\breve{\pi}\big(n, \mathsf{c}(n)\big) \mid n \in \omega\Big\}\\ &= \Big\{\Big(\breve{\pi}(n), \breve{\pi}\big(\mathsf{c}(n)\big)\Big) \mid n \in \omega\Big\}\\ &= \Big\{\Big(n, \breve{\pi}\big(\mathsf{c}(n)\big)\Big) \mid n \in \omega\Big\} \not\ni \big(k, \mathsf{c}(k)\big)\\ &\neq \mathsf{c} = \Big\{\big(n, \mathsf{c}(n)\big) \mid n \in \omega\Big\} \ni \big(k, \mathsf{c}(k)\big)\end{aligned}$$

which contradicts the fact that $\pi \in \mathit{sym}_{\mathscr{G}}(\mathsf{c})$.

□ 403

We can make use of the same model to show that there exists a model that does not satisfy König Lemma which is the following well-known result:

König Lemma ($\mathbf{AC}_\omega$)**.** *Every infinite finitely branching tree admits an infinite branch.*

Weak König Lemma ($\mathbf{AC}_\omega$)**.** *Every infinite binary branching tree admits an infinite branch.*

Proof of König Lemma: Let $T \subseteq E^{<\omega}$ be infinite. Since T is finitely branching, for every integer n, $T \cap {}^nE$ is finite, hence T is countable[2]. With the help of $\mathbf{AC}_\omega$ we can equip T with a well-ordering $<_T$.

Given any finite sequence $s \in T$, we use the notation $T_{[s]}$ to denote the subtree rooted at s, namely:

$$T_{[s]} = \left\{ s' \in E^{<\omega} \mid s^\frown s' \in T \right\}.$$

Notice first that given any infinite finitely branching tree T, the following set is non-empty:

$$\left\{ s \in {}^1E \mid T_{[s]} \text{ is infinite} \right\}.$$

By recursion on the integers we define $b : \omega \to T$ such that for each n we have $b(n) \in T \cap {}^nE$.

- $b(0) = \varnothing$
- $b(n+1) = <_T$ -least element in $\left\{ s \in T \cap {}^{n+1}E \mid \left(s \restriction n = b(n) \wedge T_{[s]} \text{ is infinite} \right) \right\}$.

Then $(b_n)_{n\in\omega}$ is the desired infinite branch of T.

□ König Lemma

As we already said, Theorem 406 typically contradicts Weak König Lemma:

Theorem 406. *Let $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ be the second Fraenkel model from Definition 401, whose set of atoms is made up of*
$\mathbb{A} = \bigcup_{n\in\omega} P_n$*, where each P_n is a pair,*

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \text{“there is an infinite binary tree without any infinite branch”}.$$

i.e.,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \not\models \text{“Weak König Lemma”}.$$

[2]Notice that since we assume $\mathbf{AC}_\omega$, we have that a countable union of countable sets is countable.

Proof of Theorem 406: We set

$$T = \bigcup_{n \in \omega} \Big\{ s \in {}^{n}\mathbb{A} \mid \forall k \in n \; s(k) \in P_k \Big\}.$$

Any infinite branch would yield a choice function picking, for each integer n, an element inside P_n, hence contradicting Theorem 403.

□ 406

23.5 The Ordered Mostowski Model

Definition 407 (Ordered Mostowski Model)**.** *Let $\boldsymbol{\mathcal{M}}$ be any transitive model of* **ZFA** *whose set of atoms is a countable set $\mathbb{A}$ equipped with a binary relation $<_{\mathbf{M}} \subseteq \mathbb{A} \times \mathbb{A}$ which is a dense order without least nor greatest element. i.e., $(\mathbb{A}, <_{\mathbf{M}})$ is isomorphic to $(\mathbb{Q}, <)$.*

We let $\mathscr{G}$ be the group of all order preserving permutations of $\mathbb{A}$. i.e.,

$$\mathscr{G} = \Big\{\pi : \mathbb{A} \xrightarrow{bij.} \mathbb{A} \mid \forall \mathbb{a} \in \mathbb{A}\, \forall \mathbb{b} \in \mathbb{A}\ \big(\mathbb{a} <_{\mathbf{M}} \mathbb{b} \longleftrightarrow \pi(\mathbb{a}) <_{\mathbf{M}} \pi(\mathbb{b})\big)\Big\}.$$

Let $\mathscr{F} \subseteq \mathcal{P}(\mathscr{G})$ be the normal filter generated by

$$\big\{fix_{\mathscr{G}}(F) \subseteq \mathscr{G} \mid F \in \mathcal{P}_{fin}(\mathbb{A})\big\},$$

which can be proved to be normal.

The ordered Mostowski model $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$ is the corresponding permutation model.

Notation 408. *Given any set y, we call **support** of y any $F_y \in \mathcal{P}_{fin}(\mathbb{A})$ which satisfies*

$$fix_{\mathscr{G}}(F_y) \subseteq sym_{\mathscr{G}}(y)\,.$$

Notice that if F_y is a support of y and $F_y \subseteq F \in \mathcal{P}_{fin}(\mathbb{A})$ holds, then F is also a support of y since we have

$$fix_{\mathscr{G}}(F) \subseteq fix_{\mathscr{G}}(F_y) \subseteq sym_{\mathscr{G}}(y)\,.$$

Lemma 409. *Let $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$ be the ordered Mostowski model described in Definition 407.*

(1) The order $<_{\mathbf{M}}$ belongs to $\boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$, where

$$<_{\mathbf{M}} = \big\{(\mathbb{a}, \mathbb{b}) \in \mathbb{A} \times \mathbb{A} \mid \mathbb{a} <_{\mathbf{M}} \mathbb{b}\big\}.$$

(2) (a) If F and F' are two supports of y, then $F \cap F'$ is also a support of y.

(b) For each set $x \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$, there exists some $\subseteq$-least support of x.

(c) The following class is symmetric:

$$\big\{(x, E) \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \times \mathcal{P}_{fin}(\mathbb{A}) \mid E \text{ is the } \subseteq\text{-least support of } x\big\}.$$

(3) For all $F \in \mathcal{P}_{fin}(\mathbb{A})$, if F has n elements, then there exist exactly 2^{2n+1} sets S belonging to $\mathcal{P}(\mathbb{A})$ such that F is a support of S.

Proof of Lemma 409:

(1) For every permutation $\pi \in \mathscr{G}$ and every $(\mathbb{a}, \mathbb{b}) \in \mathbb{A} \times \mathbb{A}$ we have

$$\begin{aligned}(\mathbb{a}, \mathbb{b}) \in <_{\mathbf{M}} &\iff \mathbb{a} <_{\mathbf{M}} \mathbb{b} \\ &\iff \pi(\mathbb{a}) <_{\mathbf{M}} \pi(\mathbb{b}) \\ &\iff \big(\pi(\mathbb{a}), \pi(\mathbb{b})\big) \in <_{\mathbf{M}}) \\ &\iff \breve{\pi}\big(\mathbb{a}, \mathbb{b}\big) \in <_{\mathbf{M}} .\end{aligned}$$

Hence, we have shown that $\breve{\pi}\big(<_{\mathbf{M}} \big) = <_{\mathbf{M}}$ holds for every permutation $\pi \in \mathscr{G}$, which yields $sym_{\mathscr{G}}\big(<_{\mathbf{M}} \big) = \mathscr{G} \in \mathscr{F}$, thus $<_{\mathbf{M}} \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$.

(2) (a) Notice that given any permutation $\pi \in \mathit{fix}_{\mathscr{G}}(F \cap F')$, there exist — for some k large enough —permutations

$$\rho_1, \ldots, \rho_k \in \mathit{fix}_{\mathscr{G}}(F) \quad \text{and} \quad \rho'_1, \ldots, \rho'_k \in \mathit{fix}_{\mathscr{G}}(F')$$

such that

$$\rho_1 \circ \rho'_1 \circ \rho_2 \circ \rho'_2 \circ \ldots \rho_k \circ \rho'_k = \pi.$$

This is better seen on an example: assume $F = \{\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3, \mathbb{a}_4\}$ and $F' = \{\mathbb{a}_1, \mathbb{b}_2, \mathbb{a}_4\}$ with $F \cap F' = \{\mathbb{a}_1, \mathbb{a}_4\}$ and

$$\mathbb{a}_1 <_{\mathbf{M}} \mathbb{a}_2 <_{\mathbf{M}} \mathbb{b}_2 <_{\mathbf{M}} \mathbb{a}_3 <_{\mathbf{M}} \mathbb{a}_4$$

Assume π satisfies $\mathbb{a}_2 <_{\mathbf{M}} \pi(\mathbb{a}_2) <_{\mathbf{M}} \mathbb{b}_2 <_{\mathbf{M}} \pi(\mathbb{b}_2) <_{\mathbf{M}} \pi(\mathbb{a}_3) <_{\mathbf{M}} \mathbb{a}_3$, then take:

(A) ρ' defined by

- on $]-\infty, \mathbb{a}_2]$, $\rho' = \pi$
- on $]\mathbb{a}_2, \mathbb{b}_2[$, $\rho' = \theta$ for some (any) order isomorphism between $]\mathbb{a}_2, \mathbb{b}_2[$ and $]\pi(\mathbb{a}_2), \mathbb{b}_2[$
- $\rho'(\mathbb{b}_2) = \mathbb{b}_2$
- on $]\mathbb{b}_2, \mathbb{a}_3[$, $\rho' = \delta$ for some (any) order isomorphism between $]\mathbb{b}_2, \mathbb{a}_3[$ and $]\mathbb{b}_2, \pi(\mathbb{a}_3)[$
- $\rho'(\mathbb{a}_3) = \pi(\mathbb{a}_3)$
- on $]\mathbb{a}_3, +\infty]$, $\rho' = \pi$

(B) ρ defined by

- on $]-\infty, \pi(\mathbb{a}_2)]$, $\rho = id$
- on $]\pi(\mathbb{a}_2), \mathbb{b}_2[$, ρ satisfies $\theta \circ \rho = \pi$
- $\rho(\mathbb{b}_2) = \pi(\mathbb{b}_2)$
- on $]\mathbb{b}_2, \mathbb{a}_3[$, ρ satisfies $\delta \circ \rho = \pi$
- $\rho(\mathbb{a}_3) = \mathbb{a}_3$

- on $]\mathbb{a}_3, +\infty]$, $\rho = id$

Notice that $\rho' \in \mathit{fix}_{\mathscr{G}}(F')$ and $\rho \in \mathit{fix}_{\mathscr{G}}(F)$ and $\rho \circ \rho' = \pi$.

(b) Take any $F \in \mathcal{P}_{\mathit{fin}}(\mathbb{A})$ such that $\mathit{fix}_{\mathscr{G}}(F) \subseteq \mathit{sym}_{\mathscr{G}}(x)$ and consider

$$E = \bigcap \{ F' \subseteq F \mid \mathit{fix}_{\mathscr{G}}(F') \subseteq \mathit{sym}_{\mathscr{G}}(x) \}.$$

Clearly $\mathit{fix}_{\mathscr{G}}(E) \subseteq \mathit{sym}_{\mathscr{G}}(x)$ and E is $\subseteq$-minimal.

(c) For any $\pi \in \mathscr{G}$ we have $\breve{\pi}(x, E) = (\breve{\pi}(x), \breve{\pi}(E))$. Moreover, $\mathit{fix}_{\mathscr{G}}(\breve{\pi}(E)) = \pi \circ \mathit{fix}_{\mathscr{G}}(E) \circ \pi^{-1}$ and $\mathit{sym}_{\mathscr{G}}(\breve{\pi}(x)) = \pi \circ \mathit{sym}_{\mathscr{G}}(x) \circ \pi^{-1}$. So, if E is the $\subseteq$-least support of x, then $\breve{\pi}(E)$ is the $\subseteq$-least support of $\breve{\pi}(x)$. Therefore, we have shown that for all $\pi \in \mathscr{G}$,

$$\mathit{sym}_{\mathscr{G}}\Big(\{(x, E) \in \boldsymbol{\mathcal{M}}^{\mathbf{HS}_{\mathscr{F}}}_{\mathit{ost.}} \times \mathcal{P}_{\mathit{fin}}(\mathbb{A}) \mid E \text{ is least support of } x\}\Big) = \mathscr{G} \in \mathscr{F}.$$

(3) Assume $F = \{\mathbb{a}_1, \ldots, \mathbb{a}_n\}$ with $\mathbb{a}_1 <_{\mathbf{M}} \ldots <_{\mathbf{M}} \mathbb{a}_n$ and F is a support of S. We have for every $\mathbb{b} \in S$:

(a) if $\mathbb{b} <_{\mathbf{M}} \mathbb{a}_1$, then $\{\mathbb{c} \in \mathbb{A} \mid \mathbb{c} <_{\mathbf{M}} \mathbb{a}_1\} \subseteq S$ holds since for any $\mathbb{c} <_{\mathbf{M}} \mathbb{a}_1$ there exists some mapping $\pi \in \mathit{fix}_{\mathscr{G}}(F)$ which satisfies $\pi(\mathbb{b}) = \mathbb{c}$. So, we have

$$\begin{aligned} \mathbb{b} \in S &\Longrightarrow \pi(\mathbb{b}) \in \breve{\pi}(S) \\ &\Longrightarrow \mathbb{c} \in S. \end{aligned}$$

(b) if $\mathbb{a}_n <_{\mathbf{M}} \mathbb{b}$, then $\{\mathbb{c} \in \mathbb{A} \mid \mathbb{a}_n <_{\mathbf{M}} \mathbb{c}\} \subseteq S$ since for any $\mathbb{a}_n <_{\mathbf{M}} \mathbb{c}$ there exists some mapping $\pi \in \mathit{fix}_{\mathscr{G}}(F)$ which satisfies $\pi(\mathbb{b}) = \mathbb{c}$. So, we have

$$\begin{aligned} \mathbb{b} \in S &\Longrightarrow \pi(\mathbb{b}) \in \breve{\pi}(S) \\ &\Longrightarrow \mathbb{c} \in S. \end{aligned}$$

(c) if $\mathbb{a}_i <_{\mathbf{M}} \mathbb{b} <_{\mathbf{M}} \mathbb{a}_{i+1}$ then $\{\mathbb{c} \in \mathbb{A} \mid \mathbb{a}_i <_{\mathbf{M}} \mathbb{c} <_{\mathbf{M}} \mathbb{a}_{i+1}\} \subseteq S$ since for any $\mathbb{a}_i <_{\mathbf{M}} \mathbb{c} <_{\mathbf{M}} \mathbb{a}_{i+1}$ there exists some mapping $\pi \in \mathit{fix}_{\mathscr{G}}(F)$ which satisfies $\pi(\mathbb{b}) = \mathbb{c}$. So, we have

$$\begin{aligned} \mathbb{b} \in S &\Longrightarrow \pi(\mathbb{b}) \in \breve{\pi}(S) \\ &\Longrightarrow \mathbb{c} \in S. \end{aligned}$$

So, there are

- exactly $n + 1$ such intervals, such that, for each of them, either it entirely belongs to S or it is disjoint from S.
- exactly n atoms in F, each of which may or may not belong to S.

So, there are as many sets of the form $S \subseteq \mathbb{A}$ with F as support, as there are mappings from $n+1+n$ to $\{0,1\}$ which makes a total of 2^{2n+1} different subsets of $\mathbb{A}$ whose support is a finite subset of $\mathbb{A}$ with n many atoms.

□ 409

Theorem 410. *Let $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$ be the ordered Mostowski model described in Definition 407. (In particular, its set of atoms is a countable set $\mathbb{A}$ equipped with a binary relation $<_{\mathbf{M}} \subseteq \mathbb{A} \times \mathbb{A}$ which is a dense order without least nor greatest element.)*

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models \text{“ there exists some mapping } f : \mathcal{P}_{fin}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A}) \text{ ”}.$$

Proof of Theorem 410:

(1) For all support $F = \{\mathbb{a}_1, \ldots, \mathbb{a}_n\}$ with $\mathbb{a}_1 <_{\mathbf{M}} \ldots <_{\mathbf{M}} \mathbb{a}_n$, define a mapping

$$\begin{array}{cccc} \mathscr{S}: & {}^{2n+1}2 & \longrightarrow & \mathcal{P}(\mathbb{A}) \\ & \chi & \mapsto & \mathscr{S}(\chi) \end{array}$$

so that $\left\{\mathscr{S}(\chi) \mid \chi \in {}^{2n+1}2\right\}$ is the set of all subsets of $\mathbb{A}$ whose support is F. Namely,

$$\mathscr{S}(\chi) = \bigcup \left\{I_k \subseteq \mathbb{A} \mid 0 \leqslant k \leqslant n \ \wedge \ \chi(2k) = 1\right\} \cup \left\{a_k \in \mathbb{A} \mid 1 \leqslant k \leqslant n \ \wedge \ \chi(2k-1) = 1\right\}$$

where

- $I_0 =]-\infty, \mathbb{a}_1[= \left\{\mathbb{b} \in \mathbb{A} \mid \mathbb{b} <_{\mathbf{M}} \mathbb{a}_1\right\}$
- $I_k =]\mathbb{a}_k, \mathbb{a}_{k+1}[= \left\{\mathbb{b} \in \mathbb{A} \mid \mathbb{a}_k <_{\mathbf{M}} \mathbb{b} <_{\mathbf{M}} \mathbb{a}_{k+1}\right\}$ (any $1 \leqslant k < n$)
- $I_n =]\mathbb{a}_n, +\infty[= \left\{\mathbb{b} \in \mathbb{A} \mid \mathbb{a}_n <_{\mathbf{M}} \mathbb{b}\right\}$.

(2) We are now able to show that inside the *Mostowski model* $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.}$ there exists some mapping

$$f : \mathcal{P}_{fin}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A}).$$

We equip $2^{<\omega}$ with the lexicographic ordering $<_{lex.}$ defined by

$$\chi <_{lex.} \chi' \iff \exists i \left(\chi(i) = 0 \ \wedge \ \chi(i) = 1 \ \wedge \ \forall j < i \ \chi(j) = \chi'(j)\right).$$

For every sequence $\chi \in 2^{<\omega}$, we write $\overset{\leftrightsquigarrow}{\chi}$ for the sequence obtained from χ by swapping 0's and 1's. Namely, $\overset{\leftrightsquigarrow}{\chi}$ has the same length as χ, and for every integer $n < lh(\chi)$, $\chi(n) = 1 - \overset{\leftrightsquigarrow}{\chi}(n)$ holds.

We define a mapping $g : 2^{<\omega} \to 2^{<\omega}$ by $g(\varnothing) = \varnothing$ and for any non-empty sequence χ,

$$\begin{array}{rcll} g(\chi) & = & \chi & \text{if} \quad \chi(0) = 0 \\ & = & \overset{\leftrightsquigarrow}{\chi} & \text{if} \quad \chi(0) = 1 \end{array}$$

So, $g(\chi)$ is the only sequence inside $\{\chi, \overset{\leftrightsquigarrow}{\chi}\}$ which starts with a 0.

For every integer n and every sequence $\chi \in 2^n$ we write $\chi^\frown \langle 0 \rangle$ for the sequence in 2^{n+1} which satisfies $\chi^\frown \langle 0 \rangle \upharpoonright n = \chi$ and $\chi^\frown \langle 0 \rangle\, (n) = 0$.

We define an ordering $\prec_n$ on ${}^{2n+1}2$ by

$$\chi \prec_n \chi' \iff g(\chi^\frown \langle 0 \rangle) <_{lex.} g(\chi'^\frown \langle 0 \rangle).$$

and denote by

$$\begin{array}{rcl} \chi_{(n,-)} : 2^{2n+1} & \xrightarrow{onto} & {}^{2n+1}2 \\ i & \mapsto & \chi_{(n,i)} \end{array}$$

the enumeration of ${}^{2n+1}2$ along $\prec_n$. i.e., we have

$$\chi_{(n,0)} \prec_n \chi_{(n,1)} \prec_n \ldots \prec_n \chi_{(n,2^{2n+1}-1)}.$$

We finally define the surjection by

$$\begin{array}{rcl} f : \mathcal{P}_{\mathit{fin}}(\mathbb{A}) & \xrightarrow{onto} & \mathcal{P}(\mathbb{A}) \\ F \neq \varnothing & \mapsto & (\chi_{(|F|,|F|)}) \\ \varnothing & \mapsto & \varnothing. \end{array}$$

So, if the cardinality of F is n, then $\chi_{(|F|,|F|)}$ is the n^{th} mapping — with regard to the ordering $\prec_n$ — of the form $\chi : 2n+1 \to \{0,1\}$.

This mapping belongs to the *Mostowski model* $\mathcal{M}_{ost.}^{\mathbf{HS}_{\mathscr{F}}}$, essentially because, as a permutation model, it satisfies **ZFA** .

It remains to show that f is onto. For this purpose, take any $S \in \mathcal{P}(\mathbb{A}) \smallsetminus \varnothing$. Assume the $\subseteq$-least support of S is some $F \in \mathcal{P}_{\mathit{fin}}(\mathbb{A})$ with $|F| = n$. By the construction presented in (1) on page 382 together with the enumeration above, there exists some integer $i < 2^{2n+1}$ such that

$$\mathscr{S}(\chi_{(n,i)}) = S.$$

Because of the whole construction and mainly the following two different facts, we have $i \geqslant n$.

- F being the $\subseteq$-least support of S, there can be neither 3 consecutive 0's nor 3 consecutive 1's in $\chi_{(n,i)}$. Otherwise one could eliminate at least one atom from F while still preserving the fact that F is a support of

$$S = \mathscr{S}(\chi_{(n,i)}) = \begin{pmatrix} \bigcup \{ I_k \subseteq \mathbb{A} \mid 0 \leqslant k \leqslant n \ \wedge \ \chi_{(n,i)}(2k) = 1 \} \\ \cup \\ \{ a_k \in \mathbb{A} \mid 1 \leqslant k \leqslant n \ \wedge \ \chi_{(n,i)}(2k-1) = 1 \}. \end{pmatrix}$$

- By construction of the ordering $\prec_n$, the fact that it relies on the mapping $g(\chi)$ which picks the only sequence inside $\{\chi, \overset{\leftrightsquigarrow}{\chi}\}$ which starts with a 0, and compares not the sequences $g(\chi)$ lexicographically, but the sequences $g(\chi^\frown \langle 0 \rangle)$, as shown by

$$\chi \prec_n \chi' \iff g(\chi^\frown \langle 0 \rangle) <_{lex.} g(\chi'^\frown \langle 0 \rangle)$$

  guarantees that $i \geqslant n$ holds.

So, if $i = n$, then we are done.

Otherwise, it is tedious but straightforward to check that F can be extended into a set $E \supseteq F$ which satisfies $|E| = i$ and $\mathscr{S}(\chi_{(i,i)}) = \mathscr{S}(\chi_{(n,i)})$, which this time gives the result.

□ 410

Chapter 24

Simulating Permutation Models by Symmetric Models

The main result in this chapter is that one can simulate arbitrary large fragments of permutation models by symmetric submodels of generic extensions. Indeed, we may have $\boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A})$, for γ as large as needed, embed into some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathscr{F}}$. This way, most of the results obtained in the context of permutation models, henceforth in the realm of **ZFA** as opposed to **ZF**, may now be transferred to proper models of **ZF**. This is the case, for instance, of Proposition 399 which states that the basic Fraenkel model cannot recognize that it is built from some infinite and countable set of atoms. This is also the case of Theorem 403 which says that there is a countable family of pairs for which no choice function succeeds in picking exactly one element in each pair.

24.1 The Jech-Sochor Embedding Theorem

The Jech-Sochor Embedding Theorem. *Let $\mathcal{Z}$ be any model of* **ZFA** *with $\mathbb{A}$ as set of atoms, $\mathcal{G}_{\mathbb{A}}$ any subgroup of the group of permutations of $\mathbb{A}$, and $\mathcal{F}_{\mathbb{A}}$ any normal filter on $\mathcal{G}_{\mathbb{A}}$. Let $\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ be the permutation model induced by $\mathcal{Z}$ and $\mathcal{F}_{\mathbb{A}}$. Let also γ be any ordinal and*

$$\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \mathbf{ZFA} + \left(\mathbf{AC}\right)^{\boldsymbol{\mathcal{P}}^{\alpha}(\varnothing)}.$$

There exist

- *a symmetric model*

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$$

- *an embedding:*

$$\begin{array}{rcl} (\underset{\sim}{\ })_G : \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} & \xrightarrow{1-1} & \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \\ x & \mapsto & (\underset{\sim}{x})_G \end{array}$$

○ *whose restriction to* $\mathcal{P}^{\gamma}(\mathbb{A}) \cap \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ *is an* $\in$*-isomorphism:*

$$\begin{aligned}\left(\underset{\sim}{\ }\right)_G : \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \cap \mathcal{P}^{\gamma}(\mathbb{A}) &\xleftrightarrow{\in-isom.} \left(\mathcal{P}^{\gamma}\left(\left(\underset{\sim}{\mathbb{A}}\right)_G\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathbb{A}}}} \\ x &\mapsto \left(\underset{\sim}{x}\right)_G .\end{aligned}$$

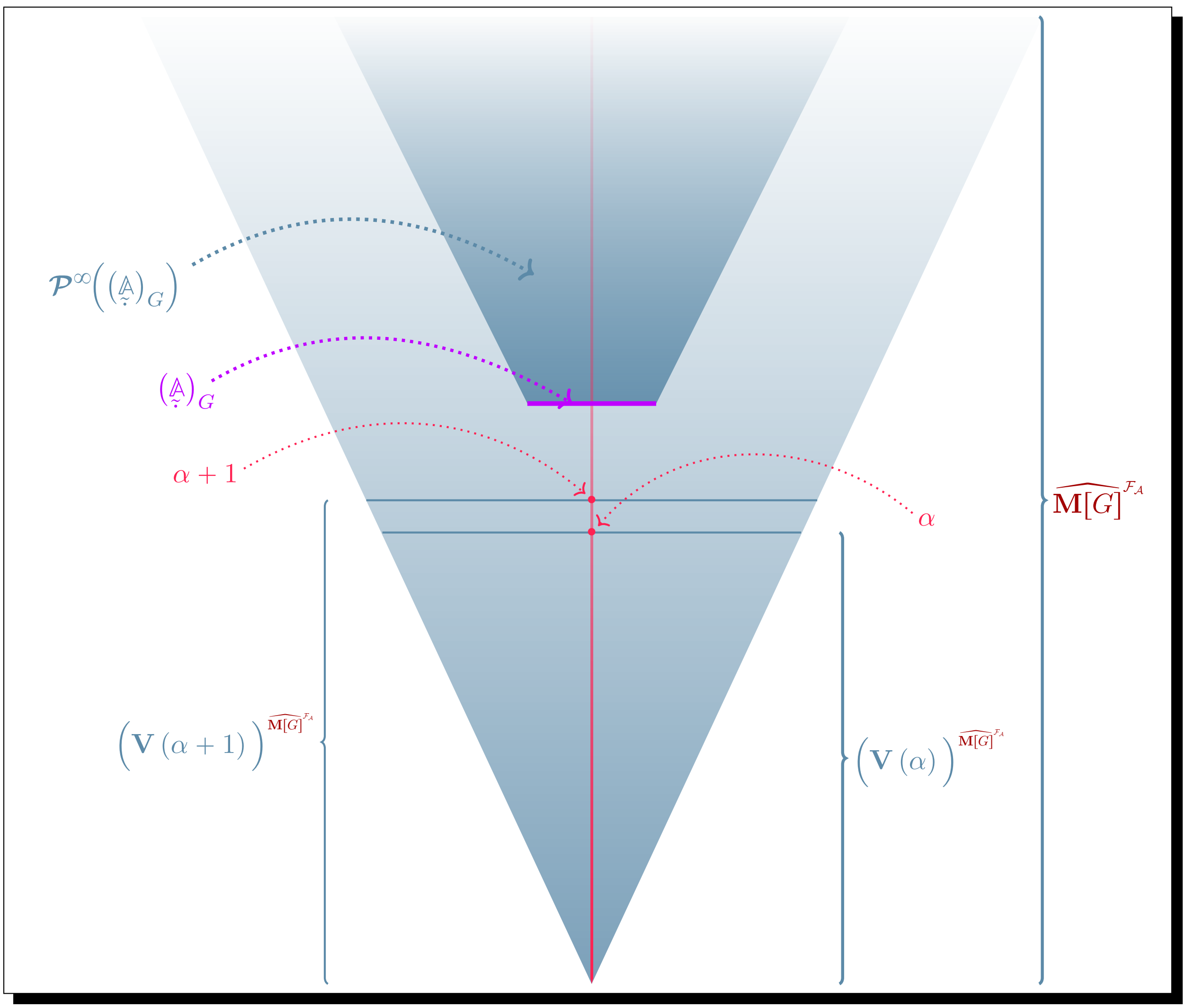


Figure 24.1: The Class $\mathcal{P}^{\infty}(\mathbb{A})$ embedded inside a symmetric submodel of $\mathbf{M}[G]$.

Proof of the Jech-Sochor Embedding Theorem: We let $\mathbf{M}$ be the kernel of the model of **ZFA**:

$$\mathbf{M} = \boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \cap \boldsymbol{\mathcal{Z}}.$$

Inside $\boldsymbol{\mathcal{Z}}$, we pick any set $\mathcal{A}$ that belongs to the kernel ($\mathcal{A} \in \mathbf{M}$) and satisfies $|\mathcal{A}| = |\mathbb{A}|$, together with any bijection $\iota : \mathbb{A} \overset{\text{bij.}}{\longleftrightarrow} \mathcal{A}$ that witnesses that $\mathcal{A}$ and $\mathbb{A}$ have same cardinalities.
Inside $\mathbf{M}$ we choose a cardinal κ large enough so that the following holds:

$$\mathbf{M} \models \text{“}\kappa \text{ is a regular cardinal”} \quad \text{and} \quad |\boldsymbol{\mathcal{P}}^{\gamma}(\mathcal{A})| < \kappa.$$

We force with $(\mathbb{P}, \leqslant, \mathbb{1})$ defined by

$$\mathbb{P} = \left\{p : \big(\mathcal{A} \times \kappa \times \kappa\big) \to \{0,1\} \,\Big|\, |dom\,(p)\,| < \kappa\right\}$$

$$p \leqslant q \iff p \supseteq q$$

$$\mathbb{1} = \varnothing$$

.

By choice of κ, $\mathbb{P}$ is a κ-closed notion of forcing. i.e., if $\langle p_\xi / \xi < \delta\rangle$ is a $\leqslant$-decreasing sequence for some $\delta < \kappa$, then $\left|\bigcup_{\xi<\delta} dom\,(p_\xi)\right| < \kappa$, hence $p = \bigcup_{\xi<\delta} p_\xi \in \mathbb{P}$.

We then define, for each element z of $\boldsymbol{\mathcal{Z}}$, some canonical $\mathbb{P}$-name $\underset{\sim}{z}$ which belong to $\mathbf{M}$ as follows. For each $\mathbb{a} \in \mathbb{A}$ and $\xi < \kappa$, we set

(1)
$$\underset{\sim}{x}_{\mathbb{a},\xi} = \left\{(\check{\zeta}, p) \mid p \in \mathbb{P} \wedge p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1\right\}$$

(2)
$$\underset{\sim}{\mathbb{a}} = \left\{(\underset{\sim}{x}_{\mathbb{a},\xi}, \mathbb{1}) \mid \xi \in \kappa\right\}$$

(3)
$$\underset{\sim}{\mathbb{A}} = \left\{\underset{\sim}{\mathbb{a}} \mid \mathbb{a} \in \mathbb{A}\right\}.$$

Finally, for each set $z \in \boldsymbol{\mathcal{Z}}$, by recursion, we define

$$\begin{array}{rcll} \underset{\sim}{z} & = & \left\{(\underset{\sim}{y}, \mathbb{1}) \mid y \in z\right\} & \text{if} \quad z \notin \mathbb{A}. \\ & = & \underset{\sim}{\mathbb{a}} & \text{if} \quad z = \mathbb{a} \in \mathbb{A}. \end{array}$$

Claim 412. *Let G which is $\mathbb{P}$-generic over $\mathbf{M}$. For all $x \in \boldsymbol{\mathcal{Z}}$, for all $\mathbb{a}, \mathbb{a}' \in \mathbb{A}$, and for all $\xi < \kappa$, we have*

(1) $\mathbb{a} \neq \mathbb{a}' \Rightarrow \left(\underset{\sim}{\mathbb{a}}\right)_G \neq \left(\underset{\sim}{\mathbb{a}}'\right)_G$

(2) $\left(\underset{\sim}{x}\right)_G \notin \left(\underset{\sim}{\mathbb{a}}\right)_G$

(3) $\left(\underset{\sim}{x}\right)_G \neq \left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G$.

Proof of Claim 412:

(1) $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G = \left(\underset{\sim}{x}_{\mathbb{a}',\xi'}\right)_G \iff \mathbb{a} = \mathbb{a}'$ and $\xi = \xi'$.

($\Longrightarrow$) We show ($\mathbb{a} \neq \mathbb{a}'$ or $\xi \neq \xi'$) $\Longrightarrow$ $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G \neq \left(\underset{\sim}{x}_{\mathbb{a}',\xi'}\right)_G$. So, we assume that either $\mathbb{a} \neq \mathbb{a}'$ or $\xi \neq \xi'$ holds and first show that the following set is dense:

$$S = \Big\{p \in \mathbb{P} \mid \exists \zeta < \kappa \;\; p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1 \wedge p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 0\Big\}$$

Indeed, take any $q \in \mathbb{P}$. Since $|dom(q)| < \kappa$ and $cof(\kappa) = \kappa$, we have

$$\big\{\zeta < \kappa \mid (\iota(\mathbb{a}), \xi, \zeta) \notin dom(q)\big\} \cap \big\{\zeta < \kappa \mid (\iota(\mathbb{a}'), \xi', \zeta) \notin dom(q)\big\} \neq \varnothing$$

Take any ordinal ζ in this set and form p such that

$$dom(p) = dom(q) \cup \big\{(\iota(\mathbb{a}), \xi, \zeta), (\iota(\mathbb{a}'), \xi', \zeta)\big\}$$

with $p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1$ and $p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 0$.
Since $S \in \mathbf{M}$ and S is dense, there exists some $p \in S \cap G$. Therefore, for some ordinal $\zeta < \kappa$, one has $(\check{\zeta}, p) \in \underset{\sim}{x}_{\mathbb{a},\xi}$ and $(\check{\zeta}, p) \notin \underset{\sim}{x}_{\mathbb{a}',\xi'}$. Henceforth, $\underset{\sim}{x}_{\mathbb{a},\xi} \neq \underset{\sim}{x}_{\mathbb{a}',\xi'}$.

($\Longleftarrow$) is immediate.

(2) $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G \neq \left(\underset{\sim}{x}\right)_G$ holds for all $x \in \boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \cap \boldsymbol{\mathcal{Z}}$.
Notice first that for any $x \in \boldsymbol{\mathcal{P}}^{\infty}(\varnothing) \cap \boldsymbol{\mathcal{Z}}$, by construction we precisely have $\underset{\sim}{x} = \check{x}$, so that $(\check{x})_G = x$. So, it is enough to show that $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G \notin \mathbf{M}$. Towards a contradiction, we assume $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G \in \mathbf{M}$. Then, the following set also belongs to $\mathbf{M}$:

$$D = \left\{p \in \mathbb{P} \mid \exists \zeta < \kappa \;\; \Big(\big(\iota(\mathbb{a}), \xi, \zeta\big) \in dom(p) \wedge \Big(p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1 \longleftrightarrow \zeta \notin \left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G\Big)\Big)\right\}$$

We show that D is dense. Indeed, given any $p \in \mathbb{P}$, there exists some $\zeta < \kappa$, with $\big(\iota(\mathbb{a}), \xi, \zeta\big) \notin dom(p)$ so that we can extend p by q and r the following way:

- $\circ$ $dom\,(q) = dom\,(r) = dom\,(p) \cup \{\, (\iota(\mathbb{a}), \xi, \zeta) \,\}$,
- $\circ$ $q \restriction dom\,(p) = r \restriction dom\,(p) = p$,
- $\circ$ $q\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1$ and $r\big(\iota(\mathbb{a}), \xi, \zeta\big) = 0$.

Since we have $q \perp r$, we have either $q \in D$ or $r \in D$, which shows D is dense.

Now, since D is dense, we take any $p \in D \cap G$, and any ζ which satisfy

$$p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1 \longleftrightarrow \zeta \notin \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G .$$

Then, the definition of $\big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G$ leads to the following contradiction.

$$\zeta \in \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G \longleftrightarrow p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1 \longleftrightarrow \zeta \notin \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G .$$

So, we have shown $\big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G \notin \mathbf{M}$.

(3) For all $x \in \boldsymbol{\mathcal{Z}}$ and all $\mathbb{a} \in \mathbb{A}$, $\big(\underset{\sim}{x}\big)_G \notin \big(\underset{\sim}{\mathbb{a}}\big)_G$.

We recall that

$$\big(\underset{\sim}{\mathbb{a}}\big)_G = \Big(\Big\{(\underset{\sim}{x}_{\mathbb{a},\xi}, \mathbb{1}) \mid \xi \in \kappa\Big\}\Big)_G = \Big\{(\underset{\sim}{x}_{\mathbb{a},\xi})_G \mid \xi \in \kappa.\Big\}$$

Towards a contradiction, we assume that there exists some $\xi \in \kappa$ such that $\big(\underset{\sim}{x}\big)_G = \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G$. i.e.,

$$\begin{aligned}
\big(\underset{\sim}{x}\big)_G &= \big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G \\
&= \Big(\Big\{(\check{\zeta}, p) \mid p \in \mathbb{P} \wedge p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1\Big\}\Big)_G \\
&= \Big\{\, \big(\check{\zeta}\big)_G \mid \exists p \in G \;\; p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1\Big\} \\
&= \Big\{\zeta < \kappa \mid \exists p \in G \;\; p\big(\iota(\mathbb{a}), \xi, \zeta\big) = 1\Big\}.
\end{aligned}$$

(a) If $x = \mathbb{a}' \in \mathbb{A}$, then $\underset{\sim}{x} = \underset{\sim}{\mathbb{a}}' = \Big\{(\underset{\sim}{x}_{\mathbb{a}',\xi'}, \mathbb{1}) \mid \xi' \in \kappa\Big\}$ and

$$\begin{aligned}
\big(\underset{\sim}{x}\big)_G &= \big(\underset{\sim}{\mathbb{a}}'\big)_G \\
&= \Big(\big\{(\underset{\sim}{x}_{\mathbb{a}',\xi'}, \mathbb{1}) \mid \xi' \in \kappa\big\}\Big)_G \\
&= \Big\{(\underset{\sim}{x}_{\mathbb{a}',\xi'})_G \mid \xi' \in \kappa\Big\} \\
&= \Big\{\Big\{\, \big(\check{\zeta}\big)_G \mid \exists p \in G \;\; p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 1\Big\} \mid \xi' \in \kappa\Big\} \\
&= \Big\{\, \underbrace{\Big\{\zeta < \kappa \mid \exists p \in G \;\; p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 1\Big\}}_{\mathscr{S}(\xi')} \mid \xi' \in \kappa\Big\}
\end{aligned}$$

Now, it is enough to show that no set $\mathscr{S}(\xi') = \left\{\zeta < \kappa \mid \exists p \in G \;\; p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 1\right\}$ is an ordinal to get a contradiction. To see this, simply notice that the following set is trivially dense in $\mathbb{P}$:

$$\left\{ p \in \mathbb{P} \;\mid\; \exists \zeta < \zeta' < \kappa \begin{pmatrix} \big(\iota(\mathbb{a}'), \xi', \zeta\big) \in dom\,(p) & \wedge & p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 0 \\ & \text{and} & \\ \big(\iota(\mathbb{a}'), \xi', \zeta'\big) \in dom\,(p) & \wedge & p\big(\iota(\mathbb{a}'), \xi', \zeta\big) = 1 \end{pmatrix} \right\}$$

which implies that $\mathscr{S}(\xi')$ is a non-empty set of ordinals which is not an initial segment of the ordinals, henceforth it is not an ordinal.

(b) If $x \notin \mathbb{A}$, then $\underset{\sim}{x} = \left\{(\underset{\sim}{y}, \mathbb{1}) \mid y \in x\right\}$ and $\big(\underset{\sim}{x}\big)_G = \left\{\big(\underset{\sim}{y}\big)_G \mid y \in x\right\}$.

- If $tc\,(x)$ does not contain any atom (x belongs to the kernel), then by construction $\underset{\sim}{x} = \check{x}$ and by Claim 412 (2) we have $\big(\underset{\sim}{x}_{\mathbb{a},\xi}\big)_G \neq (\check{x})_G$.
- If $tc\,(x)$ contains an atom $\mathbb{a}'$, then $tc\,((x)_G)$ contains $(\mathbb{a}')_G$ which is impossible by case (3)(a).

□ 412

Claim 413. *Let G be $\mathbb{P}$-generic over* $\mathbf{M}$. *For all $x, y \in \boldsymbol{\mathcal{Z}}$, we have*

(1) $\big(x \in y\big)^{\boldsymbol{\mathcal{Z}}} \iff \Big(\big(\underset{\sim}{x}\big)_G \in \big(\underset{\sim}{y}\big)_G\Big)^{\mathbf{M}[G]}$

(2) $\big(x = y\big)^{\boldsymbol{\mathcal{Z}}} \iff \Big(\big(\underset{\sim}{x}\big)_G = \big(\underset{\sim}{y}\big)_G\Big)^{\mathbf{M}[G]}$.

Proof of Claim 413: The proof is by induction on $rk_{\mathcal{P}^\infty(\mathbb{A})}(y)$. We prove simultaneously (1) and (2).

$\boldsymbol{rk_{\mathcal{P}^\infty(\mathbb{A})}(y) = 0}$**:** corresponds to y being an atom of the form $\mathbb{a} \in \mathbb{A}$.

(1) $(\Longrightarrow)$ $\big(x \in \mathbb{a}\big)^{\boldsymbol{\mathcal{Z}}}$ never holds, so the result is immediate.

$(\Longleftarrow)$ $\Big(\big(\underset{\sim}{x}\big)_G \in \big(\underset{\sim}{\mathbb{a}}\big)_G\Big)^{\mathbf{M}[G]}$ never holds, as we saw above, so the result is immediate.

(2) $(\Longrightarrow)$ is trivial.

$(\Longleftarrow)$
- If $x = \mathbb{a}' \in \mathbb{A}$, then $\big(\underset{\sim}{\mathbb{a}}\big)_G = \big(\underset{\sim}{\mathbb{a}}'\big)_G \Longrightarrow \mathbb{a} = \mathbb{a}'$.
- If $x \notin \mathbb{A}$, then $\underset{\sim}{x} = \left\{(\underset{\sim}{z}, \mathbb{1}) \mid z \in x\right\}$ and $\big(\underset{\sim}{x}\big)_G = \big(\underset{\sim}{\mathbb{a}}\big)_G$ implies $\big(\underset{\sim}{z}\big)_G \in \big(\underset{\sim}{\mathbb{a}}\big)_G$ holds for some $z \in x$, which contradicts Claim 412 (2).

$\boldsymbol{rk_{\mathcal{P}^\infty(\mathbb{A})}(y) > 0}$**:** corresponds to y being an atom of the form $\mathbb{a} \in \mathbb{A}$.

(1) $(\Longrightarrow)$ follows by definition of $\underset{\sim}{x}$ and $\underset{\sim}{y}$.

($\Longleftarrow$) by construction of $\underset{\sim}{y} = \{(\underset{\sim}{z}, \mathbb{1}) \mid z \in y\}$, there exists some $z \in y$ such that $(\underset{\sim}{x})_G = (\underset{\sim}{z})_G$. By induction hypothesis, we obtain $(x = z)^{\mathcal{Z}}$, hence $(x \in y)^{\mathcal{Z}}$.

(2) ($\Longrightarrow$) is trivial.

($\Longleftarrow$) If $(x \neq y)^{\mathcal{Z}}$, then by symmetry, there exists $(z \in y \wedge z \notin x)^{\mathcal{Z}}$ and by (24.1) we obtain $(\underset{\sim}{z})_G \in (\underset{\sim}{y})_G$ and $(\underset{\sim}{z})_G \notin (\underset{\sim}{x})_G$ which yields $(\underset{\sim}{x})_G \neq (\underset{\sim}{y})_G$.

$\square$ 413

We then associate to every permutation $\rho \in \mathcal{G}_{\mathbb{A}}$ (where $\rho : \mathbb{A} \xrightarrow{bij.} \mathbb{A}$ and $\mathcal{G}_{\mathbb{A}}$ is the subgroup of the group of permutations of $\mathbb{A}$) the following set Π_ρ of permutations $\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa$:

$$\begin{aligned}\Pi_\rho &= \left\{\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa \mid \forall \mathbb{a} \in \mathbb{A}\ \forall \xi < \kappa\ \exists \zeta < \kappa\ \ \pi\big(\iota(\mathbb{a}), \xi\big) = \Big(\iota\big(\rho(\mathbb{a})\big), \zeta\Big)\right\} \\ &= \left\{\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa \mid \forall \mathbb{a} \in \mathbb{A}\ \ \pi\Big[\{\underbrace{\iota(\mathbb{a})}_{\mathtt{a}}\} \times \kappa\Big] = \{\underbrace{\iota(\overbrace{\rho(\mathbb{a})}^{\mathbb{b}})}_{\mathtt{b}}\} \times \kappa\right\}.\end{aligned}$$

The intuition behind all this is that $\mathcal{A} \times \kappa$ should be regarded as as many disjoint copies of κ as there are atoms ($\mathbb{A}$-many or equivalently $\mathcal{A}$-many). Then, every permutation $\rho_{\mathbb{A}} : \mathbb{A} \xrightarrow{bij.} \mathbb{A}$ induces a permutation $\rho_{\mathcal{A}} : \mathcal{A} \xrightarrow{bij.} \mathcal{A}$ via the bijection $\iota : \mathbb{A} \xleftrightarrow{\text{bij.}} \mathcal{A}$. Now, we only consider the permutations $\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa$ which, for every $\mathtt{a} \in \mathcal{A}$, map $\{\mathtt{a}\} \times \kappa$ to $\{\mathtt{b}\} \times \kappa$ — where the relation between $\mathtt{a}$ and $\mathtt{b}$ is given by $\iota(\mathbb{a}) = \mathtt{a}$ and $\iota \circ \rho(\mathbb{a}) = \mathtt{b}$.
So, as shown in Figure 24.2, each permutation in Π_ρ can be regarded as as many permutations of κ as there are atoms, since for every $\mathtt{a} \in \mathcal{A}$:

$$\pi_{\restriction \{\mathtt{a}\} \times \kappa} : \{\mathtt{a}\} \times \kappa \xrightarrow{bij.} \{\mathtt{b}\} \times \kappa.$$

We set

$$\mathcal{G}_{\mathcal{A}} = \bigcup \{\Pi_\rho \mid \rho \in \mathcal{G}_{\mathbb{A}}\}.$$

For every subgroup $\mathcal{H}_{\mathbb{A}} \subseteq \mathcal{G}_{\mathbb{A}}$, we set

$$\mathcal{H}_{\mathcal{A}} = \bigcup \{\Pi_\rho \mid \rho \in \mathcal{H}_{\mathbb{A}}\}.$$

Now, every permutation $\pi : \mathcal{A} \times \kappa \xrightarrow{bij.} \mathcal{A} \times \kappa$ induces an automorphism $\pi_{\mathbb{P}} : \mathbb{P} \xrightarrow{bij.} \mathbb{P}$ defined by

$$\pi_{\mathbb{P}}\big(\pi(\mathtt{a}, \xi), \zeta\big) = \big(\mathtt{a}, \xi, \zeta\big)$$

or, to say it differently:

$$\pi_{\mathbb{P}}\big(\mathtt{a}, \xi, \zeta\big) = \big(\pi^{-1}(\mathtt{a}, \xi), \zeta\big)$$

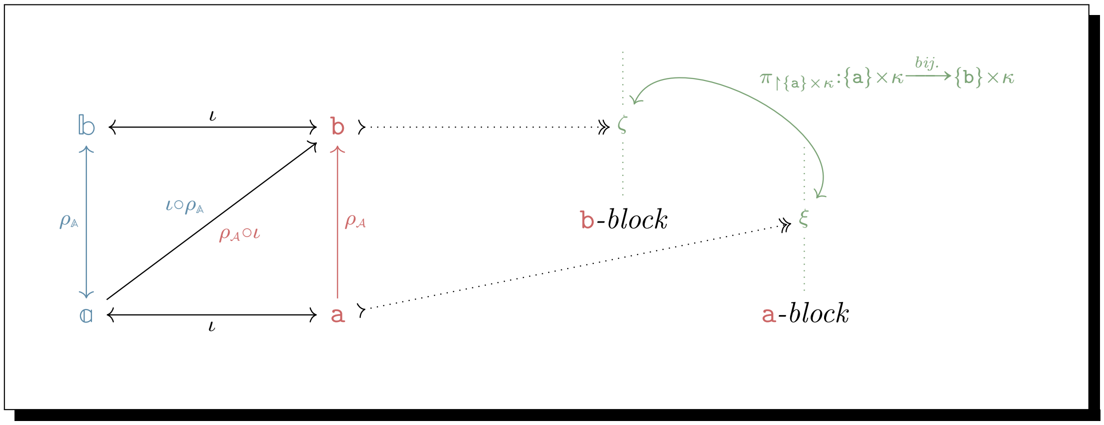


Figure 24.2: The permutation of blocks induced by $\pi \in \Pi_\rho$.

Following this, we regard every permutation $\pi \in \mathcal{G}_{\mathcal{A}}$ as the automorphism $\pi_{\mathbb{P}}$ of $\mathbb{P}$ that it induces. For every $F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$ we set

$$fix_{\mathcal{G}_{\mathcal{A}}}(F) = \left\{\pi \in \mathcal{G}_{\mathcal{A}} \mid \forall (\mathtt{a}, \xi) \in F \ \ \pi(\mathtt{a}, \xi) = (\mathtt{a}, \xi)\right\}$$

Finally, we set $\mathcal{F}_{\mathcal{A}}$ is the filter generated by

$$\left\{\mathcal{H}_{\mathcal{A}} \mid \mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}\right\} \cup \left\{fix_{\mathcal{G}_{\mathcal{A}}}(F) \mid F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)\right\}.$$

i.e.,

$$\mathcal{H} \in \mathcal{F}_{\mathcal{A}} \iff \begin{cases} \mathcal{H}_1 \cap \ldots \cap \mathcal{H}_n \subseteq \mathcal{H} \subseteq \mathcal{G}_{\mathcal{A}} \text{ and for some } \mathcal{H}_1, \ldots, \mathcal{H}_n \in \\ \quad \left\{\mathcal{H}_{\mathcal{A}} \mid \mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}\right\} \cup \left\{fix_{\mathcal{G}_{\mathcal{A}}}(F) \mid F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)\right\}. \end{cases}$$

Clearly, $\left\{\mathcal{H}_{\mathcal{A}} \mid \mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}\right\}$ is closed under finite intersections because $\mathcal{F}_{\mathbb{A}}$ is closed under finite intersections. Also, $\left\{fix_{\mathcal{G}_{\mathcal{A}}}(F) \mid F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)\right\}$ is closed under finite intersections because for $F_1, \ldots, F_n \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$, we have

$$fix_{\mathcal{G}_{\mathcal{A}}}(F_1) \cap \ldots \cap fix_{\mathcal{G}_{\mathcal{A}}}(F_n) = fix_{\mathcal{G}_{\mathcal{A}}}(F_1 \cup \ldots \cup F_n).$$

Therefore,

$$\mathcal{J} \in \mathcal{F}_{\mathcal{A}} \iff \mathcal{H}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(F) \subseteq \mathcal{J} \subseteq \mathcal{G}_{\mathcal{A}} \text{ for some } \mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}} \text{ and } F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa).$$

We check that $\mathcal{F}_{\mathcal{A}}$ is a normal filter on $\mathcal{G}_{\mathcal{A}}$. i.e., $\mathcal{F}_{\mathcal{A}}$ is a set of subgroups of $\mathcal{G}_{\mathcal{A}}$ such that for all subgroups $\mathcal{H}_{\mathcal{A}}$, $\mathcal{K}$ of $\mathcal{G}$ and all $\pi \in \mathcal{G}_{\mathcal{A}}$:

(1) $\mathcal{G}_{\mathcal{A}} \in \mathcal{F}_{\mathcal{A}}$: we have $\mathcal{G}_{\mathcal{A}} = \bigcup\left\{\Pi_\rho \mid \rho \in \mathcal{G}_{\mathbb{A}}\right\}$ and $\mathcal{G}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}$ so, $\mathcal{G}_{\mathcal{A}} \in \left\{\mathcal{H}_{\mathcal{A}} \mid \mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}\right\} \subseteq \mathcal{F}_{\mathcal{A}}$.

(2) if $\mathcal{H} \in \mathcal{F}_{\mathcal{A}}$ and $\mathcal{H} \subseteq \mathcal{K}$, then $\mathcal{K} \in \mathcal{F}_{\mathcal{A}}$: This is by the very definition of $\mathcal{F}_{\mathcal{A}}$.

(3) if $\mathcal{H} \in \mathcal{F}_{\mathcal{A}}$ and $\mathcal{K} \in \mathcal{F}_{\mathcal{A}}$, then $\mathcal{H} \cap \mathcal{K} = \mathcal{J} \in \mathcal{F}_{\mathcal{A}}$: assume for some $\mathcal{H}_{\mathbb{A}}, \mathcal{K}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}$ and $H, K \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$

$$\mathcal{H}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(H) \subseteq \mathcal{H} \text{ and } \mathcal{K}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(K) \subseteq \mathcal{K}.$$

Then

$$\mathcal{H}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(H) \cap \mathcal{K}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(K) \subseteq \mathcal{H} \cap \mathcal{K}$$

i.e.,

$$\underbrace{\mathcal{H}_{\mathcal{A}} \cap \mathcal{K}_{\mathcal{A}}}_{\substack{\mathcal{J}_{\mathcal{A}} \\ \mathit{with} \\ \mathcal{J}_{\mathbb{A}} = \mathcal{H}_{\mathbb{A}} \cap \mathcal{K}_{\mathbb{A}}}} \cap \underbrace{\mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(H) \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(K)}_{\mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(H \cup K)} \subseteq \mathcal{H} \cap \mathcal{K}$$

(4) if $\mathcal{H} \in \mathcal{F}_{\mathcal{A}}$, then $\pi \circ \mathcal{H} \circ \pi^{-1} \in \mathcal{F}_{\mathcal{A}}$: assume for some $\mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}$ and $H \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$

$$\mathcal{H}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(H) \subseteq \mathcal{H}.$$

Then, since $\pi \in \mathcal{G}_{\mathcal{A}}$, there exists some $\rho \in \mathcal{G}_{\mathbb{A}}$ such that $\pi \in \Pi_{\rho}$. Now,

$$\pi \circ \mathcal{H}_{\mathcal{A}} \circ \pi^{-1} = \mathcal{H}_{\mathcal{A}}{}' \text{ where } \mathcal{H}_{\mathbb{A}}{}' = \rho \circ \mathcal{H}_{\mathbb{A}} \circ \rho^{-1} \in \mathcal{F}_{\mathbb{A}}$$

and

$$\begin{aligned} \pi \circ \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(H) \circ \pi^{-1} &= \left\{ \pi \circ \tau \circ \pi^{-1} \mid \tau \in \mathcal{G}_{\mathcal{A}} \ \mathit{and} \ \forall (\mathtt{a}, \xi) \in H \ \ \tau(\mathtt{a}, \xi) = (\mathtt{a}, \xi) \right\} \\ &= \left\{ \tau \in \mathcal{G}_{\mathcal{A}} \mid \forall (\mathtt{b}, \xi) \in \pi[H] \ \ \tau(\mathtt{b}, \xi) = (\mathtt{b}, \xi) \right\} \in \mathcal{F}_{\mathcal{A}} \end{aligned}$$

Now, since $\mathcal{H}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(H) \subseteq \mathcal{H}$, we have

$$\pi \circ \mathcal{H}_{\mathcal{A}} \circ \pi^{-1} \cap \pi \circ \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(H) \circ \pi^{-1} \subseteq \pi \circ \mathcal{H} \circ \pi^{-1} \in \mathcal{F}_{\mathcal{A}}.$$

We let the set of all hereditarily symmetric $\mathbb{P}$-names $\mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} \subseteq \mathbf{M}^{\mathbb{P}}$

$$\mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} = \left\{ \tau \in \mathbf{M}^{\mathbb{P}} \mid \mathit{sym}_{\mathcal{G}_{\mathcal{A}}}(\tau) \in \mathcal{F}_{\mathcal{A}} \text{ and } \{ \sigma \mid \exists p \in \mathbb{P} \ (\sigma, p) \in \tau \} \subseteq \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} \right\}.$$

i.e.,

$$\tau \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} \iff \mathit{sym}_{\mathcal{G}_{\mathcal{A}}}(\tau) \in \mathcal{F}_{\mathcal{A}} \text{ and } \{ \sigma \mid \exists p \in \mathbb{P} \ (\sigma, p) \in \tau \} \subseteq \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}.$$

We denote the symmetric submodel of the generic extension $\mathbf{M}[G]$ by

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} = \left\{ (\tau)_G \in \mathbf{M}[G] \mid \tau \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} \right\}.$$

We notice that the following sets belong to the symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$:

- $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G$ (for all $\mathbb{a} \in \mathbb{A}$ and all $\xi < \kappa$) because
$$sym_{\mathcal{G}_\mathcal{A}}\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right) = fix_{\mathcal{G}_\mathcal{A}}\left(\{\, (\iota(\mathbb{a}), \xi)\,\}\right).$$
- $\left(\underset{\sim}{\mathbb{a}}\right)_G$ (for all $\mathbb{a} \in \mathbb{A}$) because
$$sym_{\mathcal{G}_\mathcal{A}}\left(\underset{\sim}{\mathbb{a}}\right) = \bigcup\left\{\Pi_\rho \mid \rho \in sym_{\mathcal{G}_\mathbb{A}}(\mathbb{a})\right\}.$$
- $\left(\underset{\sim}{\mathbb{A}}\right)_G$ because
$$sym_{\mathcal{G}_\mathcal{A}}\left(\underset{\sim}{\mathbb{A}}\right) = \mathcal{G}_\mathcal{A}.$$

So, we already have $\left(\underset{\sim}{\mathbb{A}}\right)_G$, each $\left(\underset{\sim}{\mathbb{a}}\right)_G$, and each $\left(\underset{\sim}{x}_{\mathbb{a},\xi}\right)_G$ all belong to the symmetric submodel.

We now show that x belongs to the permutation model $\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_\mathbb{A}}}$ if and only if $\left(\underset{\sim}{x}\right)_G$ belongs to the symmetric submodel of the generic extension $\widehat{\mathbf{M}[G]}^{\mathcal{F}_\mathcal{A}}$.

Claim 414. *For all $x \in \mathcal{Z}$,*
$$x \in \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_\mathbb{A}}} \iff \underset{\sim}{x} \in \mathbf{HS}_{\mathcal{F}_\mathcal{A}}.$$

Proof of Claim 414: This comes down to proving
$$sym_{\mathcal{G}_\mathbb{A}}(x) \in \mathcal{F}_\mathbb{A} \iff sym_{\mathcal{G}_\mathcal{A}}\left(\underset{\sim}{x}\right) \in \mathcal{F}_\mathcal{A}.$$

($\Longrightarrow$) We have $sym_{\mathcal{G}_\mathcal{A}}\left(\underset{\sim}{x}\right) = \bigcup\left\{\Pi_\rho \mid \rho \in sym_{\mathcal{G}_\mathbb{A}}(x)\right\}$. So, if $sym_{\mathcal{G}_\mathbb{A}}(x) \in \mathcal{F}_\mathbb{A}$, then, by definition,
$$\bigcup\left\{\Pi_\rho \mid \rho \in sym_{\mathcal{G}_\mathbb{A}}(x)\right\} = sym_{\mathcal{G}_\mathcal{A}}\left(\underset{\sim}{x}\right) \in \mathcal{F}_\mathcal{A}.$$

($\Longleftarrow$) If $sym_{\mathcal{G}_\mathcal{A}}\left(\underset{\sim}{x}\right) \in \mathcal{F}_\mathcal{A}$, then, for some $\mathcal{H}_\mathbb{A} \in \mathcal{F}_\mathbb{A}$ and $F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$, we have
$$\mathcal{H}_\mathcal{A} \cap fix_{\mathcal{G}_\mathcal{A}}(F) \subseteq sym_{\mathcal{G}_\mathcal{A}}\left(\underset{\sim}{x}\right) \subseteq \mathcal{G}_\mathcal{A}$$
where $\mathcal{H}_\mathcal{A} = \bigcup\left\{\Pi_\rho \mid \rho \in \mathcal{H}_\mathbb{A}\right\}$. We set
$$\mathbb{F} = \left\{\mathbb{a} \mid \exists \xi < \kappa \quad \left(\iota(\mathbb{a}), \xi\right) \in F\right\}.$$
We have
$$\underbrace{\underbrace{\mathcal{H}_\mathbb{A}}_{\in \mathcal{F}_\mathbb{A}} \cap \bigcap_{\mathbb{a} \in \mathbb{F}} \underbrace{fix_{\mathcal{G}_\mathbb{A}}(\mathbb{a})}_{\in \mathcal{F}_\mathbb{A}}}_{\in \mathcal{F}_\mathbb{A}} \subseteq sym_{\mathcal{G}_\mathbb{A}}(x) \subseteq \mathcal{G}_\mathbb{A}$$

since $\mathcal{F}_{\mathbb{A}}$ is finite, this shows that $sym_{\mathcal{G}_{\mathbb{A}}}(x) \in \mathcal{F}_{\mathbb{A}}$.

□ 414

Claim 415. *For all $x \in \boldsymbol{\mathcal{Z}}$,*

$$x \in \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \iff (\underset{\sim}{x})_G \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$$

Proof of Claim 415:

($\Longrightarrow$) This is a consequence of Claim 414 since $x \in \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \Longrightarrow \underset{\sim}{x} \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}} \Longrightarrow (\underset{\sim}{x})_G \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$.

($\Longleftarrow$) We proceed by contradiction, assuming there exists $x \in \boldsymbol{\mathcal{Z}}$ such that $(\underset{\sim}{x})_G \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ but $x \notin \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$. We assume x be the $\in$-least such set in the sense that $y \in \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ holds for all $\left(\underset{\sim}{y}\right)_G \in (\underset{\sim}{x})_G$.

Since $(\underset{\sim}{x})_G \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$, there exists some $\mathbb{P}$-name $\underset{\sim}{z} \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}$ and some forcing condition $p_z \in G$ such that

$$p_z \Vdash \underset{\sim}{z} = \underset{\sim}{x}.$$

Since $\underset{\sim}{z} \in \mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}$, there exist both $\mathcal{H}_{\mathbb{A}} \in \mathcal{F}_{\mathbb{A}}$ and $F \in \mathcal{P}_{fin}(\mathcal{A} \times \kappa)$ such that

$$\mathcal{H}_{\mathcal{A}} \cap fix_{\mathcal{G}_{\mathcal{A}}}(F) \subseteq sym_{\mathcal{G}_{\mathcal{A}}}(\underset{\sim}{z}) \subseteq \mathcal{G}_{\mathcal{A}}$$

Since, $sym_{\mathcal{G}_{\mathbb{A}}}(x) \notin \mathcal{F}_{\mathbb{A}}$, we have, for $\mathbb{F} = \left\{ \mathbb{a} \in \mathbb{A} \mid \exists \xi < \kappa \ \ \left(\iota(\mathbb{a}), \xi\right) \in F \right\}$,

$$\underbrace{\underbrace{\mathcal{H}_{\mathbb{A}}}_{\in \mathcal{F}_{\mathbb{A}}} \cap \bigcap_{\mathbb{a} \in \mathbb{F}} \underbrace{fix_{\mathcal{G}_{\mathbb{A}}}(\mathbb{a})}_{\in \mathcal{F}_{\mathbb{A}}}}_{\in \mathcal{F}_{\mathbb{A}}} \nsubseteq sym_{\mathcal{G}_{\mathbb{A}}}(x) \subseteq \mathcal{G}_{\mathbb{A}}.$$

Therefore, there exists

$$\rho \in \left(\mathcal{H}_{\mathbb{A}} \cap \bigcap_{\mathbb{a} \in \mathbb{F}} fix_{\mathcal{G}_{\mathbb{A}}}(\mathbb{a})\right) \smallsetminus sym_{\mathcal{G}_{\mathbb{A}}}(x).$$

In particular, we have $\rho(x) \neq x$.

Since $|dom(p_z)| < \kappa$, there exists some $\delta < \kappa$ such that

$$\left\{(a, \xi) \in \mathcal{A} \times \kappa \mid \delta < \xi\right\} \cap \left(F \cup dom(p_z) \upharpoonright \mathcal{A} \times \kappa\right) = \varnothing.$$

So, in order to have some $\pi \in \Pi_\rho$ satisfy both

(1) $\pi \in \mathcal{H}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(F)$ (2) $\pi(p_z)$ and p_z are compatible.

we can define π as follows:

$$\begin{aligned}
&\text{for } \mathbb{a} \in \mathbb{F} \text{ and } \xi < \kappa : && \pi\big(\iota(\mathbb{a}), \xi\big) &&= \big(\iota(\mathbb{a}), \xi\big)\\
&\text{for } \mathbb{a} \notin \mathbb{F} \text{ and } \xi < \delta : && \pi\big(\iota(\mathbb{a}), \xi\big) &&= \big(\iota \circ \rho(\mathbb{a}), \delta + \xi\big)\\
& && \pi\big(\iota(\mathbb{a}), \delta + \xi\big) &&= \big(\iota \circ \rho(\mathbb{a}), \xi\big)\\
&\text{for } \mathbb{a} \notin \mathbb{F} \text{ and } \delta < \xi + 1 < \kappa : && \pi\big(\iota(\mathbb{a}), \delta + \xi\big) &&= \big(\iota \circ \rho(\mathbb{a}), \delta + \xi\big).
\end{aligned}$$

We then have

- $\tilde{\pi}(\underset{\sim}{z}) = \underset{\sim}{z}$ (because $\pi \in \mathcal{H}_{\mathcal{A}} \cap \mathit{fix}_{\mathcal{G}_{\mathcal{A}}}(F)$);
- $p_z \Vdash \tilde{\pi}(\underset{\sim}{x}) \neq \underset{\sim}{x}$ (because $\boldsymbol{\mathcal{Z}} \models \rho(x) \neq x$ and by Claim 413 (2)

$$\boldsymbol{\mathcal{Z}} \models \rho(x) \neq x \iff \mathbf{M}[G] \models \Big(\underset{\sim}{\rho(x)}\Big)_G \neq (\underset{\sim}{x})_G$$

$$\iff \mathbf{M}[G] \models \big(\tilde{\pi}(\underset{\sim}{x})\big)_G \neq (\underset{\sim}{x})_G .)$$

- there exists $q \in \mathbb{P}$ such that $q \leqslant p_z$ and $q \leqslant \pi(p_z)$ which leads to the following contradiction:

$$q \Vdash \underset{\sim}{z} = \underset{\sim}{x} \quad\text{and}\quad q \Vdash \tilde{\pi}(\underset{\sim}{x}) \neq \underset{\sim}{x} \quad\text{and}\quad q \Vdash \tilde{\pi}(\underset{\sim}{x}) = \underset{\sim}{z}.$$

□ 415

Claim 416. *For all $x \in \boldsymbol{\mathcal{Z}}$, and all ordinal γ,*

$$\big\{ (\underset{\sim}{x})_G \mid x \in \boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}} \big\} = \Big(\boldsymbol{\mathcal{P}}^{\gamma}\big((\underset{\sim}{\mathbb{A}})_G \big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}.$$

Proof of Claim 416:

($\subseteq$) is immediate.

($\supseteq$) The proof is by $\in$-induction. We let $x \in \boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathcal{A}}}}$ with $(\underset{\sim}{x})_G \in \Big(\boldsymbol{\mathcal{P}}^{\gamma}\big((\underset{\sim}{\mathbb{A}})_G \big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}$ and $y \in \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ be such that $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models y \in (\underset{\sim}{x})_G$. We consider $\underset{\sim}{y}$ any $\mathbb{P}$-name for y. Now, for each $u \in x$, the following set is dense:

$$D_u = \{q \in \mathbb{P} \mid q \Vdash \underset{\sim}{u} \in \underset{\sim}{y} \ \text{ or } \ q \Vdash \underset{\sim}{u} \notin \underset{\sim}{y}\}.$$

Since $|\boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}| < \kappa$, we have $|x| < \kappa$, hence $|\{D_u \mid u \in x\}| < \kappa$ and, since $\mathbb{P}$ is κ-closed, there exists some forcing condition $p \in \mathbb{P}$ which "decides" for each $u \in x$, whether $\underset{\sim}{u} \in \underset{\sim}{y}$ or $\underset{\sim}{u} \notin \underset{\sim}{y}$ holds. Namely,

$$p \in G \cap \bigcap_{u \in x} \{q \in \mathbb{P} \mid q \Vdash \underset{\sim}{u} \in \underset{\sim}{y} \ \text{ or } \ q \Vdash \underset{\sim}{u} \notin \underset{\sim}{y}\}.$$

We take $z = \{u \in x \mid p \Vdash \underset{\sim}{u} \in \underset{\sim}{y}\}$ so that we have $(\underset{\sim}{z})_G = y$ and since $(\underset{\sim}{z})_G$ belongs to $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$, we also have $z \in \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ by Claim 415.

□ 416

Claim 416 yields that the embedding $\begin{array}{ccc} \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} & \to & \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \\ x & \mapsto & (\underset{\sim}{x})_G \end{array}$ satisfies

$$\{ (\underset{\sim}{x})_G \mid x \in \boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \} = \Big(\boldsymbol{\mathcal{P}}^{\gamma}\big((\underset{\sim}{\mathbb{A}})_G \big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}$$

and for all $x, y \in \boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ we have

$$\boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models y \in x \iff \widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models (\underset{\sim}{y})_G \in (\underset{\sim}{x})_G\,.$$

So, it follows that the mapping $x \mapsto (\underset{\sim}{x})_G$ is an $\in$-isomorphism between $\boldsymbol{\mathcal{P}}^{\gamma}(\mathbb{A}) \cap \boldsymbol{\mathcal{Z}}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ and $\Big(\boldsymbol{\mathcal{P}}^{\gamma}\big((\underset{\sim}{\mathbb{A}})_G \big)\Big)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}$.

□ Jech-Sochor Embedding Theorem

24.2 Some applications of the Jech-Sochor Embedding Theorem

Corollary 417. *Let* $\boldsymbol{\mathcal{Z}}$ *be any model of* **ZFA** *with*

- $\mathbb{A}$ *as set of atoms,*
- $\mathcal{G}_{\mathbb{A}}$ *any subgroup of the group of permutations of* $\mathbb{A}$*,*
- $\mathcal{F}_{\mathbb{A}}$ *any normal filter on* $\mathcal{G}_{\mathbb{A}}$*,*

such that the permutation model $\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ *induced by* $\mathcal{Z}$ *and* $\mathcal{F}_{\mathbb{A}}$ *satisfies*

$$\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \mathbf{ZFA} + (\mathbf{AC})^{\mathcal{P}^{\alpha}(\varnothing)}.$$

Let also α *be any ordinal and* φ *be any formula of the form*

$$\varphi := \exists x \underbrace{\psi(x)}_{\Delta_0^{0-rud}}$$

where ψ *is some* Δ_0^{0-rud}*-formula whose quantifiers are all bounded by* $\mathcal{P}^{\alpha}(x)$*.*

If $\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \varphi$*, then there exists a symmetric model* $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ *such that* $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \varphi$*.*

Proof of Corollary 417: Since $\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \exists x \ \psi(x)$, we fix

- any $B \in \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}}$ such that $\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \psi(B)$ and
- any large enough ordinal γ such that $\mathcal{P}^{\alpha}(B) \subseteq \mathcal{P}^{\gamma}(\mathbb{A})$.

By the Jech-Sochor Embedding Theorem (on page 385) there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ together with an $\in$-isomorphism:

$$\mathscr{I} : \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \cap \mathcal{P}^{\gamma}(\mathbb{A}) \xleftrightarrow{\in\text{-isomorphism}} \left(\mathcal{P}^{\gamma}\left((\underset{\sim}{\mathbb{A}})_G\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}}.$$

hence

$$\begin{aligned}
\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \varphi \iff & \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \ \exists x \ \psi(x) \\
\implies & \exists B \in \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \ \exists \gamma \in \mathbf{On} \ \left(\mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \models \psi(B) \ \text{ and } \ \mathcal{P}^{\alpha}(B) \subseteq \mathcal{P}^{\gamma}(\mathbb{A})\right) \\
\implies & \mathcal{Z}^{\mathbf{HS}_{\mathcal{F}_{\mathbb{A}}}} \cap \mathcal{P}^{\gamma}(\mathbb{A}) \models \ \psi(B) \\
\iff & \left(\mathcal{P}^{\gamma}\left((\underset{\sim}{\mathbb{A}})_G\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}} \models \ \psi(\mathscr{I}(B)) \\
\implies & \left(\mathcal{P}^{\gamma}\left((\underset{\sim}{\mathbb{A}})_G\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}} \models \ \exists x \ \psi(x). \\
\implies & \left(\mathcal{P}^{\gamma}\left((\underset{\sim}{\mathbb{A}})_G\right)\right)^{\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}} \models \ \varphi.
\end{aligned}$$

□ 417

Corollary 418. *If* **ZF** *is consistent, then the following theories are consistent as well:*

(1) **ZF**+ *" there exists some infinite Dedekind-finite set*[1] *".*

(2) **ZF**+ *" there exists some infinite set $\mathcal{A}$ such that $\mathcal{P}(\mathcal{A})$ is Dedekind-finite ".*

(3) **ZF**+ *" there exists some countable family of pairs which does not admit any choice function ".*

(4) **ZF**+ *" there exists some infinite binary tree without any infinite branch ".*

(5) **ZF**+ *" there exists an infinite set $\mathbb{A}$ and a mapping $f : \mathcal{P}_{fin}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A})$ ".*

Proof of Corollary 418: First, notice that **ZF** and **ZFC** are equiconsistent. Then,

(1) By Proposition 399, the basic Fraenkel model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ which was defined on page 372 contains $\mathbb{A}$, an infinite set of atoms, which is Dedekind-finite:

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \aleph_0 \overset{1-1}{\not\lesssim} \mathbb{A}.$$

For any integer n and any atom $\mathbb{a}$, we have

$$(n, \mathbb{a}) = \left\{ \{n\}, \{n, \mathbb{a}\} \right\} \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \cap \mathcal{P}^{n+3}(\mathbb{A}),$$

as well as

$$(\mathbb{a}, n) = \left\{ \{\mathbb{a}\}, \{\mathbb{a}, n\} \right\} \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \cap \mathcal{P}^{n+3}(\mathbb{A}),$$

Hence, if f is of the form $f : \omega \to \mathbb{A}$ and f belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$, then it belongs to $\mathcal{P}^{\omega}(\mathbb{A})$. So, we have

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \left\{ f \subseteq \omega \times \mathbb{A} \mid f : \omega \to \mathbb{A} \right\} \subseteq \mathcal{P}^{\omega+1}(\mathbb{A}).$$

Similarly, g is of the form $f : \mathbb{A} \to n$ for some integer n, and g belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$, then it belongs to $\mathcal{P}^{n+4}(\mathbb{A})$. So, we have

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \left\{ g \subseteq \omega \times \mathbb{A} \mid \exists n \in \omega \;\; g : \mathbb{A} \to n \right\} \subseteq \mathcal{P}^{\omega}(\mathbb{A}).$$

Moreover,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \left(\mathbf{AC}\right)^{\mathcal{P}^{\alpha}(\varnothing)}.$$

[1]See Definition 353 on page 307, where it was stated that A is Dedekind-finite if $\omega \overset{1-1}{\lesssim} A$ does not hold.

Now,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \exists x \ \Big(\forall n \in \omega \ \text{“ there is no } g : x \xrightarrow{1-1} n\text{”} \quad \wedge \quad \text{“ there is no } f : \omega \xrightarrow{1-1} x\text{”}\Big)$$

$$\Longleftrightarrow$$

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \exists x \ \Big(\forall n, g \in \boldsymbol{\mathcal{P}}^{\omega+1}(x) \ \big(n \in \omega \longrightarrow \neg g : x \xrightarrow{1-1} n\big) \ \wedge \ \forall f \in \boldsymbol{\mathcal{P}}^{\omega+1}(x) \ \neg f : \omega \xrightarrow{1-1} x\Big)$$

$$\Longleftrightarrow$$

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \exists x \ \Big(\neg \exists n, \, g \in \boldsymbol{\mathcal{P}}^{\omega+1}(x) \ \big(n \in \omega \ \wedge \ g : x \xrightarrow{1-1} n\big) \ \wedge \ \neg \exists f \in \boldsymbol{\mathcal{P}}^{\omega+1}(x) \ f : \omega \xrightarrow{1-1} x\Big).$$

By Corollary 417, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ such that

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \exists x \ \Big(\neg \exists n, \, g \ \big(n \in \omega \ \wedge \ g : x \xrightarrow{1-1} n\big) \ \wedge \ \neg \exists f \ \ f : \omega \xrightarrow{1-1} x\Big),$$

which is equivalent to

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \exists x \ \text{“} x \text{ is infinite and there is no } f : \omega \xrightarrow{1-1} x\text{”}.$$

(2) *mutatis mutandis*, the proof is the same as for (1): By Proposition 400, the basic Fraenkel model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0}$ contains an infinite set of atoms $\mathbb{A}$ which satisfies:

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \aleph_0 \overset{1-1}{\not\leq} \mathcal{P}(\mathbb{A}).$$

For any integer n and any set of atom $\mathbb{B} \subseteq \mathbb{A}$, we have

$$(n, \mathbb{B}) \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \quad \Longrightarrow \quad (n, \mathbb{B}) \in \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \cap \boldsymbol{\mathcal{P}}^{\omega+3}(\mathbb{A}).$$

Hence,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models {}^{\omega}\mathcal{P}(\mathbb{A}) \subseteq \boldsymbol{\mathcal{P}}^{\omega+4}(\mathbb{A}).$$

we have,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \text{“ there is no } f : \omega \xrightarrow{1-1} \mathcal{P}(\mathbb{A})\text{”} \iff \mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \big(\text{“ there is no } f : \omega \xrightarrow{1-1} \mathcal{P}(\mathbb{A})\text{”}\big)^{\boldsymbol{\mathcal{P}}^{\omega+4}(\mathbb{A})}.$$

So, we have

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \exists x \ \Big(\forall n, g \in \boldsymbol{\mathcal{P}}^{\omega+4}(x) \ \big(n \in \omega \longrightarrow \neg g : x \xrightarrow{1-1} n\big) \ \wedge \ \forall f \in \boldsymbol{\mathcal{P}}^{\omega+4}(x) \ \neg f : \omega \xrightarrow{1-1} \mathcal{P}(x)\Big)$$

$$\Longleftrightarrow$$

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \exists x \left(\neg \exists n, g \in \mathcal{P}^{\omega+4}(x) \left(n \in \omega \wedge g : x \xrightarrow{1-1} n\right) \wedge \neg \exists f \in \mathcal{P}^{\omega+4}(x)\ f : \omega \xrightarrow{1-1} \mathcal{P}(x)\right).$$

Since, $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models (\mathbf{AC})^{\mathcal{P}^{\infty}(\varnothing)}$ holds, by Corollary 417, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ such that the following holds as well:

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \exists x \left(\neg \exists n, g \left(n \in \omega \quad \wedge \quad g : x \xrightarrow{1-1} n\right) \quad \wedge \quad \neg \exists f \ \ f : \omega \xrightarrow{1-1} \mathcal{P}(x)\right),$$

which is equivalent to

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \exists x \ \ \text{“}x \text{ is infinite and there is no } f : \omega \xrightarrow{1-1} \mathcal{P}(x)\text{”}.$$

(3) By Theorem 403, the second Fraenkel model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ defined on page 374 whose set of atoms is $\mathbb{A} = \bigcup_{n\in\omega} P_n$, where each P_n is a pair of two distinct atoms, satisfies not only that the set of all pairs of atoms $\{P_n \mid n \in \omega\}$ belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$, but also that

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \text{“}\{P_n \mid n \in \omega\} \text{ does not admit any choice function”}.$$

Now, a choice function is an element $f \in {}^{\omega}\mathbb{A}$ which satisfies

$$\forall n \in \omega \quad f(n) \in P_n.$$

Since we have,

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models {}^{\omega}\mathbb{A} \subseteq \mathcal{P}^{\omega+1}(\mathbb{A}),$$

we obtain

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \text{“ there is no choice function for } \{P_n \mid n \in \omega\}\text{”}$$

$$\Longleftrightarrow$$

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \left(\text{“ there is no choice function for } \{P_n \mid n \in \omega\}\text{”}\right)^{\mathcal{P}^{\omega+1}(\mathbb{A})}.$$

So, we have

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models \left(\text{“ there is no choice function for } \{P_n \mid n \in \omega\}\text{”}\right)^{\mathcal{P}^{\omega+1}(\mathbb{A})}.$$

Notice also that it was shown in Lemma **??** that the mapping $f = \{(n, P_n) \mid n \in \omega\}$ belongs to $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$, hence it belongs to $\mathcal{P}^{\omega+1}(\mathbb{A})$.

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_0} \models \exists x \ \exists f,\, y \in \mathcal{P}^{\omega+1}(x) \begin{pmatrix} \text{“}y \text{ is a set of disjoint pairs of elements from } x\text{”} \\ \bigwedge \\ f : \omega \xrightarrow{bij.} y \\ \bigwedge \\ \neg\exists \mathsf{c} \in \mathcal{P}^{\omega+1}(x) \ \left(\mathsf{c} : \omega \to x \ \wedge \ \forall n \in \omega \ \ \mathsf{c}(n) \in f(n)\right) \end{pmatrix}.$$

Since, $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2} \models (\mathbf{AC})^{\mathcal{P}^{\infty}(\varnothing)}$, by Corollary 417, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathbb{A}}}$ such that

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathbb{A}}} \models \exists x \ \exists f,\, y \begin{pmatrix} \text{“}y \text{ is a set of disjoint pairs of elements from } x\text{”} \\ \bigwedge \\ f : \omega \xrightarrow{bij.} y \\ \bigwedge \\ \neg\exists \mathsf{c} \ \left(\mathsf{c} : \omega \to x \ \wedge \ \forall n \in \omega \ \ \mathsf{c}(n) \in f(n)\right) \end{pmatrix}$$

or equivalently,

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathbb{A}}} \models \text{“ there is a countable set of pairs with no choice function ”}.$$

(4) By Theorem 406, Weak König Lemma fails inside the second Fraenkel model $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{\mathcal{F}_2}$ with $\mathbb{A} = \bigcup_{n\in\omega} P_n$ and $P_n = \{a_n, b_n\}$ as set of atoms. Because the infinite binary tree

$$T = \bigcup_{n\in\omega} \left\{ s \in {}^{n}\mathbb{A} \mid \forall k \in n \ s(k) \in P_k \right\}.$$

does not have any infinite branch (for the reason such an infinite branch would yield a choice function that would contradict Theorem 403).

Now, every element of this tree belongs to some $\mathcal{P}^k(\mathbb{A})$ for some integer k large enough. Hence, $T \subseteq \mathcal{P}^\omega(\mathbb{A})$ which yields

$$\mathcal{M}_{\mathcal{F}_2}^{\mathbf{HS}_{\mathscr{F}}} \models T \in \mathcal{P}^{\omega+1}(\mathbb{A}).$$

Moreover, an infinite branch of T would be a mapping $b : \omega \to T$, hence would satisfy $b \subseteq \omega \times T$, hence would belong to $\mathcal{P}^{\omega+1}(\mathbb{A})$.

Therefore we obtain

$$\mathcal{M}_{\mathcal{F}_2}^{\mathbf{HS}_{\mathscr{F}}} \models \Big(\text{“ there exists an infinite binary tree on } \mathbb{A} \text{ with no infinite branch ”}\Big)^{\mathcal{P}^{\omega+1}(\mathbb{A})}.$$

So, we have

$$\mathcal{M}_{\mathcal{F}_2}^{\mathbf{HS}_{\mathscr{F}}} \models \exists x \;\; \exists T \in \mathcal{P}^{\omega+1}(x) \left(\begin{array}{c} \text{“}T \text{ is an infinite binary tree on } x\text{”} \\ \bigwedge \\ \neg\exists b \in \mathcal{P}^{\omega+1}(x)\,\text{“}b \text{ is an infinite branch of } T\text{”} \end{array}\right)$$

Since, $\mathcal{M}_{\mathcal{F}_2}^{\mathbf{HS}_{\mathscr{F}}} \models (\mathbf{AC})^{\mathcal{P}^{\infty}(\varnothing)}$, by Corollary 417, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_A}$ such that

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_A} \models \exists x \;\; \exists T \left(\begin{array}{c} \text{“}T \text{ is an infinite binary tree on } x\text{”} \\ \bigwedge \\ \neg\exists b\text{“}b \text{ is an infinite branch of } T\text{”}c \end{array}\right)$$

or equivalently,

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_A} \models \text{“ there exists an infinite binary tree with no infinite branch ”}.$$

(5) On page 379, Definition 407 presented the ordered Mostowski model $\mathcal{M}_{ost.}^{\mathbf{HS}_{\mathscr{F}}}$ which comes with a countable set of atoms $\mathbb{A}$ equipped with a binary relation $<_{\mathbf{M}} \subseteq \mathbb{A} \times \mathbb{A}$ which makes it

a dense ordering without least nor greatest element — so that it isomorphic to $(\mathbb{Q}, <_{(} \mathbb{Q})))$.

In Theorem 410, we proved that this permutation model satisfies

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models \text{" there exists a mapping } f : \mathcal{P}_{fin}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A}) \text{"}$$

Now, every element of this mapping is of the form $(\mathbb{F}, \mathbb{B})$ for some $\mathbb{F} \subseteq \mathcal{P}_{fin}(\mathbb{A})$ and $\mathbb{B} \subseteq \mathcal{P}(\mathbb{A})$. Since both $\mathbb{F}$ and $\mathbb{B}$ belong to $\boldsymbol{\mathcal{P}}^{1}(\mathbb{A})$ we have $(\mathbb{F}, \mathbb{B}) = \big\{ \{\mathbb{F}\}, (\mathbb{F}, \mathbb{B}) \big\}$ belongs to $\boldsymbol{\mathcal{P}}^{3}(\mathbb{A})$. This shows that the mapping f belongs to $\boldsymbol{\mathcal{P}}^{4}(\mathbb{A})$.
Therefore we obtain

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models \Big(\text{" there exist an infinite set } \mathbb{A} \text{ and a mapping } f : \mathcal{P}_{fin}(\mathbb{A}) \xrightarrow{onto} \mathcal{P}(\mathbb{A}) \text{ "} \Big)^{\boldsymbol{\mathcal{P}}^{4}(\mathbb{A})}.$$

So, we have

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models \exists x \ \Big(\text{" } x \text{ is infinite "} \quad \wedge \quad \exists f \in \boldsymbol{\mathcal{P}}^{4}(x) \ \ f : \mathcal{P}_{fin}(x) \xrightarrow{onto} \mathcal{P}(x) \Big).$$

Translating " x is infinite " by the formula $\forall n, g \in \boldsymbol{\mathcal{P}}^{\omega}(x) \ \Big(n \in \omega \longrightarrow \neg g : x \xrightarrow{1-1} n \Big)$ yields

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models \exists x \left(\begin{array}{c} \forall n, g \in \boldsymbol{\mathcal{P}}^{\omega}(x) \ \Big(n \in \omega \longrightarrow \neg g : x \xrightarrow{1-1} n \Big) \\ \\ \wedge \\ \\ \exists f \in \boldsymbol{\mathcal{P}}^{\omega}(x) \quad f : \mathcal{P}_{fin}(x) \xrightarrow{onto} \mathcal{P}(x) \end{array} \right).$$

Since, $\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models (\mathbf{AC})^{\boldsymbol{\mathcal{P}}^{\infty}(\varnothing)}$, by Corollary 417, there exists some symmetric submodel $\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}}$ such that

$$\mathcal{M}^{\mathbf{HS}_{\mathscr{F}}}_{ost.} \models \exists x \ \Big(\text{" } x \text{ is infinite "} \quad \wedge \quad \exists f \in \boldsymbol{\mathcal{P}}^{4}(x) \ \ f : \mathcal{P}_{fin}(x) \xrightarrow{onto} \mathcal{P}(x) \Big).$$

which gives

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \exists x \left(\begin{array}{c} \forall n, g \ \Big(n \in \omega \longrightarrow \neg g : x \xrightarrow{1-1} n \Big) \\ \\ \wedge \\ \\ \exists f \quad f : \mathcal{P}_{fin}(x) \xrightarrow{onto} \mathcal{P}(x) \end{array} \right).$$

or equivalently,

$$\widehat{\mathbf{M}[G]}^{\mathcal{F}_{\mathcal{A}}} \models \text{“ there exist an infinite set } x \text{ and a mapping } f : \mathcal{P}_{fin}(x) \xrightarrow{onto} \mathcal{P}(x) \text{ ”.}$$

□ 418