Part IV

Gödel's Constructible Universe

Chapter 11

The Constructible Sets

11.1 Definability

Definition 259 (Definability)**.** *Given any set Y, we say that $X \subseteq Y$ is definable over Y if there exists some $\mathcal{L}_{\mathrm{ST}}$-formula $\varphi := \varphi(x, x_1, \ldots, x_n)$ whose free variables are among $x, x_1, \ldots, x_n$ and parameters $a_1, \ldots, a_n \in Y$ such that*

$$X = \left\{ x \in Y \mid \Big(\varphi(x, a_1/x_1, \ldots, a_n/x_n)\Big)^{Y} \right\}.$$

Definition 260 (Definable subsets)**.** *Let Y be any set. The set of the definable subsets of Y is defined as*

$$\{X \subseteq Y \mid X \text{ is definable over } Y\}.$$

Notice that this definition does not fall under the strict framework of set theory. As such it quantifies over first order formulas which are not members of set theory. So, there are two options here in order to properly define it.
We can easily define a way of coding $\mathcal{L}_{\mathrm{ST}}$-formulas and proofs within **ZF** (or **ZFC**, or etc.) such that — among others — the following sets are $\mathcal{P}rim.\ \mathcal{R}ec.$:

- The set of all codes of $\mathcal{L}_{\mathrm{ST}}$-formulas

$$\left\{ \ulcorner\varphi\urcorner \mid \varphi \text{ is a formula from } \mathcal{L}_{\mathrm{ST}} \right\}$$

- The set of all codes of formulas from $\mathcal{L}_{\text{ST}}$ that contain the variable x_n

$$\mathcal{F}_{\checkmark x} = \left\{ \left(\ulcorner \varphi \urcorner, n \right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\text{ST}} \textit{ and } \varphi \textit{ contains } x_n \right\}$$

- The set of all codes of formulas from $\mathcal{L}_{\text{ST}}$ that do not contain the variable x_n

$$\mathcal{F}_{\times x} = \left\{ \left(\ulcorner \varphi \urcorner, n \right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\text{ST}} \textit{ and } \varphi \textit{ does not contain } x_n \right\}$$

- The set of all codes of formulas from $\mathcal{L}_{\text{ST}}$ that contain x_n as a free variable

$$\mathcal{F}_{\checkmark x \textit{ free}} = \left\{ \left(\ulcorner \varphi \urcorner, n \right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\text{ST}} \textit{ and } x_n \textit{ is free in } \varphi \right\}$$

- The set of all codes of formulas from $\mathcal{L}_{\text{ST}}$ that contain x_n as a bound variable

$$\mathcal{F}_{\checkmark x \textit{ bound}} = \left\{ \left(\ulcorner \varphi \urcorner, n \right) \mid \varphi \textit{ is a formula from } \mathcal{L}_{\text{ST}} \textit{ and } x_n \textit{ is bound in } \varphi \right\}$$

- The set of all codes of closed formulas from $\mathcal{L}_{\text{ST}}$

$$\mathcal{F}_{\checkmark \textit{closed}} = \left\{ \ulcorner \varphi \urcorner \mid \varphi \textit{ is a closed formula from } \mathcal{L}_{\text{ST}} \right\}$$

We define a class-relation $\texttt{Correct} \subseteq \omega \times \mathbf{V} \times \mathbf{V}$ by

$$\texttt{Correct}\,(k, S, Y) \iff \begin{cases} \text{“} k = \ulcorner \varphi \urcorner \text{ codes an } \mathcal{L}_{\text{ST}}\text{-formula } \varphi \text{”} \\ \wedge \\ \text{“} S \text{ is a mapping from some (finite) set of integers to } Y \text{”} \\ \wedge \\ \text{“ for every integer } n \text{ s.t. } x_n \text{ is free in } \varphi,\ n \in \textit{dom}\,(S) \text{ ”.} \end{cases}$$

We also define a class-relation $\texttt{Holds} \subseteq \omega \times \mathbf{V} \times \mathbf{V}$ by induction on the integers by

$$\texttt{Holds}(k,S,Y) \iff \left\{ \begin{array}{c} k = \ulcorner\varphi\urcorner \ \wedge\ \texttt{Correct}(\ulcorner\varphi\urcorner, S, Y) \\ \bigwedge \\ \left(\begin{array}{c} \ulcorner\varphi\urcorner = \ulcorner x_i = x_j\urcorner \ \wedge\ S(i) = S(j) \\ \vee \\ \ulcorner\varphi\urcorner = \ulcorner x_i \in x_j\urcorner \ \wedge\ S(i) \in S(j) \\ \vee \\ \left(\ulcorner\varphi\urcorner = \ulcorner\neg\psi\urcorner \ \wedge\ \neg\texttt{Holds}(\ulcorner\psi\urcorner, S, Y)\right) \\ \vee \\ \left(\ulcorner\varphi\urcorner = \ulcorner(\psi \wedge \theta)\urcorner \ \wedge\ \texttt{Holds}(\ulcorner\psi\urcorner, S, Y) \ \wedge\ \texttt{Holds}(\ulcorner\theta\urcorner, S, Y)\right) \\ \vee \\ \ulcorner\varphi\urcorner = \ulcorner\exists x_i\ \psi\urcorner \ \wedge\ \exists y \in Y\ \texttt{Holds}\left(\ulcorner\psi\urcorner, (i,y) \cup S \restriction \left(dom(S) \smallsetminus \{i\}\right), Y\right) \end{array} \right) \end{array} \right.$$

Definition 261 (Definability defined inside set theory)**.** *We define a "set-like" class-relation* $\textit{Definable\_Over} \subseteq \mathbf{V} \times \mathbf{V}$

$$\textit{Definable\_Over}(X,Y)$$

$$\iff$$

$$\exists n \in \omega\ \exists \ulcorner\varphi\urcorner \left(\begin{array}{c} \text{"}\varphi \text{ has exactly } x_0, x_1, \ldots, x_n \text{ as free variables "} \\ \wedge \\ \exists S \text{ "}S \text{ is a mapping from } \{1,\ldots,n\} \text{ to } Y\text{"} \\ \forall x_0 \in Y \ \left(x_0 \in X \longleftrightarrow \textit{Holds}(\ulcorner\varphi\urcorner, S \cup \{(0,x_0)\}, Y)\right). \end{array} \right)$$

Definition 262 (Definable subsets defined inside set theory)**.** *Let Y be any set The set* $\textit{Def}(Y)$ *of the definable subsets of Y is defined as*

$$\textit{Def}(Y) = \{X \subseteq Y \mid \textit{Definable\_Over}(X,Y)\}.$$

Definition 263 (Definability defined outside set theory)**.** *Given any set Y, we say that $X \subseteq Y$ is definable over Y if there exists some $\mathcal{L}_{\text{ST}}$-formula $\varphi := \varphi(x, x_1, \ldots, x_n)$ whose free variables are among $x, x_1, \ldots, x_n$ and parameters $a_1, \ldots, a_n \in Y$ such that*

$$X = \left\{x \in Y \mid \varphi(x, a_1/x_1, \ldots, a_n/x_n)^Y\right\}.$$

Remark 264. (Equivalence of the definition of definability outside set theory and inside set theory). Given any sets $X \subseteq Y$,

$$X \text{ is definable over } Y$$

$$\Longleftrightarrow$$

$$\exists n \in \omega\ \exists \ulcorner\varphi\urcorner \left(\begin{array}{c} \text{“}\varphi \text{ has exactly } x_0, x_1, \ldots, x_n \text{ as free variables ”} \\ \wedge \\ \exists S \text{ “} S \text{ is a mapping from } \{1, \ldots, n\} \text{ to } Y\text{”} \\ \forall x_0 \in Y \quad \Big(x_0 \in X \longleftrightarrow \texttt{Holds}\left(\ulcorner\varphi\urcorner, S \cup \{(0, x_0)\}, Y\right)\Big). \end{array}\right)$$

Proof of Remark 264: Exercise. □ 264

Notation 265. *Given any set A, we denote by $\mathscr{P}_{\mathit{fin.}}(A)$ the set of all finite subsets of A.*

Lemma 266 (**ZF**)**.** *Let Y be any set.*

(1) $Y \in \mathtt{Def}(Y)$

(2) $\mathscr{P}_{\mathit{fin.}}(Y) \subseteq \mathtt{Def}(Y) \subseteq \mathscr{P}(Y)$

(3) Y *transitive* $\implies Y \subseteq \mathtt{Def}(Y)$

(4) *(**AC**)* $|Y| \geqslant \aleph_0 \implies |\mathtt{Def}(Y)| = |Y|$.

Proof of Lemma 266:

(1) Clearly,

$$Y = \{x \in Y \mid x = x\} = \left\{x \in Y \mid (x = x)^Y\right\}.$$

(2) Clearly $\varnothing \in \mathtt{Def}(Y)$. If $\varnothing \neq X \in \mathscr{P}_{\text{fin.}}(Y)$, then there exists $a_1, \ldots, a_n$ such that $X = \{a_1, \ldots, a_n\}$. One has

$$X = \left\{x \in Y \mid \bigvee_{1 \leqslant i \leqslant n} x = a_i\right\} = \left\{x \in Y \mid \left(\bigvee_{1 \leqslant i \leqslant n} x = a_i\right)^Y\right\}.$$

(3) Take any $y \in Y$. Since Y is transitive, it follows $y \subseteq Y$, hence

$$y = \{x \in Y \mid x \in y\} = \left\{x \in Y \mid (x \in y)^Y\right\}.$$

(4) One has $\mathscr{P}_{\text{fin.}}(Y) \subseteq \mathtt{Def}(Y)$, hence $|Y| = |\mathscr{P}_{\text{fin.}}(Y)| \leqslant |\mathtt{Def}(Y)|$. Moreover, since there are countably many $\mathcal{L}_{\text{ST}}$-formulas and $|Y^{<\omega}| = |Y|$, one has

$$|\mathtt{Def}(Y)| \leqslant \aleph_0 \cdot |Y^{<\omega}| = \aleph_0 \cdot |Y| = |Y|.$$

□ 266

11.2 The Constructible Sets

Definition 267 (Gödel's Constructible Universe)**.** *By transfinite recursion on $\alpha \in \mathbf{On}$ we define the sets $\mathbf{L}(\alpha)$ by:*

- $\mathbf{L}(0) = \varnothing$
- $\mathbf{L}(\alpha + 1) = \mathit{Def}(\mathbf{L}(\alpha))$
- $\mathbf{L}(\alpha) = \bigcup_{\xi < \alpha} \mathbf{L}(\xi)$ *(α a limit ordinal).*

We also define Gödel's Constructible Universe as the class

$$\mathbf{L} = \bigcup_{\alpha \in \mathbf{On}} \mathbf{L}(\alpha).$$

Definition 268. *If $x \in \mathbf{L}$, then*

$$rk_{\mathbf{L}}(x) = \text{ the least } \alpha \in \mathbf{On} \text{ s.t. } x \in \mathbf{L}(\alpha + 1).$$

We list a few properties of the constructible hierarchy that will prove helpful.

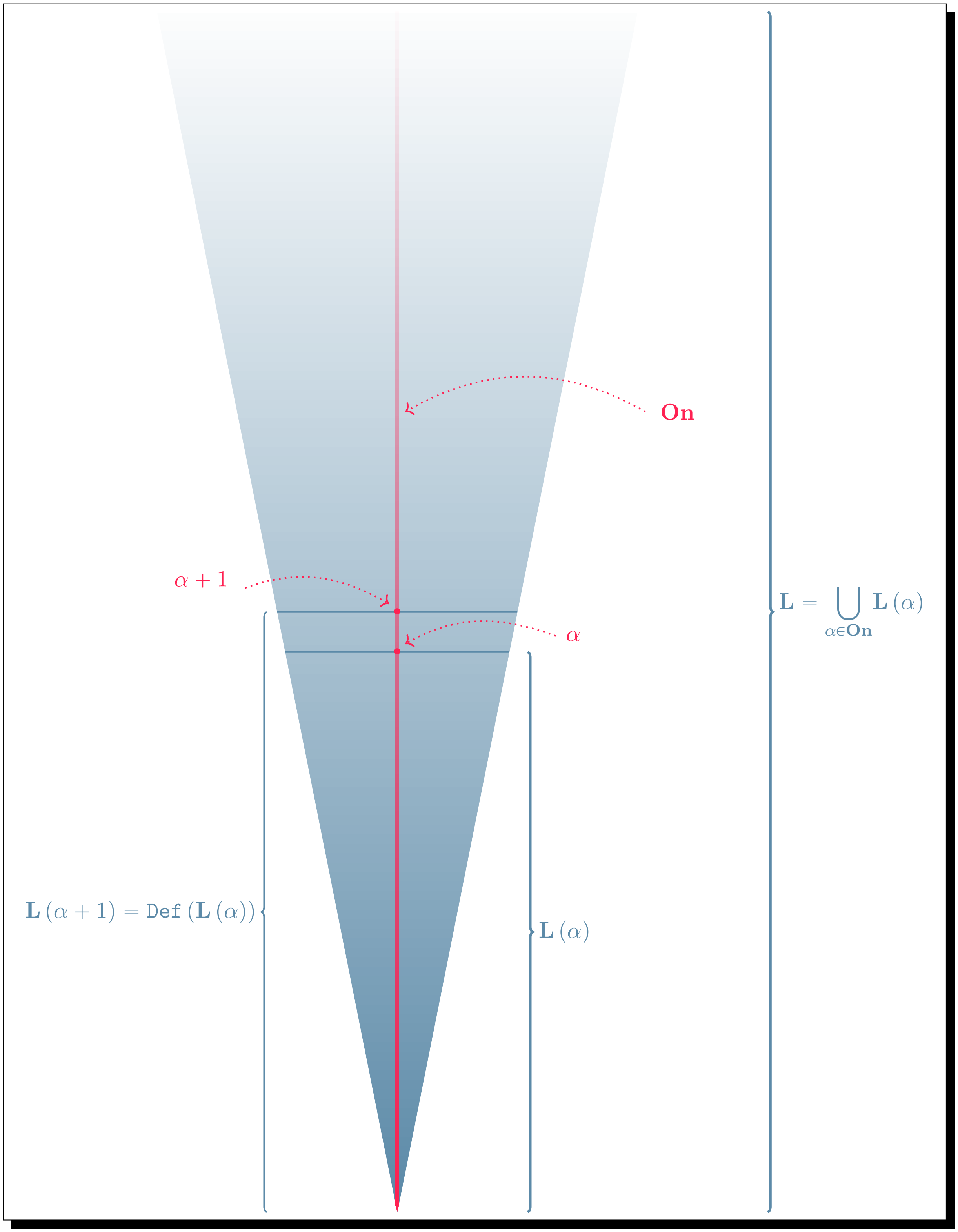

Figure 11.1: The Universe $\mathbf{L} = \bigcup_{\alpha \in \mathbf{On}} \mathbf{L}(\alpha)$.

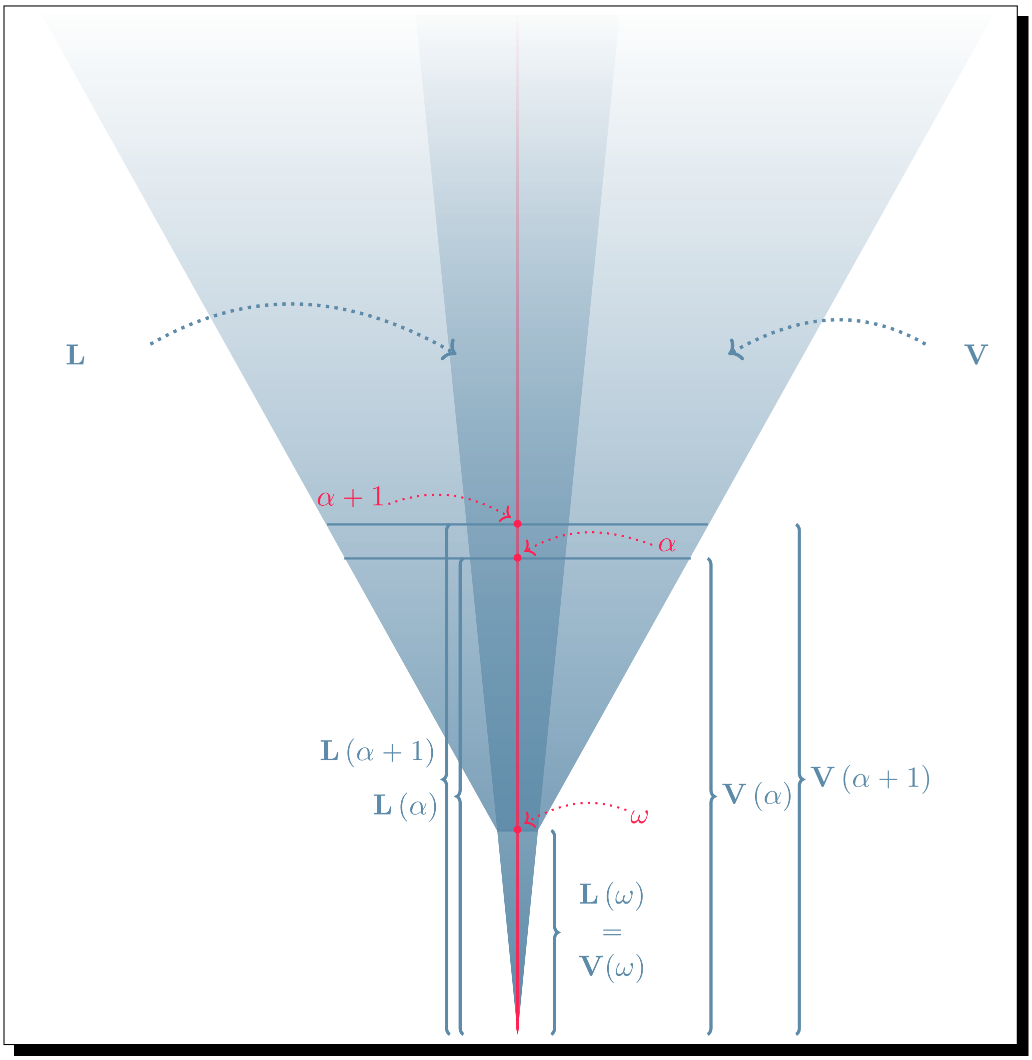

Figure 11.2: The Classes $\mathbf{L} = \bigcup_{\alpha \in \mathbf{On}} \mathbf{L}(\alpha)$ and $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} \mathbf{V}(\alpha)$.

Lemma 269 (ZF). *Given any ordinals $\xi < \alpha$,*

(1) $\mathbf{L}(\alpha) \subseteq \mathbf{V}(\alpha)$

(2) *For all* $\alpha \leqslant \omega$, $\mathbf{L}(\alpha) = \mathbf{V}(\alpha)$

(3) $\mathbf{L}(\alpha)$ *is a transitive set*

(4) $\mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha)$

(5) $\mathbf{L}(\xi) \in \mathbf{L}(\alpha)$

(6) $\mathbf{L}(\alpha) = \{x \in \mathbf{L} \mid rk_{\mathbf{L}}(x) < \alpha\}$

(7) $\alpha \in \big(\mathbf{L}(\alpha+1) \setminus \mathbf{L}(\alpha)\big)$.

Proof of Lemma 269: The different proofs all go by induction on α

(1) is obvious by definition of both $\mathbf{L}(\alpha)$ and $\mathbf{V}(\alpha)$.

(2) is immediate by Lemma 266 (2).

(3) $\boldsymbol{\alpha := 0}$ immediate since $\mathbf{L}(0) = \varnothing$;

$\boldsymbol{\alpha := \alpha + 1}$ if $x \in X \in \mathbf{L}(\alpha+1)$, then $x \in X \subseteq \mathbf{L}(\alpha)$. So, $x \in \mathbf{L}(\alpha)$ and also $x \subseteq \mathbf{L}(\alpha)$ since by induction hypothesis $\mathbf{L}(\alpha)$ is transitive. Then, one has

$$x = \{y \in \mathbf{L}(\alpha) \mid y \in x\} = \Big\{y \in \mathbf{L}(\alpha) \mid (y \in x)^{\mathbf{L}(\alpha)}\Big\} \in \mathbf{L}(\alpha+1).$$

$\boldsymbol{\alpha}$ **limit** If $x \in X \in \mathbf{L}(\alpha)$, then $x \in X \in \mathbf{L}(\xi)$ holds for some $\xi < \alpha$. By induction hypothesis, $\mathbf{L}(\xi)$ is transitive which yields $x \in \mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha)$.

(4) By induction on α.

$\boldsymbol{\alpha := 0}$ immediate since there is no $\xi < \alpha$.

$\boldsymbol{\alpha := \alpha + 1}$ One has

(a) $\mathbf{L}(\alpha) \subseteq \mathbf{L}(\alpha+1)$ — by Lemma 266 (3) —

(b) $\mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha)$ — by induction hypothesis —

which yields

$$\mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha) \subseteq \mathbf{L}(\alpha+1),$$

hence, $\mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha+1)$.

$\boldsymbol{\alpha}$ **limit** Immediate since $\mathbf{L}(\alpha) = \bigcup_{\xi<\alpha} \mathbf{L}(\xi)$.

(5) One has

(a) $\mathbf{L}(\xi+1) \subseteq \mathbf{L}(\alpha)$ — by Lemma 269 (4)

(b) $\mathbf{L}(\xi) \in \mathbf{L}(\xi+1)$ — by Lemma 266 (1)

So, all together, one obtains $\mathbf{L}(\xi) \in \mathbf{L}(\alpha)$.

(6) Clearly, one has

(a) $\left(x \in \mathbf{L}(\alpha) \implies rk_{\mathbf{L}}(x) < \alpha\right)$: since $\mathbf{L}(\alpha) = \bigcup_{\xi \leqslant \alpha} \mathbf{L}(\xi)$ hods by Lemma 269 (4)

(b) $\left(rk_{\mathbf{L}}(x) < \alpha \implies x \in \mathbf{L}(\alpha)\right)$: since $rk_{\mathbf{L}}(x) = \xi < \alpha \implies x \in \mathbf{L}(\xi+1) \subseteq \mathbf{L}(\alpha)$.

(7) First, notice that $rk(\alpha) = \alpha$, hence $\alpha \in \left(\mathbf{V}(\alpha+1) \smallsetminus \mathbf{V}(\alpha)\right)$.

(a) $\alpha \notin \mathbf{L}(\alpha)$ holds since $\mathbf{L}(\alpha) \subseteq \mathbf{V}(\alpha)$ and $\alpha \notin \mathbf{V}(\alpha)$.

(b) $\alpha \in \mathbf{L}(\alpha+1)$ is shown by induction on α.

$\boldsymbol{\alpha := 0}$ is immediate since $\mathbf{L}(1) = \{\varnothing\}$.

$\boldsymbol{\alpha := \alpha + 1}$ for every ordinal $\xi \leqslant \alpha$

$$\xi \in \mathbf{L}(\alpha+1)$$

holds since we have $\left(\xi \in \mathbf{L}(\xi+1)\right)$ by induction hypothesis, and $\left(\mathbf{L}(\xi+1) \subseteq \mathbf{L}(\alpha)\right)$ by Lemma 269 (4). So, one has both

- $\alpha \subseteq \mathbf{L}(\alpha+1)$
- $\alpha \in \mathbf{L}(\alpha+1)$

which yields

$$\begin{aligned}\alpha+1 &= \left\{y \in \mathbf{L}(\alpha+1) \mid y \in \alpha \vee y = \alpha\right\} \\ &= \left\{y \in \mathbf{L}(\alpha+1) \mid \left(y \in \alpha \vee y = \alpha\right)^{\mathbf{L}(\alpha+1)}\right\} \in \mathbf{L}(\alpha+2).\end{aligned}$$

$\boldsymbol{\alpha}$ **limit** for every $\xi < \alpha$, the induction hypothesis gives

$$\xi \in \mathbf{L}(\xi+1)$$

which yields

$$\mathbf{L}(\alpha) \cap \mathbf{On} = \alpha.$$

Using the fact that "x is an ordinal" is a Δ_0^{0-rud}-formula, hence absolute for transitive classes (see Lemma 199) we obtain:

$$\begin{aligned}\alpha &= \left\{x \in \mathbf{L}(\alpha) \mid \text{“}x \text{ is an ordinal”}\right\} \\ &= \left\{x \in \mathbf{L}(\alpha) \mid \left(\text{“}x \text{ is an ordinal”}\right)^{\mathbf{L}(\alpha)}\right\} \in \mathbf{L}(\alpha+1).\end{aligned}$$

□ 269

Since for all integer n, we have $\mathbf{V}(n) = \mathbf{L}(n)$, we notice that $|\mathbf{L}(0)| = |\mathbf{V}(0)| = |\varnothing| = 0$ and for each n $|\mathbf{L}(n+1)| = |\mathbf{V}(n+1)| = 2^n$. Therefore $|\mathbf{V}(\omega)| = |\mathbf{L}(\omega)| = \aleph_0$ holds. But as soon as $\omega < \alpha$, the whole picture of the cardinality of $\mathbf{L}(\alpha)$ becomes very different from the one of $\mathbf{V}(\alpha)$. Indeed, assuming **AC**, the cardinality of $\mathbf{V}(\omega+\alpha)$ is $\beth_\alpha$ (see Definition 122) whereas we will see that the cardinality of $\mathbf{L}(\omega+\alpha)$ is simply the cardinality of α: compare $\beth_\alpha$ with $|\alpha|$!

Lemma 270 (**ZFC**)**.** *Given any $\omega \leqslant \alpha \in \mathbf{On}$,*

$$|\mathbf{L}(\alpha)| = |\alpha|.$$

Proof of Lemma 270: By induction on $\alpha \geqslant \omega$.

$\boldsymbol{\alpha := \omega}$ In this case, $\mathbf{L}(\omega) = \mathbf{V}(\omega)$, hence $|\mathbf{L}(\omega)| = |\mathbf{V}(\omega)| = \aleph_0 = |\omega|$.

$\boldsymbol{\alpha := \alpha + 1}$ since $|\alpha+1| = |\alpha|$, it is enough to show $|\mathbf{L}(\alpha)| = |\mathbf{L}(\alpha+1)|$.

(1) $|\mathbf{L}(\alpha)| \leqslant |\mathbf{L}(\alpha+1)|$: immediate from $\mathbf{L}(\alpha) \subseteq \mathbf{L}(\alpha+1)$.
(2) $|\mathbf{L}(\alpha)| \geqslant |\mathbf{L}(\alpha+1)|$:

$$\begin{aligned}\mathbf{L}(\alpha+1) &= \mathtt{Def}\big(\mathbf{L}(\alpha)\big)\\ &= \{X \subseteq \mathbf{L}(\alpha) \mid X \text{ is definable over } \mathbf{L}(\alpha)\}\\ &= \Big\{X \subseteq \mathbf{L}(\alpha) \mid X = \big\{x \in \mathbf{L}(\alpha) \mid \varphi_{(x,a_1/x_1,\ldots,a_n/x_n)}{}^{\mathbf{L}(\alpha)}\big\}\Big\}\\ &\quad \text{for some } \varphi_{(x,x_1,\ldots,x_n)} \text{ and } \langle a_1,\ldots,a_n\rangle \in \mathbf{L}(\alpha)^{<\omega}.\end{aligned}$$

Since there are $\aleph_0$-many $\mathcal{L}_{\text{ST}}$-formulas and $|\mathbf{L}(\alpha)^{<\omega}| = |\mathbf{L}(\alpha)|$, we obtain

$$|\mathbf{L}(\alpha+1)| \leqslant \aleph_0 \cdot |\mathbf{L}(\alpha)| = |\mathbf{L}(\alpha)|.$$

$\boldsymbol{\alpha}$ **limit** Since $\mathbf{L}(\alpha) = \bigcup_{\xi<\alpha} \mathbf{L}(\xi)$, the induction hypothesis and Lemma 104 yield

$$|\mathbf{L}(\alpha)| = \left|\bigcup_{\xi<\alpha} \mathbf{L}(\xi)\right| \leqslant |\alpha|.$$

Also, for each $\xi < \alpha$, we have $\mathbf{L}(\xi) \subseteq \mathbf{L}(\alpha)$ and $\xi = |\mathbf{L}(\xi)| \leqslant |\mathbf{L}(\alpha)|$, hence

$$|\alpha| = |\sup_{\xi<\alpha} \xi| \leqslant |\mathbf{L}(\alpha)|.$$

So, we end up with $|\mathbf{L}(\alpha)| = |\alpha|$.

□ 270

11.3 The Constructible Universe Satisfies ZF

This section is devoted to showing that $\mathbf{L} \models \mathbf{ZF}$. This means that for each formula[1] $\varphi \in \mathbf{ZF}$ we need to show that $\mathbf{L} \models \varphi$. The proof is done within $\mathbf{ZF}$, i.e., we show $\mathbf{ZF} \vdash_c (\varphi)^{\mathbf{L}}$.

Theorem 271.

$$\mathbf{ZF} \vdash_c (\mathbf{ZF})^{\mathbf{L}}.$$

Proof of Theorem 271:

(1) $(\textbf{Extensionality})^{\mathbf{L}}$ since $\mathbf{L}$ is transitive (see Lemma 187).

(2) $(\textbf{Comprehension Schema})^{\mathbf{L}}$: at first glance, we may think of using the condition stated as a special case in Lemma 188 which assures that if $\mathbf{M}$ is closed under the powerset operation which maps x to $\mathscr{P}(x)$, then $(\textbf{Comprehension Schema})^{\mathbf{M}}$. But we cannot show that $\mathbf{L}$ is closed under this powerset operation. In fact, if it were the case then we would have in particular that for each ordinal α, $\mathbf{V}(\alpha) \subseteq \mathbf{L}$ would hold, which would yield $\mathbf{V} = \mathbf{L}$.

We are then left with — the main condition of Lemma 188, i.e., — proving that for each $\varphi(x, X, z_1, \ldots, z_k)$ with free variables among $\{x, X, z_1, \ldots, z_k\}$, one has

$$\forall X \in \mathbf{L}\ \forall z_1 \in \mathbf{L}\ \ldots \forall z_k \in \mathbf{L}\ \left\{x \in X \mid \big(\varphi(x, X, z_1, \ldots, z_k)\big)^{\mathbf{L}}\right\} \in \mathbf{L}.$$

In order to complete the proof we need a very general result known as as a reflection principle.

to be continued...

11.4 A Reflection Principle for L

We first need to prove a *reflection principle for* $\mathbf{L}$ which is a copy of the reflection principle for $\mathbf{V}$ due to Azriel Lévy and Richard Montague [27].

Reflection Principle (Lévy & Montague)**.** *Let $\varphi_0, \ldots, \varphi_n$ be any $\mathcal{L}_{\text{ST}}$-formulas.*

$$\mathbf{ZF} \vdash_c \forall \alpha \in \mathbf{On}\ \exists \beta > \alpha \quad \text{“}\varphi_0, \ldots, \varphi_n \text{ are absolute for } \mathbf{V}(\beta),\ \mathbf{V}.\text{”}$$

Proof of the Reflection Principle: Identical to the proof of Theorem 273, *mutatis mutandis.*

□ Reflection Principle

[1] Each axiom or instance of axiom schema.

In particular, the Reflection Principle states that given any finite subtheory Δ of **ZF** and any ordinal α, there exists some ordinal β (way larger than α) such that $\mathbf{V}(\beta) \models \Delta$. In particular, $\mathbf{V}(\beta)$ is a set which is a model of Δ. So, for every finite subtheory of $\Delta \subseteq \mathbf{ZF}$, we have $\mathbf{ZF} \vdash_c$ "Δ *has a model*". Notice that **ZFC** proves the compactness theorem which says that given any first order theory $\mathscr{T}$, the following holds:

$$\mathscr{T} \text{ has a model if and only if every finite subtheory of } \mathscr{T} \text{ has a model.}$$

So, at first glance it seems that a consequence is that **ZF** has a model, which contradicts "$\mathbf{ZF} \nvdash_c cons(\mathbf{ZF})$". But what is required to be able to apply the compactness theorem is not just that for for every finite subtheory of $\Delta \subseteq \mathbf{ZF}$, we have $\mathbf{ZF} \vdash_c$ "Δ *has a model*", but rather **ZF** proves that for all finite subtheory of $\Delta \subseteq \mathbf{ZF}$, $\mathbf{ZF} \vdash_c$ "Δ *has a model*". This is the difference between for each instance of a problem schema, proving that particular instance and proving the problem schema.

In particular, a consequence of the Reflection Principle is that, assuming **ZF** is consistent, **ZF** is not finitely axiomatizable. Otherwise, there would exist some formula $\varphi_{\mathbf{ZF}}$ such that

- $\mathbf{ZF} \vdash_c \varphi_{\mathbf{ZF}}$
- $\varphi_{\mathbf{ZF}} \vdash_c \psi$, holds for every $\psi \in \mathbf{ZF}$
- $\mathbf{V}(\beta) \models \varphi_{\mathbf{ZF}}$ holds for some (infinitely many indeed!) ordinal β.

Hence, $\mathbf{V}(\beta) \models \mathbf{ZF}$ would hold, contradicting Gödel's second incompleteness theorem.

Theorem 273 (**Reflection Principle for L**). *Let $\varphi_0, \ldots, \varphi_n$ be any $\mathcal{L}_{\mathrm{ST}}$-formulas.*

$$\mathbf{ZF} \vdash_c \forall\alpha \in \mathbf{On}\ \exists\beta > \alpha \quad \text{“}\varphi_0, \ldots, \varphi_n \ \textit{ are absolute for } \mathbf{L}(\beta), \mathbf{L}.\text{”}$$

Proof of Theorem 273: First, without loss of generality we may assume that the set of formulas $\{\varphi_0, \ldots, \varphi_n\}$ is closed under sub-formulas and only contains formulas using $\neg, \wedge$ as connectors and $\exists$ as quantifiers.

For each integer $i \leqslant n$ such that φ_i is of the form $\exists x\ \varphi_j(x, y_1, \ldots, y_{k_i})$, we define a class-function $\mathbf{G}_i : \underbrace{\mathbf{L} \times \ldots \times \mathbf{L}}_{k_i} \to \mathbf{On}$ by

$$\begin{aligned} \mathbf{G}_i(y_1, \ldots, y_{k_i}) &= 0 \text{ if } \Big(\neg\exists x\ \varphi_j(x, y_1, \ldots, y_{k_i})\Big)^{\mathbf{L}} \\ &= \text{least } \theta \text{ s.t. } \exists x \in \mathbf{L}(\theta)\ \ \big(\varphi_j(x, y_1, \ldots, y_{k_i})\big)^{\mathbf{L}} \text{ otherwise.} \end{aligned}$$

Then, for each integer $i \leqslant n$ we define a class-function $\mathbf{F}_i : \mathbf{On} \to \mathbf{On}$ by

$$\mathbf{F}_i(\xi) = \sup\{\mathbf{G}_i(y_1, \ldots, y_{k_i}) \mid y_1, \ldots, y_{k_i} \in \mathbf{L}(\xi)\} \text{ if } \mathbf{G}_i \text{ is defined}$$

$$\mathbf{F}_i(\xi) = 0 \text{ otherwise.}$$

Given any ordinal α, one defines the strictly increasing sequence $(\beta_k)_{n\in\omega}$ and a limit ordinal β by:

- $\beta_0 = \alpha$
- $\beta_{k+1} = \sup\left\{\beta_k + 1, \mathbf{F}_1(\beta_k), \dots, \mathbf{F}_n(\beta_k)\right\}$
- $\beta = \sup_{k\in\omega} \beta_k$

We show — by induction on the height of the formula — that for each integer $i \leqslant n$, one has

$$\forall y_1 \in \mathbf{L}(\beta) \dots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\varphi_i(y_1, \dots, y_{k_i})^{\mathbf{L}(\beta)} \longleftrightarrow \varphi_i(y_1, \dots, y_{k_i})^{\mathbf{L}}\right) \tag{11.1}$$

If φ_i is an atomic formula:

- If φ_i is $y_1 = y_2$, then one has $(y_1 = y_2)^{\mathbf{L}(\beta)} = (y_1 = y_2)^{\mathbf{L}} = (y_1 = y_2)$, hence

$$\forall y_1 \in \mathbf{L}(\beta)\ \forall y_2 \in \mathbf{L}(\beta)\ \left((y_1 = y_2)^{\mathbf{L}(\beta)} \longleftrightarrow (y_1 = y_2)^{\mathbf{L}}\right)$$

  comes down to

$$\forall y_1 \in \mathbf{L}(\beta)\ \forall y_2 \in \mathbf{L}(\beta)\ \left(y_1 = y_2 \longleftrightarrow y_1 = y_2\right)$$

  which trivially holds
- If φ_i is $y_1 \in y_2$, then one has $(y_1 \in y_2)^{\mathbf{L}(\beta)} = (y_1 \in y_2)^{\mathbf{L}} = (y_1 = y_2)$, hence

$$\forall y_1 \in \mathbf{L}(\beta)\ \forall y_2 \in \mathbf{L}(\beta)\ \left((y_1 \in y_2)^{\mathbf{L}(\beta)} \longleftrightarrow (y_1 \in y_2)^{\mathbf{L}}\right)$$

  comes down to

$$\forall y_1 \in \mathbf{L}(\beta)\ \forall y_2 \in \mathbf{L}(\beta)\ \left(y_1 \in y_2 \longleftrightarrow y_1 \in y_2\right)$$

  which trivially holds as well.
- If φ_i is either $y_1 = y_1$ or $y_1 \in y_1$, theses cases are taken care of by the previous cases by taking $y_2 = y_1$.

So, in any case, when φ_i is an atomic formula, the formula 11.1 is satisfied.

If $\varphi_i := \neg\varphi_j(y_1, \dots, y_{k_i})$: by induction hypothesis, one has

$$\forall y_1 \in \mathbf{L}(\beta) \dots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\varphi_j(y_1, \dots, y_{k_i})^{\mathbf{L}(\beta)} \longleftrightarrow \varphi_j(y_1, \dots, y_{k_i})^{\mathbf{L}}\right)$$

which yields

$$\forall y_1 \in \mathbf{L}(\beta) \dots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\neg\left(\varphi_j(y_1, \dots, y_{k_i})\right)^{\mathbf{L}(\beta)} \longleftrightarrow \neg\left(\varphi_j(y_1, \dots, y_{k_i})\right)^{\mathbf{L}}\right)$$

and finally gives

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left((\neg\varphi_j(y_1, \ldots, y_{k_i}))^{\mathbf{L}(\beta)} \longleftrightarrow (\neg\varphi_j(y_1, \ldots, y_{k_i}))^{\mathbf{L}} \right)$$

which shows that formula 13.1 is satisfied.

If $\varphi_{\mathbf{i}} := (\varphi_{\mathbf{j}}(\mathbf{y_1}, \ldots, \mathbf{y_{k_i}}) \wedge \varphi_{\mathbf{k}}(\mathbf{y_1}, \ldots, \mathbf{y_{k_i}}))$**:** by induction hypothesis, one has both

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\varphi_j(y_1, \ldots, y_{k_i})^{\mathbf{L}(\beta)} \longleftrightarrow \varphi_j(y_1, \ldots, y_{k_i})^{\mathbf{L}} \right)$$

and

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\varphi_k(y_1, \ldots, y_{k_i})^{\mathbf{L}(\beta)} \longleftrightarrow \varphi_k(y_1, \ldots, y_{k_i})^{\mathbf{L}} \right).$$

Now, given any $y_1, \ldots y_{k_i} \in \mathbf{L}(\beta)$, one has that both formulas $\varphi_j(y_1, \ldots, y_{k_i})$ and $\varphi_k(y_1, \ldots, y_{k_i})$ hold in $\mathbf{L}(\beta)$ if and only if they both hold in $\mathbf{L}$. Therefore, $\left(\varphi_j(y_1, \ldots, y_{k_i}) \wedge \varphi_k(y_1, \ldots, y_{k_i})\right)$ holds in $\mathbf{L}(\beta)$ if and only if it holds in $\mathbf{L}$. This shows that formula 11.1 is satisfied.

If $\varphi_{\mathbf{i}} := \exists \mathbf{x} \ \varphi_{\mathbf{j}}(\mathbf{x}, \mathbf{y_1}, \ldots, \mathbf{y_{k_i}})$**:** we have to check that

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left((\exists x \ \varphi_j(x, y_1, \ldots, y_{k_i}))^{\mathbf{L}(\beta)} \longleftrightarrow (\exists x \ \varphi_j(x, y_1, \ldots, y_{k_i}))^{\mathbf{L}} \right)$$

i.e., Clearly, the direction

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\exists x \in \mathbf{L}(\beta) \ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{L}(\beta)} \longrightarrow \exists x \in \mathbf{L} \ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{L}} \right)$$

is taken care of by the induction hypothesis. So, we show

$$\forall y_1 \in \mathbf{L}(\beta) \ldots \forall y_{k_i} \in \mathbf{L}(\beta) \quad \left(\exists x \in \mathbf{L} \ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{L}} \longrightarrow \exists x \in \mathbf{L}(\beta) \ \varphi_j(x, y_1, \ldots, y_{k_i})^{\mathbf{L}(\beta)} \right)$$

We fix $y_1 \in \mathbf{L}(\beta), \ldots, y_{k_i} \in \mathbf{L}(\beta)$. For some large enough integer p, one has

$$\{y_1, \ldots, y_{k_i}\} \subseteq \mathbf{L}(\beta_p).$$

By construction, there exists $x \in \mathbf{L}(\mathbf{G}_i(y_1, \ldots, y_{k_i}))$ such that $\left(\varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{L}}$. Since $\mathbf{G}_i(y_1, \ldots, y_{k_i}) \leqslant \mathbf{F}_i(\beta_p) \leqslant \beta_{p+1}$, it follows that there exists $x \in \mathbf{L}(\beta_{p+1}) \subseteq \mathbf{L}(\beta)$ such that $\left(\varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{L}}$. Finally, by induction hypothesis, there exists $x \in \mathbf{L}(\beta)$ such that $\left(\varphi_j(x, y_1, \ldots, y_{k_i})\right)^{\mathbf{L}(\beta)}$.

□ 273

We now resume the proof that the **Comprehension Schema** holds inside Gödel's constructible universe.

(2) *...proof of Theorem 271 continued:* Proving $\left(\textbf{Comprehension Schema}\right)^{\mathbf{L}}$ comes down to showing

$$\forall X \in \mathbf{L}\ \forall z_1 \in \mathbf{L}\ \ldots \forall z_k \in \mathbf{L}\ \ \left\{x \in X \mid \left(\varphi(x, X, z_1, \ldots, z_k)\right)^{\mathbf{L}}\right\} \in \mathbf{L}.$$

So, we fix $\{X, z_1, \ldots, z_k\} \subseteq \mathbf{L}$ and consider any $\alpha > \sup\{rk_{\mathbf{L}}(X), rk_{\mathbf{L}}(z_1), \ldots, rk_{\mathbf{L}}(z_k)\}$. By the Reflection Principle for $\mathbf{L}$ (Theorem 273) there exists some $\beta > \alpha$ such that the formula $\left(x \in X \wedge \varphi(x, X, z_1, \ldots, z_k)\right)$ is absolute for $\mathbf{L}(\beta), \mathbf{L}$.

Therefore,

$$\begin{aligned}\left\{x \in X \mid \varphi(x, X, z_1, \ldots, z_k)^{\mathbf{L}}\right\} &= \left\{x \in \mathbf{L} \mid \Big(x \in X \wedge \varphi(x, X, z_1, \ldots, z_k)\Big)^{\mathbf{L}}\right\} \\ &= \left\{x \in \mathbf{L}(\beta) \mid \Big(x \in X \wedge \varphi(x, X, z_1, \ldots, z_k)\Big)^{\mathbf{L}(\beta)}\right\} \in \mathbf{L}(\beta + 1).\end{aligned}$$

(3) $(\textbf{Pairing})^{\mathbf{L}}$ is almost immediate, since we only need to show

$$\forall x \in \mathbf{L}\ \forall y \in \mathbf{L}\ \exists z \in \mathbf{L}\ \ \left((x \in z\ \wedge\ y \in z)\right)^{\mathbf{L}}$$

i.e.,

$$\forall x \in \mathbf{L}\ \forall y \in \mathbf{L}\ \exists z \in \mathbf{L}\ \ (x \in z\ \wedge\ y \in z)$$

Take any $\alpha > \max\{rk_{\mathbf{L}}(x), rk_{\mathbf{L}}(y)\}$. One has $x, y \in \mathbf{L}(\alpha)$, hence

$$\begin{aligned}\{x, y\} &= \left\{z \in \mathbf{L}(\alpha) \mid (x = z\ \vee\ y = z)\right\} \\ &= \left\{z \in \mathbf{L}(\alpha) \mid \left((x = z\ \vee\ y = z)\right)^{\mathbf{L}}\right\} \in \mathbf{L}(\alpha + 1) \subseteq \mathbf{L}.\end{aligned}$$

(4) $(\textbf{Union})^{\mathbf{L}}$ is easy. We simply show that given any $X \in \mathbf{L}$, the set $\bigcup X$ also belongs to $\mathbf{L}$.

So, we assume $X \in \mathbf{L}(\alpha)$. Since $\mathbf{L}(\alpha)$ is transitive, $tc(X) \subseteq \mathbf{L}(\alpha)$ holds and

$$\begin{aligned}\bigcup X &= \left\{x \in tc(X) \mid \exists y\ (x \in y\ \wedge\ y \in X)\right\} \\ &= \left\{x \in tc(X) \mid \exists y\ \left((x \in y\ \wedge\ y \in X)\right)^{\mathbf{L}(\alpha)}\right\} \\ &= \left\{x \in tc(X) \mid \exists y \in \mathbf{L}(\alpha)\ \ \left((x \in y\ \wedge\ y \in X)\right)^{\mathbf{L}(\alpha)}\right\} \\ &= \left\{x \in \mathbf{L}(\alpha) \mid \exists y \in \mathbf{L}(\alpha)\ \ \left((x \in y\ \wedge\ y \in X)\right)^{\mathbf{L}(\alpha)}\right\} \\ &= \left\{x \in \mathbf{L}(\alpha) \mid \left(\exists y\ (x \in y\ \wedge\ y \in X)\right)^{\mathbf{L}(\alpha)}\right\} \in \mathbf{L}(\alpha + 1) \subseteq \mathbf{L}.\end{aligned}$$

(5) $(\textbf{Infinity})^{\mathbf{L}}$ is immediate since by Lemma 269 (7) $\mathbf{L} \cap \mathbf{On} = \mathbf{On}$; so in particular ω belongs to $\mathbf{L}$.

(6) (**Power Set**)$^{\mathbf{L}}$ is proved by making use of the fact that $\mathbf{L}$ is transitive and following Lemma 189 which states that it is enough to establish

$$\forall x \in \mathbf{L}\ \exists y \in \mathbf{L}\ \big(\mathscr{P}(x) \cap \mathbf{L}\big) \subseteq y.$$

Given any $x \in \mathbf{L}$ we set

$$\alpha = \sup\{rk_{\mathbf{L}}(z) + 1 \mid z \in \mathbf{L} \ \wedge\ z \subseteq x\},$$

so we obtain $\Big(\mathscr{P}(x) \cap \mathbf{L}\Big) \subseteq \mathbf{L}(\alpha)$.

(7) (**Foundation**)$^{\mathbf{L}}$ is immediate because working within **ZF** every class is $\in$-well-founded.

(8) (**Replacement Schema**)$^{\mathbf{L}}$ holds since, by Lemma 192, we need to show that given any formula $\varphi := \varphi(x, y, A, w_1, \ldots, w_n)$ whose free variables are among $x, y, A, w_1, ..., w_n$,

$\forall A \in \mathbf{L}\ \forall w_1 \in \mathbf{L} \ldots \forall w_n \in \mathbf{L}$

$$\Big(\forall x \in A \cap \mathbf{L}\ \exists! y \in \mathbf{L}\ (\varphi)^{\mathbf{L}} \longrightarrow \exists B \in \mathbf{L}\ \Big\{y \in \mathbf{L} \mid \exists x \in A \cap \mathbf{L}\ (\varphi)^{\mathbf{L}}\Big\} \subseteq B\Big)$$

Given any $A \in \mathbf{L},\ w_1 \in \mathbf{L}, \ldots, w_n \in \mathbf{L}$, we set

$$\alpha = \sup\Big(\Big\{rk_{\mathbf{L}}(y) + 1 \mid \exists x \in A\ \ (\varphi(x, y, A, w_1, \ldots, w_n))^{\mathbf{L}}\Big\} \cup \big\{rk_{\mathbf{L}}(A) + 1\big\}\Big).$$

in order to get

$$\begin{aligned}
&\Big\{y \in \mathbf{L} \mid \exists x \in A \cap \mathbf{L}\ \ (\varphi(x, y, A, w_1, \ldots, w_n))^{\mathbf{L}}\Big\} \\
&= \Big\{y \in \mathbf{L} \mid \exists x \in A\ \ (\varphi(x, y, A, w_1, \ldots, w_n))^{\mathbf{L}}\Big\} && \textit{(since } \mathbf{L} \textit{ is transitive)} \\
&= \Big\{y \in \mathbf{L} \mid \exists x \big(x \in A\ \wedge\ (\varphi(x, y, A, w_1, \ldots, w_n))^{\mathbf{L}}\big)\Big\} && \textit{(by definition of } \exists x \in A\textit{)} \\
&= \Big\{y \in \mathbf{L} \mid \exists x \big(x \in A\ \wedge\ \varphi(x, y, A, w_1, \ldots, w_n)\big)^{\mathbf{L}}\Big\} && \textit{(by definition of relativization)} \\
&= \Big\{y \in \mathbf{L} \mid \exists x \in \mathbf{L}\ \big(x \in A\ \wedge\ \varphi(x, y, A, w_1, \ldots, w_n)\big)^{\mathbf{L}}\Big\} && \textit{(since } A \subseteq \mathbf{L}\textit{)} \\
&= \Big\{y \in \mathbf{L} \mid \Big(\exists x \big(x \in A\ \wedge\ \varphi(x, y, A, w_1, \ldots, w_n)\big)\Big)^{\mathbf{L}}\Big\} && \textit{(by definition of relativization)}
\end{aligned}$$

By the Reflection Principle for $\mathbf{L}$ (on page 199), there exists some $\beta > \alpha$ such that the formula $\exists x\big(x \in A\ \wedge\ \varphi(x, y, A, w_1, \ldots, w_n)\big)$ is absolute for $\mathbf{L}(\beta), \mathbf{L}$. So, since the formula $\exists x\big(x \in A\ \wedge\ \varphi(x, y, A, w_1, \ldots, w_n)\big)$ holds in $\mathbf{L}$, it also holds in $\mathbf{L}(\beta)$. Since $A \in \mathbf{L}(\alpha) \subseteq \mathbf{L}(\beta)$ and $\mathbf{L}(\beta)$ is transitive, every element $x \in A$ belongs also to $\mathbf{L}(\beta)$. The same holds for every element y which is — so to speak — the image through the class-function $\varphi := \varphi(x, y, A, w_1, \ldots, w_n)$ of some element x of A: such a y belongs to $\mathbf{L}(\alpha) \subseteq \mathbf{L}(\beta)$. Therefore we are left with

$$\begin{aligned}
&\left\{y \in \mathbf{L} \mid \exists x \in A \ \ (\varphi(x, y, A, w_1, \ldots, w_n))^{\mathbf{L}}\right\} \\
&= \left\{y \in \mathbf{L} \mid \Big(\exists x\big(x \in A \ \wedge \ \varphi(x, y, A, w_1, \ldots, w_n)\big)\Big)^{\mathbf{L}}\right\} \\
&= \left\{y \in \mathbf{L} \mid \Big(\exists x\big(x \in A \ \wedge \ \varphi(x, y, A, w_1, \ldots, w_n)\big)\Big)^{\mathbf{L}(\beta)}\right\} \qquad \textit{(by the Reflection Principle)} \\
&= \left\{y \in \mathbf{L}(\beta) \mid \Big(\exists x\big(x \in A \ \wedge \ \varphi(x, y, A, w_1, \ldots, w_n)\big)\Big)^{\mathbf{L}(\beta)}\right\} \qquad \textit{(since } y \in \mathbf{L}(\alpha) \subseteq \mathbf{L}(\beta) \textit{)} \\
&\in \ \mathbf{L}(\beta + 1) \subseteq \mathbf{L}.
\end{aligned}$$

So, we have found a set — namely $\mathbf{L}(\beta + 1)$ — which contains all constructible images of elements of A by the class function-like formula $\varphi := \varphi(x, y, A, w_1, \ldots, w_n)$.

□ 271

Chapter 12

AC and CH inside Gödel's Constructible Universe

12.1 The Axiom of Constructibility and the axiom of choice

Not only the Constructible Universe satisfies the Axiom of **Choice** — since it satisfies the equivalent statement that every set can be well-ordered — but it satisfies that one can define a "set-like" class-relation on the whole universe of Constructible Sets that well-orders it.

Theorem 274.

$$\mathbf{ZF} \vdash_c (\mathbf{AC})^{\mathbf{L}}$$

Proof of Theorem 274: By induction on $\alpha \in \mathbf{On}$, one defines a (strict) well-ordering $\lhd_\alpha$ on $\mathbf{L}(\alpha)$ such that for all ordinals $\beta < \alpha$, $\lhd_\alpha$ extends $\lhd_\beta$. i.e., one has

$$\lhd_\alpha \cap \mathbf{L}(\beta) \times \mathbf{L}(\beta) = \lhd_\beta \text{ and for every } x \in \mathbf{L}(\beta) \text{ and } y \in \mathbf{L}(\alpha) \smallsetminus \mathbf{L}(\beta),\ x \lhd_\alpha y.$$

Before we define $\lhd_\alpha$, we need some notation.

Assuming $X \subseteq Y \subseteq \mathbf{L}(\alpha)$ and $\lhd_\alpha$ is a well-ordering on $\mathbf{L}(\alpha)$, we define the well-ordering $\lll_\alpha$ on $\omega \times \omega \times \mathbf{L}(\alpha)^{<\omega}$ as the lexicographic ordering induced by the usual ordering on the integers and the well-ordering $\lhd_\alpha$ on $\mathbf{L}(\alpha)$:

$$(n,k,S) \lll_\alpha (n',k',S') \iff \left(\begin{array}{c} n < n' \\ \vee \\ n = n' \ \wedge \ k < k' \\ \vee \\ n = n' \ \wedge \ k = k' \ \wedge \ lh(S) < lh(S') \\ \vee \\ n = n' \ \wedge \ k = k' \ \wedge \ lh(S) = lh(S') \ \wedge \ S \lhd_{lex.\alpha} S' \end{array} \right)$$

where

$$\langle a_0, a_1, \ldots, a_m \rangle \lhd_{lex.\alpha} \langle a'_0, a'_1, \ldots, a'_m \rangle$$
$$\Longleftrightarrow$$
$$\exists\, p \leqslant m \ \left(\forall i < p\ a_i = a'_i \ \wedge \ a_p \lhd_\alpha a'_p \right)$$

Given any sets X, Y such that $\texttt{Definable\_Over}(X, Y)$, we denote by $\texttt{Wittn.\_Def.\_Over}(X, Y)$ the following set:

$$\texttt{Wittn.\_Def.\_Over}(X, Y)$$
$$=$$
$$\left\{ (\ulcorner\varphi\urcorner, n, S) \in \omega \times \omega \times Y^{<\omega} \;\middle|\; \left(\begin{array}{c} \text{“}\ \varphi \text{ has exactly } x_0, x_1, \ldots, x_n \text{ as free variables ”} \\ \wedge \\ \text{“}\ S \text{ is a mapping from } \{1, \ldots, n\} \text{ to } Y\text{”} \\ \forall x_0 \in Y \ \left(x_0 \in X \longleftrightarrow \texttt{Holds}\left(\ulcorner\varphi\urcorner, S \cup \{(0, x_0)\}, Y\right) \right). \end{array} \right) \right\}$$

Finally, we define the well-orderings $\lhd_\alpha$ on $\mathbf{L}(\alpha)$ by induction on α by:

$\boldsymbol{\alpha} := \mathbf{0}$ Obviously $\lhd_0 = \varnothing$

$\boldsymbol{\alpha} := \boldsymbol{\beta} + \mathbf{1}$ $\lhd_\alpha$ is defined by $\forall x, y \in \mathbf{L}(\alpha)$,

$$x \lhd_\alpha y$$
$$\Longleftrightarrow$$
$$\left(\begin{array}{c} x, y \in \mathbf{L}(\beta) \ \wedge \ x \lhd_\beta y \\ \vee \\ x \in \mathbf{L}(\beta) \ \wedge \ y \notin \mathbf{L}(\beta) \\ \vee \\ x, y \notin \mathbf{L}(\beta) \ \wedge \ \left(\begin{array}{c} \ll_\beta\text{-least } (\ulcorner\varphi\urcorner, n, S) \in \texttt{Wittn.\_Def.\_Over}(x, \mathbf{L}(\beta)) \\ \ll_\beta \\ \ll_\beta\text{-least } (\ulcorner\varphi'\urcorner, n', S') \in \texttt{Wittn.\_Def.\_Over}(y, \mathbf{L}(\beta)) \end{array} \right) \end{array} \right)$$

$\boldsymbol{\alpha}$ limit $\lhd_\alpha$ is defined by: $\forall x, y \in \mathbf{L}(\alpha)$, let $\beta = \sup\{rk_{\mathbf{L}}(x) + 1, rk_{\mathbf{L}}(y) + 1\}$. Notice that $\beta < \alpha$ and $x, y \in \mathbf{L}(\beta)$ and set

$$x \lhd_\alpha y \iff x \lhd_\beta y.$$

So far, we have constructed for each ordinal α, a well-ordering $\lhd_\alpha$ of $\mathbf{L}(\alpha)$. It remains to show that every set $X \in \mathbf{L}$ can be well-ordered. For this, it is enough to consider $\alpha = rk_{\mathbf{L}}(X)$ since both $X \subseteq \mathbf{L}(\alpha)$ and $(\mathbf{L}(\alpha), \lhd_\alpha)$ is a well-ordering, which yields $(X, \lhd_\alpha)$.

$\square$ 274

Since for every ordinal $\beta < \alpha$, the well-ordering $\lhd_\alpha$ on $\mathbf{L}(\alpha)$ is an extension of the well-ordering $\lhd_\beta$ on $\mathbf{L}(\beta)$, we may easily define a class-relation $<_{\mathbf{L}}$ that well-orders the whole universe of constructible sets.

Definition 275 (Well-ordering of $\mathbf{L}$). *We define a class-relation $<_{\mathbf{L}}$ that well-orders $\mathbf{L}$ by*

$$x <_{\mathbf{L}} y \iff \Big(x \lhd_\alpha y \text{ and } \alpha = \max\{rk_{\mathbf{L}}(x), rk_{\mathbf{L}}(y)\} + 1\Big).$$

Definition 276 (Axiom of Constructibility)**.**

$$\mathbf{V} = \mathbf{L} \text{ is the statement } \text{“}\forall x\ \exists \alpha \in \mathbf{On}\ \ x \in \mathbf{L}(\alpha)\ \text{”}.$$

We have just proved

Theorem 277.

$$\mathbf{ZF} + \mathbf{V} = \mathbf{L} \vdash_c \mathbf{AC}.$$

12.2 The Axiom of Constructibility and the Generalized Continuum Hypothesis

We now come to **GCH** assuming $\mathbf{V} = \mathbf{L}$. This is slightly more complicated than **AC**. Our goal is to show that $\mathbf{L} \models \mathbf{GCH}$ or more precisely that $\mathbf{ZF} \vdash_c (\mathbf{GCH})^{\mathbf{L}}$ or equivalently to prove the following theorem:

Theorem 278.

$$\mathbf{ZF} + \mathbf{V} = \mathbf{L} \vdash_c \mathbf{GCH}.$$

This main theorem is a direct consequence of the following lemma which shows that if a subset of a cardinal number κ appears somewhere [1] in the construction of the Constructible Universe, then it appears in less than κ^+ steps.

[1]As opposed to never appearing anywhere, for the reason that it is simply not constructible.

Lemma 279 (**ZF**)**.** *If* $\mathbf{V} = \mathbf{L}$*, then for every infinite cardinal* κ*,*

$$\mathscr{P}(\kappa) \subseteq \mathbf{L}\left(\kappa^{+}\right).$$

Proof of Theorem 278: The result follows easily from Lemma 279 and Lemma 270 which stated that given any infinite ordinal α, one has $|\mathbf{L}(\alpha)| = |\alpha|$. Indeed, we obtain for every infinite cardinal κ,

$$\kappa < 2^{\kappa} = |\mathscr{P}(\kappa)| \leqslant \left|\mathbf{L}\left(\kappa^{+}\right)\right| = |\kappa^{+}| = \kappa^{+}.$$

□ 278

We now concentrate on proving the main lemma.
For this purpose, we make use of the notion of an $\mathcal{L}_{\text{ST}}$-*elementary submodel.*

Definition 280 (Elementary Submodel)**.** *Let* $X \subseteq Y$ *be sets.* X *is an elementary submodel of* Y *— denoted* $X \prec Y$ *— if and only if for all* $\mathcal{L}_{\text{ST}}$*-formula* $\varphi(x_1, \ldots, x_n)$ *— whose free variables are among* $x_1, \ldots, x_n$ *— and all* $a_1 \in X, \ldots, a_n \in X$*,*

$$\Big(\varphi(a_1/x_1, \ldots, a_n/x_n)\Big)^{X} \longleftrightarrow \Big(\varphi(a_1/x_1, \ldots, a_n/x_n)\Big)^{Y}.$$

In words, X is an elmentary submodel of Y if $X \subseteq Y$ and both structures satisfy the same formulas whose parameters are taken from the smaller one.
We now show that every subset of $\mathbf{L}(\alpha)$ can be extended in an elementary submodel of $\mathbf{L}(\alpha)$ whose cardinality does not exceed the one of X, provided that X is infinite.

Lemma 281 (**ZFC**)**.** *If* $\omega < \alpha$ *any limit ordinal, and* X *is any set such that* $X \subseteq \mathbf{L}(\alpha)$*. Then,*

there exists M *such that* $|M| = \sup\{|X|, \aleph_0\}$*,* $X \subseteq M$*, and* $M \prec \mathbf{L}(\alpha)$*.*

Proof of Lemma 281: We make use of the Tarski-Vaught criterion [2, 3, 4, 5, 6, 33]. This criterion states that

$$M \prec \mathbf{L}(\alpha)$$

$$\Longleftrightarrow$$

for each $\mathcal{L}_{\text{ST}}$-formula $\varphi(x_0, x_1, \ldots, x_n)$ and $a_1 \in M, \ldots, a_n \in M$

$$\big(\exists x_0\ \varphi(x_0, x_1, \ldots, x_n)\big)^{\mathbf{L}(\alpha)} \longrightarrow \big(\exists x_0\ \varphi(x_0, x_1, \ldots, x_n)\big)^{M}.$$

We construct M that satisfies the Tarski-Vaught criterion by recursion on the integers:

- $M_0 = X \cup \omega$
- $M_{n+1} = M_n \cup \left\{ x \in \mathbf{L}(\alpha) \,\middle|\, \begin{array}{c} \text{there exist } \varphi(x, x_1, \ldots, x_k), a_1 \in M_n, \ldots, a_k \in M_n \\ \big(\varphi(x, a_1, \ldots, a_k)\big)^{\mathbf{L}(\alpha)} \\ \wedge \\ \forall y \in \mathbf{L}(\alpha) \quad \Big(\big(\varphi(y, a_1, \ldots, a_k)\big)^{\mathbf{L}(\alpha)} \longrightarrow y \nless_{\mathbf{L}} x \Big) \end{array} \right\}$
- $M = \bigcup_{n \in \omega} M_n.$

It is then easy to check that M satisfies the following:

(1) $X \cup \omega \subseteq M \prec \mathbf{L}(\alpha)$.

(2) $|M_0| = |X \cup \omega| = \sup\{|X|, \aleph_0\} = |X|$.

(3) $|M_{n+1}| = |M_n|$ since $|M_n| \leqslant |M_{n+1}| \leqslant |M_n| + |M_n^{<\omega}| \cdot \aleph_0 = \sup\{|X|, \aleph_0\} = |M_n|$.

(4) $|M| = |X|$ since $|X| \leqslant |M| = |\bigcup_{n \in \omega} M_n \leqslant |M_0| \cdot \aleph_0 = |X| \cdot \aleph_0 = |X|$.

$\square$ 281

We now see that every elementary submodel of some $\mathbf{L}(\alpha)$ for some limit ordinal α is in fact isomorphic to some $\mathbf{L}(\beta)$ for some limit ordinal $\beta \leqslant \alpha$.

Lemma 282 (ZF). *Let M be any set and $\omega < \alpha$ any limit ordinal. If $M \prec \mathbf{L}(\alpha)$, then*

$$\textit{there exists } \beta \leqslant \alpha, \textit{ and an isomorphism } \pi : \big(M, \in\big) \approx \big(\mathbf{L}(\beta), \in\big).$$

Proof of Lemma 282: We simply consider the Mostowski collapse $\pi : \big(M, \in\big) \to \big(N, \in\big)$. We recall that for each $y \in M$, one has

$$\pi(y) = \big\{\pi(x) \mid x \in y\big\}.$$

Since $M \prec \mathbf{L}(\alpha)$ and $\big(\exists x\ x = a\big)^M$ holds for every element $a \in M$, we obtain $\big(\exists x\ x = a\big)^{\mathbf{L}(\alpha)}$ holds as well for every element $a \in M$. Therefore, $M \prec \mathbf{L}(\alpha)$ implies $M \subseteq \mathbf{L}(\alpha)$.

Moreover, since $\mathbf{L}(\alpha)$ is transitive, $\mathbf{L}(\alpha)$ is also extensional[2] by Lemma 187. It follows that $\in$ is also extensional on M because from $M \prec \mathbf{L}(\alpha)$ we have

$$\Big(\forall x\ \forall y\ \big(\forall z\,(z \in x \longleftrightarrow z \in y) \to x = y\big)\Big)^M \longleftrightarrow \Big(\forall x\ \forall y\ \big(\forall z\,(z \in x \longleftrightarrow z \in y) \to x = y\big)\Big)^{\mathbf{L}(\alpha)}.$$

[2] We recall that $\in$ is extensional on "$\mathbf{L}(\alpha)$" means $\Big(\forall x\ \forall y\ \big(\forall z\,(z \in x \longleftrightarrow z \in y) \to x = y\big)\Big)^{\mathbf{L}(\alpha)}$.

i.e.,

$$\left(\textbf{Extensionality}\right)^{M} \longleftrightarrow \left(\textbf{Extensionality}\right)^{\mathbf{L}(\alpha)}.$$

By Corollary 186, it follows that N is both extensional and transitive[3]. We are going to show that $N = \mathbf{L}(\beta)$ for some limit ordinal $\beta \leqslant \alpha$. For this, we first consider $\pi^{-1} : N \to M \subseteq \mathbf{L}(\alpha)$. Given any $\mathcal{L}_{\text{ST}}$-formula $\varphi(x_1, \ldots, x_n)$ whose free variables are among $x_1, \ldots, x_n$, and any $a_1, \ldots, a_n \in N$, since both $N \approx M$ and $M \prec \mathbf{L}(\alpha)$ hold, one has

$$\begin{aligned}\left(\varphi(a_1/x_1, \ldots, a_n/x_n)\right)^{N} &\longleftrightarrow \left(\varphi(\pi^{-1}(a_1)/x_1, \ldots, \pi^{-1}(a_n)/x_n)\right)^{M}\\ &\longleftrightarrow \left(\varphi(\pi^{-1}(a_1)/x_1, \ldots, \pi^{-1}(a_n)/x_n)\right)^{\mathbf{L}(\alpha)}\end{aligned}$$

which shows that $\pi^{-1} : N \to \mathbf{L}(\alpha)$ is an elementary injection: both $\pi^{-1} : N \xrightarrow{1-1} \mathbf{L}(\alpha)$ is an injective homomorphism and $\pi^{-1}[N] \prec \mathbf{L}(\alpha)$.

In particular we have

$$\forall \xi < \alpha\ \exists x \in \mathbf{L}(\alpha) \quad \underbrace{x = \mathbf{L}(\xi)}_{\exists y \in \mathbf{L}(\alpha)\ \underbrace{\psi(x, y, \xi)}_{\Delta_0^{0-rud}}}$$

where ψ is some Δ_0^{0-rud}-formula (hence absolute for all transitive classes). So, we have

$$\begin{aligned}\forall \xi < \alpha\ \exists x \in \mathbf{L}(\alpha) \quad x = \mathbf{L}(\xi) &\longleftrightarrow \forall \xi \in \mathbf{On}\ \exists x \in \mathbf{L}(\alpha)\ \ \exists y \in \mathbf{L}(\alpha)\ \ \psi(x, y, \xi)\\ &\longleftrightarrow \left(\forall \xi \in \mathbf{On}\ \exists x\ \ \exists y\ \psi(x, y, \xi)\right)^{\mathbf{L}(\alpha)}\\ &\longleftrightarrow \left(\forall \xi \in \mathbf{On}\ \exists x\ \ \exists y\ \psi(x, y, \xi)\right)^{N}\\ &\longleftrightarrow \forall \xi \in \mathbf{On} \cap N\ \exists x \in N\ \ \exists y \in N\ \left(\psi(x, y, \xi)\right)^{N}\\ &\longleftrightarrow \forall \xi \in \mathbf{On} \cap N\ \exists x \in N\ \ \underbrace{\exists y \in N\ \psi(x, y, \xi)}_{x = \mathbf{L}(\xi)}\end{aligned}$$

Set $\beta = N \cap \mathbf{On}$ (notice that β is an ordinal by transitivity of N). By the result above we have

$$\forall \xi < \beta \quad \mathbf{L}(\xi) \in N.$$

We now use the fact that α is a limit ordinal, to show that β is also a limit ordinal. For this, we make use of the fact that "$z \in \mathbf{On}$" is Δ_0^{0-rud}, hence absolute for transitive classes.

$$\begin{aligned}\alpha \text{ is a limit ordinal} &\longrightarrow \forall \xi \in \alpha\ \exists \zeta \in \alpha \quad \xi \in \zeta\\ &\longrightarrow \left(\forall \xi \in \mathbf{On}\ \exists \zeta \in \mathbf{On} \quad \xi \in \zeta\right)^{\mathbf{L}(\alpha)}\\ &\longrightarrow \left(\forall \xi \in \mathbf{On}\ \exists \zeta \in \mathbf{On} \quad \xi \in \zeta\right)^{N}.\end{aligned}$$

[3] As it is the Mostowski collapse of an extensional class (set!) M.

So we have

$$\left.\begin{array}{r}\beta \text{ is a limit ordinal} \\ \forall \xi < \beta \quad \mathbf{L}(\xi) \in N\end{array}\right\} \Longrightarrow \mathbf{L}(\beta) = \bigcup_{\xi<\beta} \mathbf{L}(\xi) \subseteq N.$$

To show $N \subseteq \mathbf{L}(\beta) = \bigcup_{\xi<\beta} \mathbf{L}(\xi)$, it is enough to notice that

$$\begin{array}{rl}
& \forall x \in \mathbf{L}(\alpha)\ \exists v \in \mathbf{L}(\alpha)\ \exists \xi \in \mathbf{On} \cap \mathbf{L}(\alpha)\ \ \big(\overbrace{v = \mathbf{L}(\xi)}^{\exists z \in \mathbf{L}(\alpha)\ \overbrace{\psi(z,\xi,v)}^{\Delta_0^{0-rud}}} \ \wedge\ x \in v\big) \\
\longleftrightarrow & \Big(\forall x\ \exists v\ \exists \xi \in \mathbf{On}\ \ \big(\exists z\ \ \psi(z,\xi,v)\ \wedge\ x \in v\big)\Big)^{\mathbf{L}(\alpha)} \\
\longleftrightarrow & \Big(\forall x\ \exists v\ \exists \xi \in \mathbf{On}\ \ \big(\exists z\ \ \psi(z,\xi,v)\ \wedge\ x \in v\big)\Big)^{N} \\
\longleftrightarrow & \forall x \in N\ \exists v \in N\ \exists \xi \in \mathbf{On} \cap N\ \ \big(\exists z \in N\ \ \psi(z,\xi,v)^{N}\ \wedge\ x \in v\big) \\
\longleftrightarrow & \forall x \in N\ \exists v \in N\ \exists \xi \in \mathbf{On} \cap N\ \ \big(\exists z \in N\ \ \psi(z,\xi,v)\ \wedge\ x \in v\big) \\
\longleftrightarrow & \forall x \in N\ \exists v \in N\ \exists \xi \in \mathbf{On} \cap N\ \ \big(v = \mathbf{L}(\xi)\ \wedge\ x \in v\big) \\
\longleftrightarrow & \forall x \in N\ \exists v \in N\ \exists \xi < \beta\ \ \big(v = \mathbf{L}(\xi)\ \wedge\ x \in v\big)
\end{array}$$

which shows that $N \subseteq \bigcup_{\xi<\beta} \mathbf{L}(\xi) = \mathbf{L}(\beta)$.

□ 282

Everything is now ready for proving the main Lemma 279 which says: if $\mathbf{V} = \mathbf{L}$, then for every infinite cardinal κ, one has $\mathscr{P}(\kappa) \subseteq \mathbf{L}(\kappa^+)$. In other words, one does not have to look farther than $\mathbf{L}(\kappa^+)$ in order to find all constructible subsets of κ.

Proof of Lemma 279: Assume $\mathbf{V} = \mathbf{L}$. Let $Y \in \mathscr{P}(\kappa)$ and α be the least limit ordinal such that $Y \in \mathbf{L}(\alpha)$. We set

$$X = \kappa \cup \{Y\}$$

By Lemma 281 there exists M such that

- $|M| = \sup\{|X|, \aleph_0\} = \kappa$
- $X \subseteq M$
- $M \prec \mathbf{L}(\alpha)$.

By Lemma 282 there exist

- an ordinal $\beta \leqslant \alpha$
- $\pi : (M, \in) \approx (\mathbf{L}(\beta), \in)$.

The Mostowski collapsing function is the identity on every transitive set. So, κ being an ordinal is transitive, hence

$$\pi \restriction \kappa = id.$$

Thus,

$$\begin{aligned}\pi(Y) &= \{\pi(\xi) \mid \xi \in Y\} \\ &= \{\xi \mid \xi \in Y\} \\ &= Y.\end{aligned}$$

So, it follows

$$\pi(Y) = Y \in \mathbf{L}(\beta).$$

Since $|M| = \kappa$ and $(M, \in) \approx (\mathbf{L}(\beta), \in)$ we obtain

$$|\mathbf{L}(\beta)| = \kappa.$$

Finally, by Lemma 270, we get

$$|\mathbf{L}(\beta)| = |\beta| = \kappa.$$

Thus we have both

$$\beta < \kappa^+ \text{ and } Y \in \mathbf{L}(\beta).$$

Se we obtain

$$Y \in \mathbf{L}(\kappa^+)$$

which shows that

$$\mathscr{P}(\kappa) \subseteq \mathbf{L}(\kappa^+).$$

□ 279

12.3 Inner Models

Definition 283 (Inner Model)**.** *A class* **M** *is an* inner model *of* **ZFC** *if*

(1) **M** *is transitive* *(2)* $\mathbf{On} \cap \mathbf{M} = \mathbf{On} \cap \mathbf{V}$ *(3)* $(\mathbf{ZFC})^{\mathbf{M}}$.

Clearly, the Gödel's Constructible Universe is an inner model. Moreover, it turns out that **L** is the ⊆-least inner model in the following sense: if **I** is an inner model, then $\mathbf{L} \subseteq \mathbf{I}$.