Part II

Relativization and Absoluteness

Chapter 6

From Inside a Class

6.1 Relativization

For each formula and class we define what this formula becomes when the sets involved are the ones that belong to the class. For this purpose, we recall[1] that a class $\mathbf{C}$ is nothing but a formula with one free variable — that may or may not have other free variables that behave as parameters — $\varphi_{\mathbf{C}}$. Notice that for every formula $\varphi(x_1, \ldots, x_k)$ whose free variable are among $x_1, \ldots, x_k$, there is a formula $\psi(y)$ with one free variable y such that $\varphi(x_1, \ldots, x_k)$ holds if and only if $\psi\big(\langle x_1, \ldots, x_k\rangle\big)$ holds; namely:

$$\psi(y) := \forall x_1 \ldots \forall x_k \,\big(y = \langle x_1, \ldots, x_k\rangle \longrightarrow \varphi(x_1, \ldots, x_k)\big).$$

Definition 164 (Relativization)**.** *Let* $\mathbf{M}$ *be any class and* φ *any formula. The formula* $(\varphi)^{\mathbf{M}}$ *is defined by induction on* $ht(\varphi)$ *by:*

- $(\exists x\, \varphi)^{\mathbf{M}} := \exists x \in \mathbf{M}\ (\varphi)^{\mathbf{M}}$
- $(\forall x\, \varphi)^{\mathbf{M}} := \forall x \in \mathbf{M}\ (\varphi)^{\mathbf{M}}$
- $(x = y)^{\mathbf{M}} := x = y$
- $(x \in y)^{\mathbf{M}} := x \in y$
- $(\neg\varphi)^{\mathbf{M}} := \neg(\varphi)^{\mathbf{M}}$
- $(\varphi_0 \wedge \varphi_1)^{\mathbf{M}} := (\varphi_0)^{\mathbf{M}} \wedge (\varphi_1)^{\mathbf{M}}$
- $(\varphi_0 \vee \varphi_1)^{\mathbf{M}} := (\varphi_0)^{\mathbf{M}} \vee (\varphi_1)^{\mathbf{M}}$
- $(\varphi_0 \rightarrow \varphi_1)^{\mathbf{M}} := (\varphi_0)^{\mathbf{M}} \rightarrow (\varphi_1)^{\mathbf{M}}$
- $(\varphi_0 \leftrightarrow \varphi_1)^{\mathbf{M}} := (\varphi_0)^{\mathbf{M}} \leftrightarrow (\varphi_1)^{\mathbf{M}}$

So, assuming that the class $\mathbf{M}$ is described by the formula $\psi_{\mathbf{M}}(x)$, we see that $(\exists x\, \varphi)^{\mathbf{M}}$ stands for $\exists x \in \mathbf{M}\ (\varphi)^{\mathbf{M}}$ which really is $\exists x\, \Big(x \in \mathbf{M}\ \wedge\ (\varphi)^{\mathbf{M}}\Big)$, i.e., $\exists x\, \Big(\psi_{\mathbf{M}}(x)\ \wedge\ (\varphi)^{\mathbf{M}}\Big)$. *Idem* with

[1] This can be found in Section 2.4.

the universal quantifier: the formula $(\forall x\,\varphi)^{\mathbf{M}}$ really is $\forall x\left(\psi_{\mathbf{M}}(x) \rightarrow (\varphi)^{\mathbf{M}}\right)$.

Remark 165. Notice that the relativization of various notions that we introduced requires to go back to the original definition. For instance,

(1) $(x \subseteq y)^{\mathbf{M}} \iff (x \cap \mathbf{M}) \subseteq y$ holds since

- $x \subseteq y \iff \forall z\ (z \in x \longrightarrow z \in y)$
- $$\begin{aligned}\left(\forall z\ (z \in x \longrightarrow z \in y)\right)^{\mathbf{M}} &:= \forall z \in \mathbf{M}\ (z \in x \longrightarrow z \in y)\\ &\iff x \cap \mathbf{M} \subseteq y.\end{aligned}$$

(2) $\left(\mathcal{P}(x)\right)^{\mathbf{M}} = \{z \in \mathbf{M} \mid z \cap \mathbf{M} \subseteq x\}$ holds.

In case $\mathbf{M}$ is transitive, $\left(\mathcal{P}(x)\right)^{\mathbf{M}} = \mathcal{P}(x) \cap \mathbf{M}$. Indeed,

- $y = \mathcal{P}(x) \iff \forall z\ (z \in y \longleftrightarrow z \subseteq x)$
- $\left(\forall z\ (z \in y \longleftrightarrow z \subseteq x)\right)^{\mathbf{M}} := \forall z \in \mathbf{M}\left(z \in y \longleftrightarrow (z \subseteq x)^{\mathbf{M}}\right)$
- $\forall z \in \mathbf{M}\left(z \in y \longleftrightarrow (z \subseteq x)^{\mathbf{M}}\right) \iff \forall z \in \mathbf{M}\ \left(z \in y \longleftrightarrow z \cap \mathbf{M} \subseteq x\right)$.

From now on, we may use expressions such as "**ZF** *proves that* φ *holds true in* $\mathbf{M}$" or "**ZFC** *proves that the theory* $\mathcal{T}$ *holds true in* $\mathbf{M}$", where each time, being true in $\mathbf{M}$ refers to the relativized formula. So,

Definition 166. *Given any formula* φ *and any theory* $\mathcal{T}$*,*

(1) "φ holds true in $\mathbf{M}$" *or* "$\mathbf{M} \models \varphi$" *stands for* "$(\varphi)^{\mathbf{M}}$"

(2) "$\mathcal{T}$ *holds true in* $\mathbf{M}$" *or equivalently* "$\mathbf{M}$ *is a model of* $\mathcal{T}$" *stands for the assumption that for every* $\varphi \in \mathcal{T}$, "$(\varphi)^{\mathbf{M}}$".

This means that when we say, for a given class $\mathbf{M}$, that

$$\text{“ } \mathbf{ZFC} \text{ proves } \mathbf{M} \models 2^{\aleph_0} = \aleph_2 \text{”},$$

what we really mean is the statement:

$$\mathbf{ZFC} \vdash_c \left(2^{\aleph_0} = \aleph_2\right)^{\mathbf{M}}.$$

For instance, we will see that " **ZF** proves $\mathbf{L} \models \forall\alpha\ \ 2^{\aleph_\alpha} = \aleph_{\alpha+1}$", which strictly speaking means

$$\mathbf{ZF} \vdash_c \left(\forall\alpha\ \ 2^{\aleph_\alpha} = \aleph_{\alpha+1}\right)^{\mathbf{L}}.$$

We will also say, working with **ZF**, that " $\mathbf{L} \models \mathbf{ZFC}$"; where what we mean is that for every axiom $\varphi \in \mathbf{ZFC}$, one has $\mathbf{ZF} \vdash_c (\varphi)^{\mathbf{L}}$.

Lemma 167. *Let φ be any closed formula and $\mathbf{M}$ be any non-empty class.*

$$\text{if } \vdash_c \varphi \text{ holds, then } \vdash_c (\varphi)^{\mathbf{M}} \text{ holds as well.}$$

Proof of Lemma 167: By the completeness Theorem, the statement comes down to

$$\models \varphi \implies \models (\varphi)^{\mathbf{M}}.$$

But, since $\models \varphi$ holds true, it follows that in any model $\mathcal{M} = \langle |\mathcal{M}|, \in_{\restriction |\mathcal{M}|} \rangle$ one has $\mathcal{M} \models \varphi$ — meaning in any set $|\mathcal{M}|$ equipped with the membership relation φ holds. So, in particular, for every set $|\mathcal{M}| \cap \mathbf{M}$, we have

$$\langle |\mathcal{M}| \cap \mathbf{M}, \in_{\restriction |\mathcal{M}| \cap \mathbf{M}} \rangle \models \varphi.$$

□ 167

6.2 Consistency and Model Existence

Lemma 168. *Let $\mathcal{S}, \mathcal{T}$ be any $\mathcal{L}^2$-theory, and $\mathbf{M}$ any non-empty class.*

$$\left.\begin{array}{c} \mathcal{S} \nvdash_c \bot \\ \textit{and} \\ \mathcal{S} \vdash_c \text{“ } \mathbf{M} \textit{ is a model of } \mathcal{T} \text{ ”} \end{array}\right\} \implies \mathcal{T} \nvdash_c \bot.$$

Proof of Lemma 168: Towards a contradiction, we assume $\mathcal{T} \vdash_c \bot$. For some (any) closed formula φ, one has

$$\mathcal{T} \vdash_c (\varphi \wedge \neg\varphi)$$

so there exist finitely many formulas $\varphi_0, \ldots, \varphi_n \in \mathcal{T}$ such that

$$\bigwedge_{0 \leqslant i \leqslant n} \varphi_i \vdash_c (\varphi \wedge \neg\varphi)$$

which gives

$$\vdash_c \left(\bigwedge_{0 \leqslant i \leqslant n} \varphi_i \longrightarrow (\varphi \wedge \neg\varphi) \right)$$

[2] $\mathcal{L}$ stands for the language of set theory. i.e., its signature is $\{\in, =\}$.

hence, by Lemma 167,

$$\vdash_c \left(\bigwedge_{0 \leqslant i \leqslant n} \varphi_i \longrightarrow (\varphi \wedge \neg\varphi) \right)^{\mathbf{M}}$$

which yields

$$\vdash_c \left(\bigwedge_{0 \leqslant i \leqslant n} (\varphi_i)^{\mathbf{M}} \longrightarrow \left((\varphi)^{\mathbf{M}} \wedge \neg(\varphi)^{\mathbf{M}}\right) \right).$$

$\mathcal{S} \vdash_c$ " $\mathbf{M}$ is a model of $\mathcal{T}$ " yields $\mathcal{S} \vdash_c (\varphi_i)^{\mathbf{M}}$ (any $i \leqslant n$), hence

$$\mathcal{S} \vdash_c \bigwedge_{0 \leqslant i \leqslant n} (\varphi_i)^{\mathbf{M}}$$

and by *modus ponens*:

$$\mathcal{S} \vdash_c \left((\varphi)^{\mathbf{M}} \wedge \neg(\varphi)^{\mathbf{M}}\right)$$

contradicting the fact that $\mathcal{S}$ is consistent.

$\square$ 168

Chapter 7

The Mostowski Collapse

7.1 Recursion on Well-founded and "Set-Like" Relations

This section is concerned with collapsing certain classes in order to render them transitive. Morever, the collapsed class and the original one are then isomorphic and the isomorphism is unique. This gives a very natural way of transforming a class — that possibly satisfies various axioms of **ZFC** — into a transitive one which is often easier to handle.

Definition 169 ($\mathbf{ZF} \smallsetminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any class, and* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$.

$$\mathbf{R} \textit{ is "set-like" on } \mathbf{M} \iff \forall y \in \mathbf{M} \ \ \mathbf{R}^{-1}[\{y\}] = \{x \in \mathbf{M} \mid x\mathbf{R}y\} \textit{ is a set.}$$

A class-relation is "set-like" on a class if the inverse image of every element is a set. Notice that it is equivalent to say that $\mathbf{R}$ *is "set-like" on* $\mathbf{M}$ if and only if, for all set $B \in \mathbf{M}$, $\mathbf{R}^{-1}[B] = \{x \in \mathbf{M} \mid \exists y \in B \ \ x\mathbf{R}y\}$ is a set.

We consider the closure of taking the predecessors of an element x along $\mathbf{R}$.

Definition 170 ($\mathbf{ZF} \smallsetminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{X} \subseteq \mathbf{M}$*,* $x \in \mathbf{X}$ *and* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$. *We define* $cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right)$ *by:*

- $cl^0\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) = \lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}} = \mathbf{X} \cap \mathbf{R}^{-1}[\{x\}] = \{z \in \mathbf{X} \mid z\mathbf{R}x\}$
- $cl^{n+1}\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) = \bigcup\left\{\lceil \underline{z} \rceil^{\mathbf{R}}_{\mathbf{X}} \mid z \in cl^n\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right)\right\}$
- $cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) = \bigcup\left\{cl^n\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) \mid n \in \omega\right\}$

Remark 171. If $\mathbf{R}$ is "set-like" on $\mathbf{M}$, then one can easily show by induction on the integers, that for each integer n,

$$cl^n\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) \text{ is a set.}$$

Hence,

$$cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) = \bigcup \left\{ cl^n\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) \mid n \in \omega \right\} \text{ is a set.}$$

Definition 172 (**ZF** $\smallsetminus$ {**AF**})**.** *Let* $\mathbf{M}$ *be any class, and* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$.

$$\mathbf{R} \textit{ is "well-founded" on } \mathbf{M} \iff \forall X \subseteq \mathbf{M} \left(X \neq \varnothing \longrightarrow \exists y \in X \, \forall x \in X \;\; \neg x \mathbf{R} y \right).$$

So, a class-relation $\mathbf{R}$ on a class $\mathbf{M}$ is well-founded if every non-empty *subset* of $\mathbf{M}$ has some $\mathbf{R}$-minimal element.

Remark 173. If $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ is both well-founded and "set-like" on $\mathbf{M}$, then for each $x \in \mathbf{M}$, the graph of $\mathbf{R}$ on x is a set $\mathcal{G} = (V, E)$ defined by:

$$V = \{x\} \cup cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right) \quad \text{and} \quad E = \left\{(a, b) \in V \times V \mid b \mathbf{R} a\right\}.$$

This directed graph is acyclic[1].

Theorem Schema 174 (**ZF** $\smallsetminus$ {**AF**})**.** *Let* $\mathbf{M}$ *be any class, and* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *be any well-founded and "set-like" relation on* $\mathbf{M}$.

$$\forall \mathbf{X} \subseteq \mathbf{M} \left(\mathbf{X} \neq \varnothing \longrightarrow \exists y \in \mathbf{X} \, \forall x \in \mathbf{X} \;\; \neg x \mathbf{R} y \right).$$

So, this theorem claims that the property that defines a well-founded class-relation $\mathbf{R}$ on a class $\mathbf{M}$, can be *lifted from non-empty subsets to non-empty classes* provided that the class-relation $\mathbf{R}$ be "set-like" on $\mathbf{M}$ in addition to being well-founded.

Proof of Theorem 174: Take any $x \in \mathbf{X}$. If x is $\mathbf{R}$-minimal in $\mathbf{X}$ we are done. Otherwise, we consider $cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right)$ which is a set since $\mathbf{R}$ is "set-like" on $\mathbf{M}$. Since $\mathbf{R}$ is well-founded on $\mathbf{M}$ and $cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right)$ is a subset of $\mathbf{M}$, it follows that $cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right)$ admits some $\mathbf{R}$-minimal element y. We show

[1] By well-foundedness of $\mathbf{R}$.

that y is also $\mathbf{R}$-minimal in $\mathbf{X}$. Indeed, since $y \in cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right)$ there exists some integer k such that $y \in cl^k\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right)$ and any $z \in \mathbf{X}$ that would satisfy $z\mathbf{R}y$ would belong to $cl^{k+1}\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right) \subseteq cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{X}}\right)$ which would contradict the $\mathbf{R}$-minimality of y.

□ 174

We saw on page 39 that one can define a class-function by transfinite induction on the ordinals. This result can easily be extended from the ordinals to any well-founded and "set-like" relation.

Theorem Schema 175 (**ZF** ∖ {**AF**}, transfinite recursion along well-founded set-like relations)**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *be any well-founded and "set-like" relation on* $\mathbf{M}$*, and* $\mathbf{F} : \mathbf{M} \times \mathbf{V} \to \mathbf{V}$ *be any class-function.*
There exists some unique $\mathbf{G} : \mathbf{M} \to \mathbf{V}$ *such that such that*

$$\forall x \in \mathbf{M} \quad \mathbf{G}(x) = \mathbf{F}\Big(x\,, \mathbf{G} \restriction_{cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)}\Big).$$

Proof of Theorem 175:

Uniqueness: Assume there exist two different class-functions $\mathbf{G_1}$ and $\mathbf{G_2}$. By Theorem 174 the non-empty class $\{x \in \mathbf{M} \mid \mathbf{G_1}(x) \neq \mathbf{G_2}(x)\}$ has an $\mathbf{R}$-least element y. By construction, one comes to the following contradiction:

$$\mathbf{G_1}(y) = \mathbf{F}\Big(y\,, \mathbf{G_1} \restriction_{cl\left(\lceil \underline{y} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)}\Big) = \mathbf{F}\Big(y\,, \mathbf{G_2} \restriction_{cl\left(\lceil \underline{y} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)}\Big) = \mathbf{G_2}(y).$$

Existence: we construct functions that are approximations of $\mathbf{G}$ on some proper initial segment of the ordinals.

i.e., for each $x \in \mathbf{M}$, we construct a function $g_x : cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right) \to \mathbf{V}$ such that

$$\forall z \in cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right) \quad g_x(z) = \mathbf{F}\Big(z\,, \mathbf{G} \restriction_{cl\left(\lceil \underline{z} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)}\Big).$$

So, since $\mathbf{R}$ is "set-like" on $\mathbf{M}$, it follows that $cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)$ is a set, hence g_x is a function with $dom\,(g_x) = cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)$ and $ran\,(g_x) = g_x\left[cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)\right]$ is a set obtained by an instance of the **Replacement Schema**.

Clearly, by the same argument as above, g_x is unique for any given $x \in \mathbf{M}$. So, it is enough to define $\mathbf{G}(x)$ by:

- If there exists some $y \in \mathbf{M}$ such that $x \in cl\left(\lceil \underline{y} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)$, then

$$\mathbf{G}(x) = g_y(x)$$

for some (any) $y \in \mathbf{M}$ such that $x \in cl\left(\lceil \underline{y} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)$.

- If there exists no $y \in \mathbf{M}$ such that $x \in cl\left(\lceil \underline{y} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)$, then

$$\mathbf{G}(x) = \mathbf{F}\Big(x \,, g_x \restriction_{cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)} \Big).$$

□ 175

7.2 The Mostowski Collapsing Functional

We define a class-function which turns Emmentaler into Gruyère (by removing all the holes!)

Definition 176 (**ZF** ∖ {**AF**})**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *be any well-founded and "set-like" relation on* $\mathbf{M}$*, the Mostowski collapsing class-function* $\mathbf{G} : \mathbf{M} \to \mathbf{V}$ *is defined by*

$$\begin{aligned} \mathbf{G}(x) &= \Big\{ \mathbf{G}(y) \mid y \in \mathbf{M} \wedge y\mathbf{R}x \Big\} \\ &= \Big\{ \mathbf{G}(y) \mid y \in \mathbf{R}^{-1}[\{x\}] \Big\} \\ &= \Big\{ \mathbf{G}(y) \in \mathbf{G}\Big[\mathbf{R}^{-1}[\{x\}]\Big] \mid y \in \mathbf{R}^{-1}[\{x\}] \Big\}. \end{aligned}$$

The class $\mathbf{N} = \mathbf{G}[\mathbf{M}]$ *is called the* ***Mostowski collapse*** *of* $\mathbf{M}$*.*

Lemma 177 (**ZF** ∖ {**AF**})**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *be any well-founded and "set-like" relation on* $\mathbf{M}$*, and* $\mathbf{G} : \mathbf{M} \to \mathbf{V}$ *the Mostowski collapsing class-function.*

(1) $\forall x \in \mathbf{M}\, \forall y \in \mathbf{M} \ \big(y\mathbf{R}x \longrightarrow \mathbf{G}(y) \in \mathbf{G}(x)\big)$

(2) $\mathbf{N} = \mathbf{G}[\mathbf{M}]$ *is transitive*

(3) $\mathbf{N} \subseteq \mathbf{WF}$*.*

Proof of Lemma 177:

(1) $\forall x, y \in \mathbf{M} \ \big(x\mathbf{R}y \longrightarrow \mathbf{G}(x) \in \mathbf{G}(y)\big)$ is immediate by definition of the Mostowski collapsing class-function $\mathbf{G}$.

(2) Take any $v \in w \in \mathbf{N} = \mathbf{G}[\mathbf{M}]$. By definition of Mostowski collapsing class-function $\mathbf{G}$, there exist $x, y \in \mathbf{M}$ such that $w = \mathbf{G}(x)$ and $v = \mathbf{G}(y) \in \mathbf{G}(x)$, hence $v \in \mathbf{N}$.

(3) Since $\mathbf{R}$ is well-founded, we show, by induction on $\mathbf{R}$, that $\mathbf{G}(x) \in \mathbf{WF}$ holds for every $x \in \mathbf{M}$.

- If $x \in \mathbf{M}$ is $\mathbf{R}$-minimal: $\mathbf{G}(x) = \left\{\mathbf{G}(y) \mid y \in \mathbf{M} \wedge y\mathbf{R}x\right\} = \varnothing \in \mathbf{W}(1) \subseteq \mathbf{WF}$.
- If $x \in \mathbf{M}$ is not $\mathbf{R}$-minimal: $\mathbf{G}(x) = \left\{\mathbf{G}(y) \mid y \in \mathbf{M} \wedge y\mathbf{R}x\right\}$. Since R is "set-like" on $\mathbf{M}$, $\mathbf{R}^{-1}[\{x\}]$ is a set. By induction hypothesis, $\mathbf{G}(y) \in \mathbf{WF}$ holds for each $y\mathbf{R}x$. Hence, by an instance of the **Replacement Schema**, one has

$$\left\{rk\big(\mathbf{G}(y)\big) \mid y \in \mathbf{M} \wedge y\mathbf{R}x\right\}$$

is a set of ordinals. Therefore

$$\alpha = \sup\left\{rk\big(\mathbf{G}(y)\big) + 1 \mid y \in \mathbf{M} \wedge y\mathbf{R}x\right\}$$

is well defined and $x \in \mathbf{W}(\alpha) \subseteq \mathbf{WF}$.

□ 177

Definition 178 ($\mathbf{ZF} \setminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *is extensional on* $\mathbf{M}$ *if*

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \ \Big(\forall z \in \mathbf{M}\ \big(z\mathbf{R}x \longleftrightarrow z\mathbf{R}y\big) \longrightarrow x = y\Big).$$

i.e.,

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \ \Big(x \neq y \longrightarrow \lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}} \neq \lceil \underline{y} \rceil^{\mathbf{R}}_{\mathbf{M}}\Big).$$

Remark 179. Notice, that when one replaces the class-relation $\mathbf{R}$ by the membership relation $\in$, this assertion becomes

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \ \Big(\forall z \in \mathbf{M}\ \big(z \in x \longleftrightarrow z \in y\big) \longrightarrow x = y\Big)$$

which is exactly

$$\Big(\forall x\ \forall y\ (\forall z\ (z \in x \longleftrightarrow z \in y) \rightarrow x = y))\Big)^{\mathbf{M}}$$

in other words:

$$(\textbf{Extensionality})^{\mathbf{M}}.$$

So, essentially, one requires a class-relation to be extensional when one wants the Mostowski collapse to satisfy the Axiom of **Extensionality**.

Lemma 180. *If* $\mathbf{M}$ *is any transitive class, then the membership relation* $\in$ *is extensional on* $\mathbf{M}$.

Proof of Lemma 180: Take any $x, y \in \mathbf{M}$ with $x \neq y$. By symmetry, there exists either $z \in x \smallsetminus y$ or $z \in y \smallsetminus x$; which can be summarized by there exists some z such that $(z \in x \longleftrightarrow z \notin y)$. Since $z \in x \in \mathbf{M}$ and $\mathbf{M}$ is transitive, one has $z \in \mathbf{M}$, which shows

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \big(x \neq y \rightarrow \exists z \in \mathbf{M}\ (z \in x \longleftrightarrow z \notin y)\big)$$

i.e.,

$$\Big(\forall x\ \forall y\ \big(x \neq y \rightarrow \exists z\ (z \in x \longleftrightarrow z \notin y)\big)\Big)^{\mathbf{M}}$$

i.e.,

$$\Big(\forall x\ \forall y\ \big(\forall z\ (z \in x \longleftrightarrow z \in y) \rightarrow x = y\big)\Big)^{\mathbf{M}}$$

in other words:

$$(\textbf{Extensionality})^{\mathbf{M}}.$$

□ 180

Lemma 181 ($\mathbf{ZF} \smallsetminus \{\mathbf{AF}\}$)**.** *Let $\mathbf{M}$ be any class, $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ be any well-founded and "set-like" relation on $\mathbf{M}$, and $\mathbf{G} : \mathbf{M} \rightarrow \mathbf{V}$ the Mostowski collapsing class-function.*
If $\mathbf{R}$ is extensional on $\mathbf{M}$, then

$$\mathbf{G} : \mathbf{M} \longrightarrow \mathbf{N} = \mathbf{G}[\mathbf{M}] \text{ is an isomorphism from } (\mathbf{M}, \mathbf{R}) \text{ to } (\mathbf{N}, \in).$$

Proof of Lemma 181:

- We show that $\mathbf{G} : \mathbf{M} \rightarrow \mathbf{N}$ is 1-1. Otherwise, we consider any x that is $\mathbf{R}$-minimal in

$$\Big\{x \in \mathbf{M} \mid \exists y \in \mathbf{M}\ \big(\mathbf{G}(x) = \mathbf{G}(y)\ \wedge\ x \neq y\big)\Big\}$$

  and for this x a fixed y.

  Since $\mathbf{R}$ is extensional on $\mathbf{M}$ and $x \neq y$ holds, one has $[\frac{x}{-}]^{\mathbf{R}}_{\mathbf{M}} \neq [\frac{y}{-}]^{\mathbf{R}}_{\mathbf{M}}$. By symmetry, we assume $[\frac{x}{-}]^{\mathbf{R}}_{\mathbf{M}} \smallsetminus [\frac{y}{-}]^{\mathbf{R}}_{\mathbf{M}} \neq \varnothing$ and pick any $z \in \big([\frac{x}{-}]^{\mathbf{R}}_{\mathbf{M}} \smallsetminus [\frac{y}{-}]^{\mathbf{R}}_{\mathbf{M}}\big)$. One has

$$\mathbf{G}(z) \in \mathbf{G}(x) \text{ since } z\mathbf{R}x \quad \text{and} \quad \mathbf{G}(z) \in \mathbf{G}(y) \text{ since } \mathbf{G}(x) = \mathbf{G}(y).$$

  Therefore, there exists some $z' \in \mathbf{M}$ such that

$$z'\mathbf{R}y \text{ and } \mathbf{G}(z) = \mathbf{G}(z') \in \mathbf{G}(y).$$

  Thus, we have found some $z \neq z'$ such that $\mathbf{G}(z) = \mathbf{G}(z')$ and $z\mathbf{R}x$, which contradicts the $\mathbf{R}$-minimality of x.

- Since $\mathbf{G} : \mathbf{M} \rightarrow \mathbf{N}$ is 1-1 and $\mathbf{N} = \mathbf{G}[\mathbf{M}]$, $\mathbf{G} : \mathbf{M} \xleftrightarrow{bij.} \mathbf{N}$. Thus, $\mathbf{G}$ being a bijection, it follows that $\mathbf{G}$ is an isomorphism since we have:

- if $x\mathbf{R}y$, then $\mathbf{G}(x) \in \mathbf{G}(y)$ holds by definition of $\mathbf{G}$;
- if $\mathbf{G}(x) \in \mathbf{G}(y)$, then $x\mathbf{R}y$ holds since, from $\mathbf{G}(x) \in \mathbf{G}(y) = \left\{\mathbf{G}(z) \mid z \in \mathbf{M} \wedge z\mathbf{R}y\right\}$, pick some $z \in \mathbf{M}$, such that $\mathbf{G}(x) = \mathbf{G}(z)$ and $z\mathbf{R}y$ and notice that $\mathbf{G}$ being 1-1, one has $z = x$ and $x\mathbf{R}y$.

□ 181

Mostowski Collapsing Theorem ($\mathbf{ZF} \setminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any class,* $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ *be any well-founded, "set-like", and extensional relation on* $\mathbf{M}$.

(1) there exists a transitive class $\mathbf{N}$*, and*

(2) an isomorphism $\mathbf{G} : (\mathbf{M}, \mathbf{R}) \overset{isom.}{\longleftrightarrow} (\mathbf{N}, \in)$*;*

(3) moreover, the isomorphism is unique.

Proof of the Mostowski Collapsing Theorem:

(1) The existence of $\mathbf{N}$ and the fact it is transitive is Lemma 177

(2) The fact $\mathbf{G} : (\mathbf{M}, \mathbf{R}) \overset{isom.}{\longleftrightarrow} (\mathbf{N}, \in)$ is an isomorphism is Lemma 181.

(3) Towards a contradiction, assume there exist isomorphisms $\mathbf{G} : \mathbf{M} \to \mathbf{N} = \mathbf{G}[\mathbf{M}]$ and $\mathbf{G}' : \mathbf{M} \to \mathbf{N}' = \mathbf{G}'[\mathbf{M}]$ between respectively $(\mathbf{M}, \mathbf{R})$ and $(\mathbf{N}, \in)$ and $(\mathbf{M}, \mathbf{R})$ and $(\mathbf{N}', \in)$. By induction on $\mathbf{R}$, we show that $\forall x \in \mathbf{M} \;\; \mathbf{G}(x) = \mathbf{G}'(x)$.

- ○ If $\mathbf{x}$ is $\mathbf{R}$-minimal, then $\mathbf{G}, \mathbf{G}'$ being isomorphisms gives

$$\forall y \in \mathbf{M} \left(\neg y\mathbf{R}x \longrightarrow \big(\mathbf{G}(y) \notin \mathbf{G}(x) \wedge \mathbf{G}'(y) \notin \mathbf{G}'(x)\big)\right).$$

Hence (since $\mathbf{G}, \mathbf{G}'$ are bijective and $\mathbf{N}, \mathbf{N}'$ are transitive)

$$\forall z \in \mathbf{N} \; z \notin \mathbf{G}(x) \wedge \forall z \in \mathbf{N}' \; z \notin \mathbf{G}'(x).$$

Which yields $\mathbf{G}(x) = \mathbf{G}'(x) = \varnothing$.

- ○ If $\mathbf{x}$ is not $\mathbf{R}$-minimal, then one has

$$\Big(\forall y \in \mathbf{M} \; \big(y\mathbf{R}x \longleftrightarrow \mathbf{G}(y) \in \mathbf{G}(x)\big) \wedge \forall y \in \mathbf{M} \; \big(y\mathbf{R}x \longleftrightarrow \mathbf{G}'(y) \in \mathbf{G}'(x)\big)\Big).$$

which leads to

$$\forall y \in \mathbf{M} \; \Big(\big(y\mathbf{R}x \longleftrightarrow \mathbf{G}(y) \in \mathbf{G}(x)\big) \wedge \big(y\mathbf{R}x \longleftrightarrow \mathbf{G}'(y) \in \mathbf{G}'(x)\big)\Big).$$

i.e.,

$$\forall y \in \mathbf{M} \left(y\mathbf{R}x \longleftrightarrow \mathbf{G}(y) \in \mathbf{G}(x) \quad \longleftrightarrow \quad \mathbf{G}'(y) \in \mathbf{G}'(x) \right).$$

By induction hypothesis, this gives

$$\forall y \in \mathbf{M} \left(y\mathbf{R}x \longleftrightarrow \mathbf{G}(y) \in \mathbf{G}(x) \quad \longleftrightarrow \quad \mathbf{G}(y) \in \mathbf{G}'(x) \right).$$

So, in particular,

$$\forall y \in \mathbf{M} \left(\mathbf{G}(y) \in \mathbf{G}(x) \quad \longleftrightarrow \quad \mathbf{G}(y) \in \mathbf{G}'(x) \right).$$

Therefore we obtain (since $\mathbf{G}, \mathbf{G}'$ are bijective and $\mathbf{N}, \mathbf{N}'$ are transitive)

$$\mathbf{G}(x) = \mathbf{G}'(x).$$

☐ Mostowski Collapsing Theorem

Remark 183. Assume $\mathbf{M}$ is a class, $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ is well-founded, "set-like", and extensional on $\mathbf{M}$. Then, we associate, to each $x \in \mathbf{M}$, the following colored tree T_x defined by:

- The colors are among $\{x\} \cup cl\left(\lceil \underline{x} \rceil^{\mathbf{R}}_{\mathbf{M}}\right)$;
- the unique root is colored with x;
- for each node $\mathtt{n}$ colored by b and for each a such that $a\mathbf{R}b$, there exists a unique node $\mathtt{m}$ which is a child of $\mathtt{n}$ and which is colored with a.

Notice that for each $x \in \mathbf{M}$:

(1) the colored tree T_x has no infinite branch (since $\mathbf{R} \subseteq \mathbf{M} \times \mathbf{M}$ is well-founded);

(2) on any branch of T_x, there is no color that appears twice (otherwise, $\mathbf{R}$ would be ill-founded);

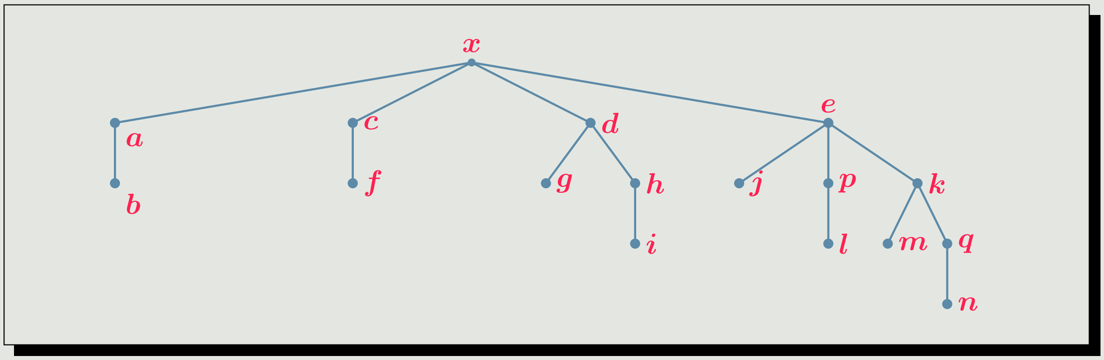
(3) if two different nodes $\mathtt{n}$ and $\mathtt{m}$ are colored with the same color a, then the colored sub-tree induced by $\mathtt{n}$ and the colored sub-tree induced by $\mathtt{m}$ are identical.

Now if we associate to any node $\mathtt{n}$ colored by a, the set

$$\widehat{a} = \left\{ \widehat{b} \mid b \text{ is the color of a child } \mathtt{c} \text{ of } \mathtt{n} \right\}$$

as we did for instance in Remark 150 and Examples 151 and 152 — we obtain exactly at the root $\mathtt{r}$, the set $\widehat{r}$ to which x is mapped to through the Mostowski collapsing class-function described in Definition 176. i.e., $\widehat{\mathtt{r}} = \mathbf{G}(x)$.

Example 184. *Below is a tree* T_x *generated as explained in the above Remark 183. Notice that the tree is well-founded.*



In a second step, let us take into account the fact $\mathbf{R}$ *is extensional. The first thing we notice is that the leaves should all have the same color since*

$$\forall a \in \mathbf{M}\ \forall a' \in \mathbf{M}\ \Big(\forall z \in \mathbf{M}\ \big(z\mathbf{R}a \longleftrightarrow z\mathbf{R}a'\big) \longrightarrow a = a'\Big).$$

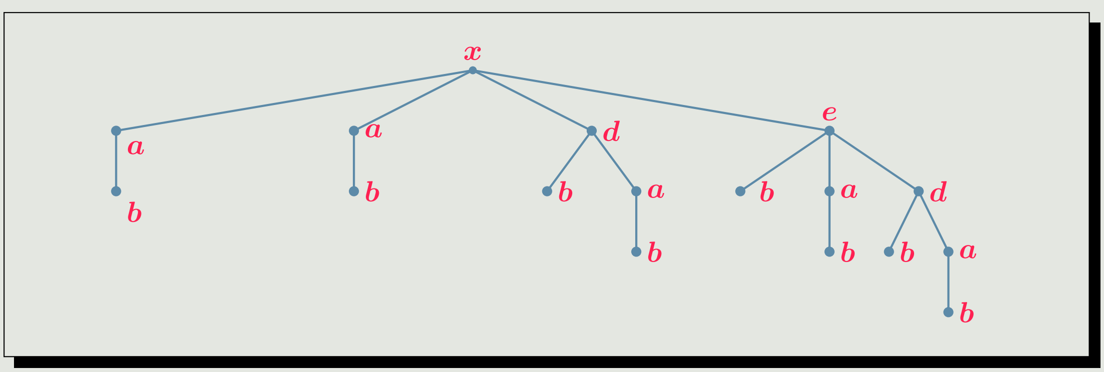
So, we should have

(1) $b = f = g = i = j = l = m = n$;

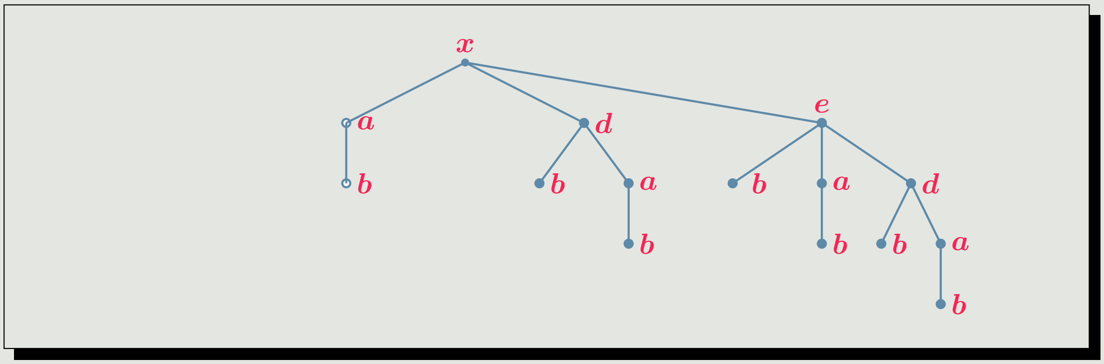
(2) $a = c = h = p = q$;

(3) $d = k$.

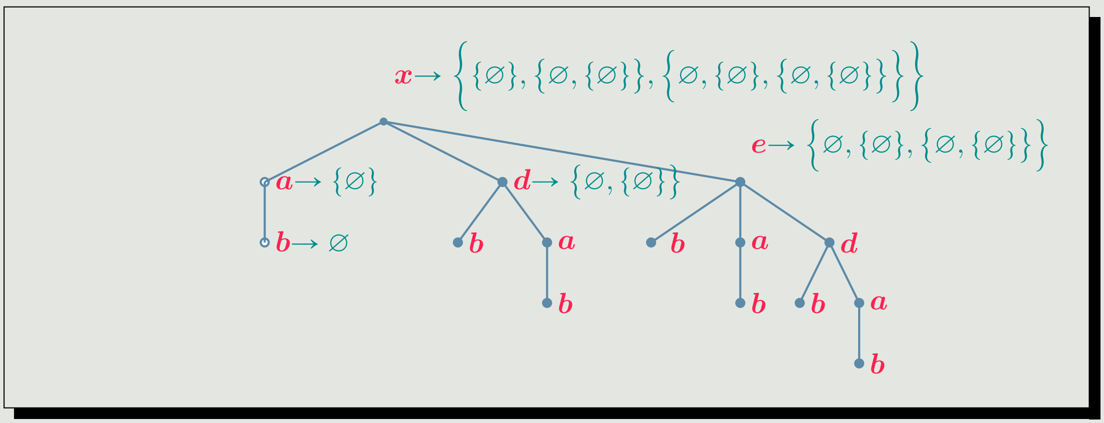
Which shows that T_x *is the following tree:*



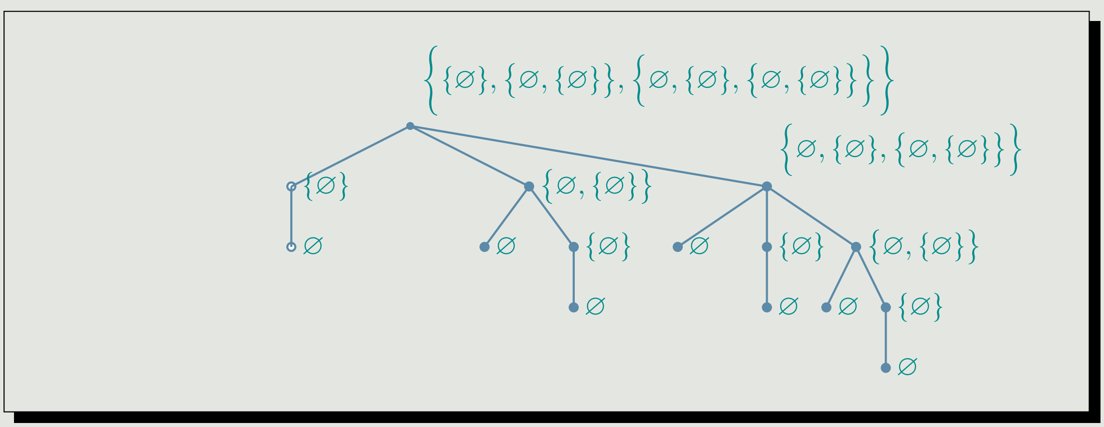
We can also get rid of one of the two copies of a *as well as its induced sub-tree, so that the tree* T_x *really looks like the one in the next picture:*



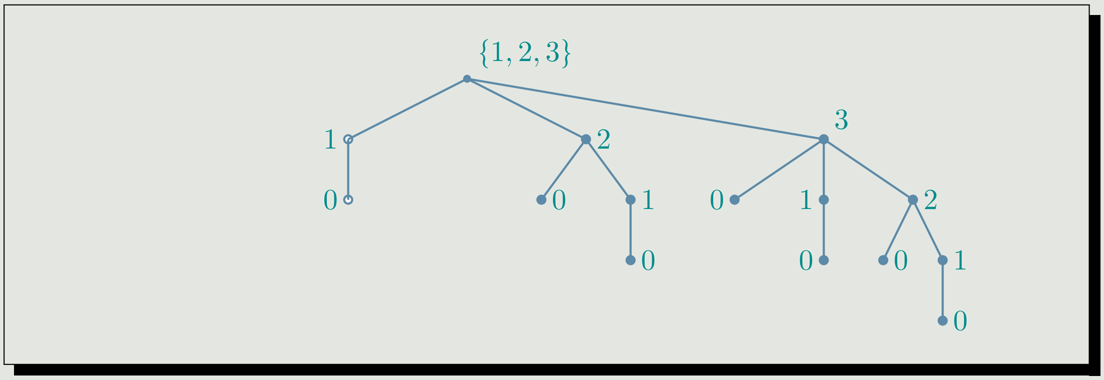
Finally, the operation $\boldsymbol{a} \mapsto \widehat{\boldsymbol{a}}$ *as described in Remark 183 on page 114 yields the following:*



We obtain in fact the following colored tree which is nothing but the way of coloring a well-founded tree that was described in Remark 150 on page 89; so it is indeed the colored tree $T_{\widehat{x}}$*:*



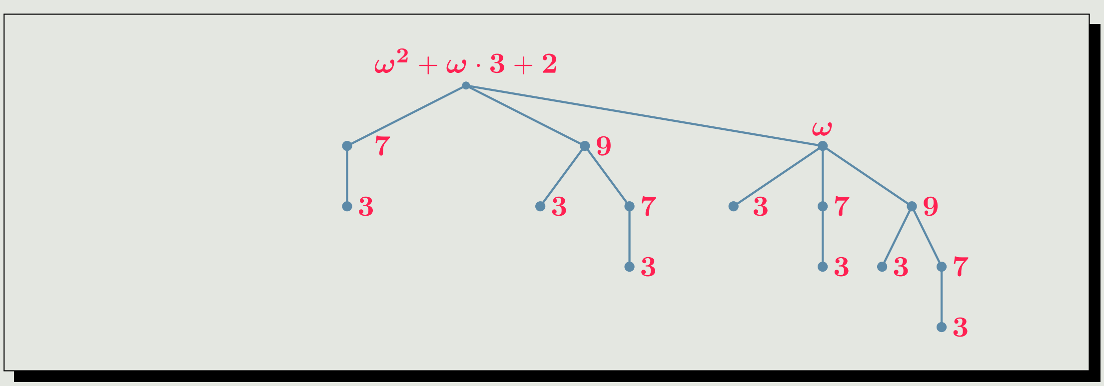
The next tree is another representation of the same colored tree, obtained by taking into account the equalities $0 = \varnothing$*,* $1 = \{\varnothing\}$*, and* $2 = \big\{\varnothing, \{\varnothing\}\big\}$*:*



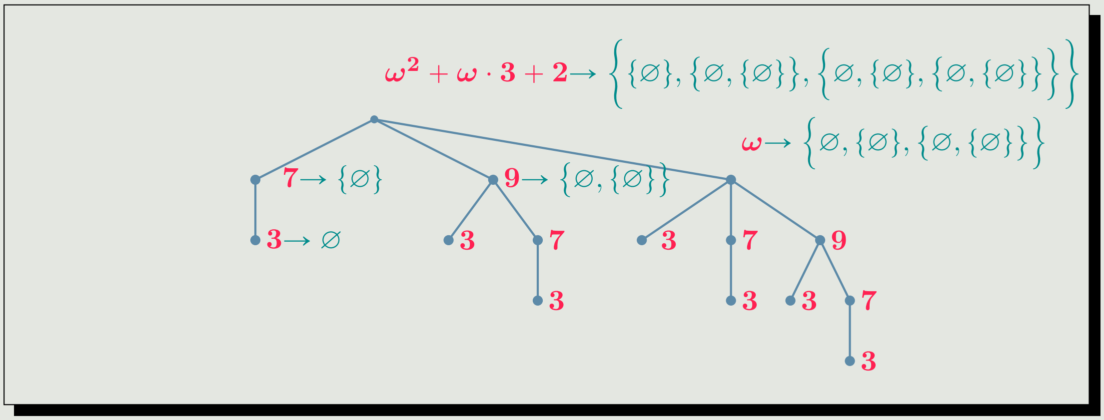
Example 185. *Assume one has a non transitive model* $\mathbf{M}$ *in which*

- *the set named* 3 *does not contain any set,*
- *the set named* 7 *only contains* 3*,*
- *the set named* 9 *only contains* $3, 7$*,*
- *the set named* ω *only contains* $3, 7, 9$*,*
- *the set named* $\omega^2 + \omega \cdot 3 + 2$ *only contains* $3, 7, 9, \omega$*.*

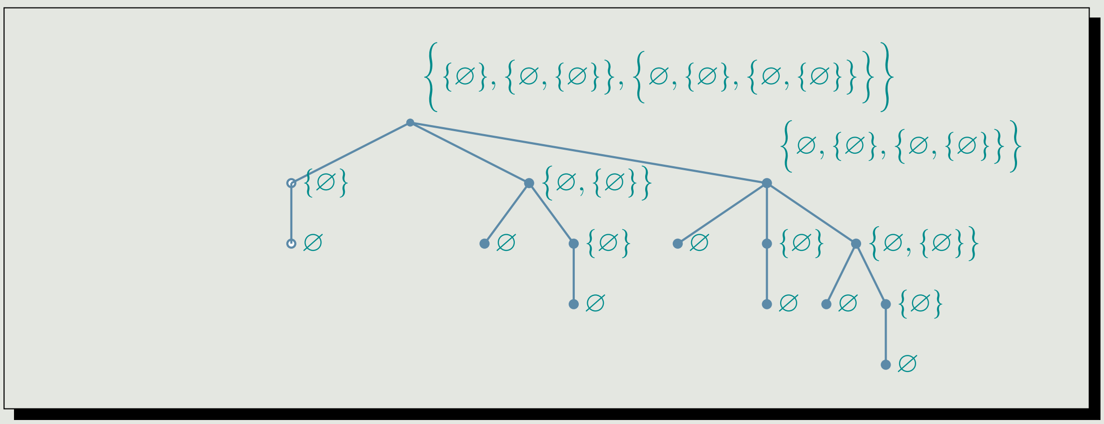
Then, the tree-like representation — as explained in Remark 183 — of the membership relation looks like:



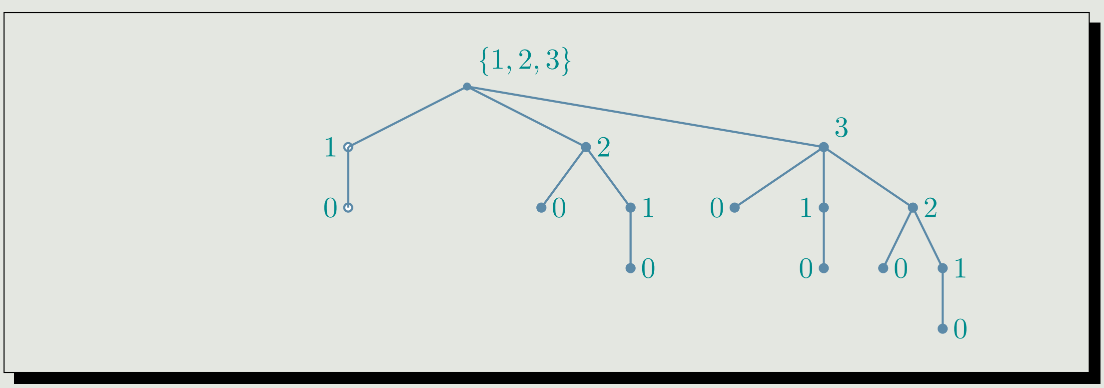
The Mostowski collapsing functionnal yields the following replacements:



We obtain in fact the following colored tree which is nothing but the way of coloring a well-founded tree that was described in Remark 150 on page 89; so it is indeed the colored tree $T_{\hat{\boldsymbol{x}}}$:



The next tree is another representation of the same colored tree, obtained by taking into account the equalities $0 = \varnothing$, $1 = \{\varnothing\}$, *and* $2 = \big\{\varnothing, \{\varnothing\}\big\}$*:*



We now go back to the Mostowski Collapsing Theorem on page 113, and notice that in the particular case where we work with the axiom of **Foundation**, the class-relation $\in \subseteq \mathbf{V} \times \mathbf{V}$ is

(1) well-founded,

(2) "set-like" [2]

(3) extensional.

Therefore, the Mostowski Collapsing Theorem immediately yields:

Corollary 186 (ZF). *If* $\mathbf{M}$ *is any class,* $\in$ *is extensional on* $\mathbf{M}$*. Then there exist*

- *a transitive class* $\mathbf{N}$*,*
- *an isomorphism* $\mathbf{G} : (\mathbf{M}, \in) \xleftrightarrow{isom.} (\mathbf{N}, \in)$*; i.e.,*

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \Big(x \in y \longleftrightarrow \mathbf{G}(x) \in \mathbf{G}(y)\Big);$$

- *moreover, the isomorphism is unique.*

Proof of Corollary 186: It is enough to notice that

(1) $\mathbf{ZF} \vdash_c$ " $\in$ is well-founded on $\mathbf{M}$ ",

(2) $\mathbf{ZF} \vdash_c$ " $\in$ is set-like on $\mathbf{M}$ ",

(3) $\mathbf{ZF} \vdash_c$ " $\in$ is extensional on $\mathbf{M}$ " ,

and apply the Mostowski Collapsing Theorem (page 113).

□ 186

[2] For any set x, $\in^{-1}[x] = \{y \in \mathbf{V} \mid y \in x\} = x$.

Chapter 8

Preservation under Relativization

8.1 Relativization of ZF

In this section, we concentrate on some of the properties which ensure that a class satisfies certain axioms of **ZFC**.

Lemma 187. *Let* $\mathbf{M}$ *be any non-empty class.*

$$\textit{If } \mathbf{M} \textit{ is transitive, then } \left(\textbf{\textit{Extensionality}}\right)^{\mathbf{M}}.$$

i.e.,

$$\left(\forall x \in \mathbf{M}\ x \subseteq \mathbf{M} \longrightarrow \ \left(\forall x\ \forall y\ \left(\forall z\left(z \in x \longleftrightarrow z \in y\right) \rightarrow x = y\right)\right)^{\mathbf{M}}\right).$$

Proof of Lemma 187: This was Lemma 180.

□ 187

Lemma 188. *Let* $\mathbf{M}$ *be any non-empty class.*

- *If for each* $\varphi_{(x,y,z_1,\ldots,z_k)}$ *with free variables among* $\{x, y, z_1, \ldots, z_k\}$*, one has*

  $$\forall y \in \mathbf{M}\ \forall z_1 \in \mathbf{M}\ \ldots \forall z_k \in \mathbf{M}\ \ \left\{x \in y \mid (\varphi)^{\mathbf{M}}{}_{(x,y,z_1,\ldots,z_k)}\right\} \in \mathbf{M}.$$

  Then $\left(\textbf{\textit{Comprehension Schema}}\right)^{\mathbf{M}}$*.*

- *In particular, if* $\mathbf{M}$ *is closed under* $\mathscr{P} : \mathbf{M} \rightarrow \mathbf{V}$ *that maps* x *to* $\mathscr{P}(x)$ *(i.e.,* $\mathscr{P}[\mathbf{M}] \subseteq \mathbf{M}$*) then* $\left(\textbf{\textit{Comprehension Schema}}\right)^{\mathbf{M}}$*.*

Proof of Lemma 188:

$$\begin{aligned}(\textbf{Comprehension Schema})^{\mathbf{M}} &= \Big(\forall y \forall z_1 \dots \forall z_n \exists X \forall x \big(x \in X \longleftrightarrow (x \in y \ \wedge \ \varphi)\big)\Big)^{\mathbf{M}} \\ &= \forall y \in \mathbf{M}\ \forall z_1 \in \mathbf{M}\ \dots \forall z_n \in \mathbf{M}\ \exists X \in \mathbf{M}\ \forall x \in \mathbf{M} \\ &\quad \big(x \in X \longleftrightarrow (x \in y \ \wedge\ \varphi)\big)^{\mathbf{M}} \\ &= \forall y \in \mathbf{M}\ \forall z_1 \in \mathbf{M}\ \dots \forall z_n \in \mathbf{M}\ \exists X \in \mathbf{M}\ \forall x \in \mathbf{M} \\ &\quad \big(x \in X \longleftrightarrow (x \in y \ \wedge\ (\varphi)^{\mathbf{M}})\big).\end{aligned}$$

So, taking $X = \left\{x \in y \mid (\varphi)^{\mathbf{M}}{}_{(x,y,z_1,\dots,z_k)}\right\}$ works since this set belongs to $\mathbf{M}$ by assumption.

□ 188

Lemma 189. *Let* $\mathbf{M}$ *be any non-empty class. If* $\mathbf{M}$ *is transitive, then*

- $$(\textbf{\textit{Power Set}})^{\mathbf{M}} \iff \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \big(\mathscr{P}(x) \cap \mathbf{M}\big) \subseteq y.$$
- *In particular, if* $\mathbf{M}$ *is closed under* $\mathscr{P} : \mathbf{M} \to \mathbf{V}$ *that maps* x *to* $\mathscr{P}(x)$ *(i.e.,* $\mathscr{P}[\mathbf{M}] \subseteq \mathbf{M}$*) then* $(\textbf{\textit{Power Set}})^{\mathbf{M}}$.

Proof of Lemma 189: We first notice that

$$\forall u \in \mathbf{M}\ \ (u \in z \to u \in x) \iff z \cap \mathbf{M} \subseteq x.$$

Also, if $\mathbf{M}$ is transitive and $z \in \mathbf{M}$, then $z \cap \mathbf{M} = z$ because $z \subseteq \mathbf{M}$.
We have:

$$\begin{aligned}(\textbf{Power Set})^{\mathbf{M}} &= \Big(\forall x\, \exists y\, \forall z \,\big(\forall u\ \ (u \in z \to u \in x) \to z \in y\big)\Big)^{\mathbf{M}} \\ &= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \big(\forall u\ \ (u \in z \to u \in x) \to z \in y\big)^{\mathbf{M}} \\ &= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \big(\forall u \in \mathbf{M}\ \ (u \in z \to u \in x) \to z \in y\big) \\ &= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \big(z \cap \mathbf{M} \subseteq x \to z \in y\big) \\ &\iff \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}\ \big(z \subseteq x \to z \in y\big) \\ &\iff \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \big(\mathscr{P}(x) \cap \mathbf{M}\big) \subseteq y.\end{aligned}$$

□ 189

Lemma 190. *Let* $\mathbf{M}$ *be any non-empty class.*

$$\forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \exists z \in \mathbf{M}\ \ (x \in z\ \wedge\ y \in z).$$

$$\Longleftrightarrow$$

$$\big(\textbf{\textit{Pairing}}\big)^{\mathbf{M}}.$$

Proof of Lemma 190: We have

$$\begin{aligned}(\textbf{Pairing})^{\mathbf{M}} &= \Big(\forall x \forall y \exists z \big(x \in z\ \wedge\ y \in z\big)\Big)^{\mathbf{M}} \\ &= \forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \exists z \in \mathbf{M}\ \ \big(x \in z\ \wedge\ y \in z\big)^{\mathbf{M}} \\ &= \forall x \in \mathbf{M}\ \forall y \in \mathbf{M}\ \exists z \in \mathbf{M}\ \ \big(x \in z\ \wedge\ y \in z\big).\end{aligned}$$

□ 190

Lemma 191. *Let* $\mathbf{M}$ *be any non-empty class.*

$$\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \ \big(\bigcup x \subseteq y\big).$$

$$\Longrightarrow$$

$$\big(\textbf{\textit{Union}}\big)^{\mathbf{M}}.$$

Proof of Lemma 191: We have

$$\begin{aligned}(\textbf{Union})^{\mathbf{M}} &= \Big(\forall x\, \exists y\, \forall a\ \forall b\ \big((a \in b\ \wedge\ b \in x) \to a \in y\big)\Big)^{\mathbf{M}} \\ &= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall a \in \mathbf{M}\ \forall b \in \mathbf{M}\ \big((a \in b\ \wedge\ b \in x) \to a \in y\big)^{\mathbf{M}} \\ &= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall a \in \mathbf{M}\ \forall b \in \mathbf{M}\ \big((a \in b\ \wedge\ b \in x) \to a \in y\big) \\ &\Leftarrow \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall a\ \forall b\ \big((a \in b\ \wedge\ b \in x) \to a \in y\big) \\ &= \forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \ \big(\bigcup x \subseteq y\big)\end{aligned}$$

□ 191

Lemma 192. *Let* $\mathbf{M}$ *be any non-empty class,* $\varphi := \varphi_{(x,y,A,w_1,\ldots,w_n)}$ *be any formula with free variables among* $x, y, A, w_1, \ldots, w_n$.

$\forall A \in \mathbf{M}\ \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M}$

$$\Big(\forall x \in A \cap \mathbf{M}\ \exists! y \in \mathbf{M}\ (\varphi)^{\mathbf{M}} \longrightarrow \exists B \in \mathbf{M}\ \Big\{y \in \mathbf{M} \mid \exists x \in A \cap \mathbf{M}\ (\varphi)^{\mathbf{M}}\Big\} \subseteq B\Big)$$

$$\Longleftrightarrow$$

$$\big(\textit{Instance of Replacement Schema for } \varphi\big)^{\mathbf{M}}$$

We recall $\exists! y\, \varphi$ abbreviates $\exists y \Big(\varphi_{(x,y,A,w_1,\ldots,w_n)} \wedge \forall z \left(\varphi(x,z,A,w_1,\ldots,w_n) \longrightarrow z = y\right)\Big)$.

Proof of Lemma 192: We have

$$\forall A\ \forall w_1 \ldots \forall w_n \Big[\forall x \left(x \in A \longrightarrow \exists! y\ \varphi\right) \longrightarrow \exists B\ \forall x \left(x \in A \longrightarrow \exists y \left(y \in B \wedge \varphi\right)\right)\Big]$$

where

$$\begin{aligned}
& \big(\textbf{Instance of Replacement Schema for } \varphi\big)^{\mathbf{M}} \\
= & \Big(\forall A\ \forall w_1 \ldots \forall w_n \Big(\forall x \left(x \in A \longrightarrow \exists! y\ \varphi\right) \longrightarrow \exists B\ \forall x \left(x \in A \longrightarrow \exists y \left(y \in B \wedge \varphi\right)\right)\Big)\Big)^{\mathbf{M}} \\
= & \ \forall A \in \mathbf{M}\ \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \\
& \Big(\forall x \in \mathbf{M}\ \big(x \in A \longrightarrow \exists! y \in \mathbf{M}\ (\varphi)^{\mathbf{M}}\big) \longrightarrow \exists B \in \mathbf{M}\ \forall x \in \mathbf{M}\ \big(x \in A \longrightarrow \exists y \in \mathbf{M}\ (y \in B \wedge (\varphi)^{\mathbf{M}})\big)\Big) \\
= & \ \forall A \in \mathbf{M}\ \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \\
& \Big(\forall x \in A \cap \mathbf{M}\ \exists! y \in \mathbf{M}\ (\varphi)^{\mathbf{M}} \longrightarrow \exists B \in \mathbf{M}\ \forall x \in A \cap \mathbf{M}\ \exists y \in B \cap \mathbf{M}\ \ (\varphi)^{\mathbf{M}}\Big) \\
\Longleftrightarrow & \ \forall A \in \mathbf{M}\ \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \\
& \Big(\forall x \in A \cap \mathbf{M}\ \exists! y \in \mathbf{M}\ (\varphi)^{\mathbf{M}} \longrightarrow \exists B \in \mathbf{M}\ \Big\{y \in \mathbf{M} \mid \exists x \in A \cap \mathbf{M}\ (\varphi)^{\mathbf{M}}\Big\} \subseteq B\Big)
\end{aligned}$$

□ 192

Lemma 193 ($\mathbf{ZF} \setminus \{\mathbf{AF}\}$)**.** *Let* $\mathbf{M}$ *be any non-empty class.*

$$\mathbf{M} \subseteq \mathbf{WF}$$

$$\Longrightarrow$$

$$\big(\textit{Foundation}\big)^{\mathbf{M}}.$$

Proof of Lemma 193: Assuming $\mathbf{M} \subseteq \mathbf{WF}$, one has $\in\restriction_{\mathbf{M}\times\mathbf{M}} \subseteq \mathbf{M} \times \mathbf{M}$ is well-founded and "set-like" on $\mathbf{M}$. So, by Theorem 174 one also has

$$\forall \mathbf{X} \subseteq \mathbf{M} \left(\mathbf{X} \neq \varnothing \longrightarrow \exists y \in \mathbf{X}\, \forall x \in \mathbf{X} \;\; x \notin y\right).$$

$$\begin{aligned}
(\textbf{Foundation})^{\mathbf{M}} &= \left(\forall X \left(\exists y\, y \in X \to \exists y \left(y \in X \;\wedge\; \neg\exists x \left(x \in X \;\wedge\; x \in y\right)\right)\right)\right)^{\mathbf{M}} \\
&= \forall X \in \mathbf{M} \left(\exists y \in \mathbf{M}\, y \in X \to \exists y \in \mathbf{M} \left(y \in X \;\wedge\; \neg\exists x \in \mathbf{M} \;(x \in X \;\wedge\; x \in y)\right)\right) \\
&= \forall X \in \mathbf{M} \left(\exists y \in \mathbf{M} \cap X \to \exists y \in \mathbf{M} \cap X\, \forall x \in \mathbf{M} \cap X \;\, x \notin y\right) \\
&\Longleftarrow \forall \mathbf{X} \subseteq \mathbf{M} \left(\mathbf{X} \neq \varnothing \longrightarrow \exists y \in \mathbf{X}\, \forall x \in \mathbf{X} \;\; x \notin y\right).
\end{aligned}$$

□ 193

Lemma 194 ($\mathbf{ZF} \setminus \{\mathbf{AF}\}$)**.**

(1) $\mathbf{ZF} \setminus \{\mathbf{AF}\} \vdash_c \left(\mathbf{ZF} \setminus \{\textit{Infinity}\}\right)^{\mathbf{W}(\omega)}$

(2) $\mathbf{ZF} \setminus \{\mathbf{AF}\} \vdash_c \left(\mathbf{ZF} \setminus \{\textit{Infinity}\}\right)^{\mathbf{WF}}$*.*

(3) $\mathbf{ZF} \setminus \{\mathbf{AF}\} \vdash_c (\mathbf{ZF})^{\mathbf{WF}}$*.*

Proof of Lemma 194: Both $\mathbf{W}(\omega)$ and $\mathbf{WF}$ are transitive classes closed under $\mathscr{P} : \mathbf{M} \to \mathbf{V}$ that maps x to $\mathscr{P}(x)$. So, one has

- $\textbf{Extensionality}^{\mathbf{W}(\omega)}$
- $\textbf{Comprehension Schema}^{\mathbf{W}(\omega)}$
- $(\textbf{Power Set})^{\mathbf{W}(\omega)}$
- $(\textbf{Pairing})^{\mathbf{W}(\omega)}$
- $(\textbf{Union})^{\mathbf{W}(\omega)}$
- $\textbf{Extensionality}^{\mathbf{WF}}$
- $\textbf{Comprehension Schema}^{\mathbf{WF}}$
- $(\textbf{Power Set})^{\mathbf{WF}}$
- $(\textbf{Pairing})^{\mathbf{WF}}$
- $(\textbf{Union})^{\mathbf{WF}}$

For the **Replacement Schema**, we consider any formula $\varphi := \varphi_{(x,y,A,w_1,\ldots,w_n)}$ with free variables among $x, y, A, w_1, \ldots, w_n$.

(1)

$$(\textbf{Instance of Replacement Schema for } \varphi)^{\mathbf{W}(\omega)}$$

$$\Longleftrightarrow$$

$$\forall A \in \mathbf{W}(\omega)\ \ \forall w_1 \in \mathbf{W}(\omega) \ldots \forall w_n \in \mathbf{W}(\omega)$$

$$\left(\forall x \in A \cap \mathbf{W}(\omega)\ \exists! y \in \mathbf{W}(\omega)\ \ (\varphi)^{\mathbf{W}(\omega)} \longrightarrow \exists B \in \mathbf{W}(\omega)\ \ \left\{y \in \mathbf{W}(\omega) \mid \exists x \in A \cap \mathbf{W}(\omega)\ \ (\varphi)^{\mathbf{W}(\omega)}\right\} \subseteq B\right)$$

$$\Longleftrightarrow$$

$$\forall A \in \mathbf{W}(\omega)\ \ \forall w_1 \in \mathbf{W}(\omega) \ldots \forall w_n \in \mathbf{W}(\omega) \qquad \textit{(since } \mathbf{W}(\omega) \textit{ is transitive)}$$

$$\left(\forall x \in A\ \exists! y \in \mathbf{W}(\omega)\ \ (\varphi)^{\mathbf{W}(\omega)} \longrightarrow \exists B \in \mathbf{W}(\omega)\ \ \left\{y \in \mathbf{W}(\omega) \mid \exists x \in A\ (\varphi)^{\mathbf{W}(\omega)}\right\} \subseteq B\right)$$

which holds since $A \in \mathbf{W}(\omega)$ implies $A \in \mathbf{W}(n)$ holds for some integer n. Thus, both

$$A \text{ and } \left\{y \in \mathbf{W}(\omega) \mid \exists x \in A\ (\varphi)^{\mathbf{W}(\omega)}\right\}$$

are finite. We set

$$k = \sup\left\{rk(y) + 1 \in \mathbf{On} \mid y \in \mathbf{W}(\omega)\ \ \wedge\ \exists x \in A\ (\varphi)^{\mathbf{W}(\omega)}\right\}$$

which leads to

$$\left\{y \in \mathbf{W}(\omega) \mid \exists x \in A\ (\varphi)^{\mathbf{W}(\omega)}\right\} \subseteq \mathbf{W}(k) \in \mathbf{W}(\omega).$$

(2)

$$(\textbf{Instance of Replacement Schema for } \varphi)^{\mathbf{WF}}$$

$$\Longleftrightarrow$$

$$\forall A \in \mathbf{WF}\ \forall w_1 \in \mathbf{WF} \ldots \forall w_n \in \mathbf{WF}$$

$$\left(\forall x \in A \cap \mathbf{WF}\ \exists! y \in \mathbf{WF}\ (\varphi)^{\mathbf{WF}} \longrightarrow \exists B \in \mathbf{WF}\ \ \left\{y \in \mathbf{WF} \mid \exists x \in A \cap \mathbf{WF}\ (\varphi)^{\mathbf{WF}}\right\} \subseteq B\right)$$

$$\Longleftrightarrow$$

$$\forall A \in \mathbf{WF}\ \forall w_1 \in \mathbf{WF} \ldots \forall w_n \in \mathbf{WF} \qquad \textit{(since } \mathbf{WF} \textit{ is transitive)}$$

$$\left(\forall x \in A\ \exists! y \in \mathbf{WF}\ (\varphi)^{\mathbf{WF}} \longrightarrow \exists B \in \mathbf{WF}\ \ \left\{y \in \mathbf{WF} \mid \exists x \in A\ (\varphi)^{\mathbf{WF}}\right\} \subseteq B\right)$$

which holds since $A \in \mathbf{WF}$ implies

$$\left\{y \in \mathbf{WF} \mid \exists x \in A\ (\varphi)^{\mathbf{WF}}\right\} \subseteq \mathbf{WF}$$

By Lemma 145 this leads to

$$\left\{y \in \mathbf{WF} \mid \exists x \in A\ (\varphi)^{\mathbf{WF}}\right\} \in \mathbf{WF}.$$

(3) ω belongs to $\mathbf{WF}$ since every ordinal α belongs to $\mathbf{WF}$ because it satisfies $\in$ is well-founded

on $tc(\alpha) = \alpha$; hence by Theorem 161, $\alpha \in \mathbf{WF}$.

Lemma 193 takes care of the Axiom of **Foundation** since obviously both $\mathbf{W}(\omega) \subseteq \mathbf{WF}$ and $\mathbf{WF} \subseteq \mathbf{WF}$ hold.

□ 194

8.2 Absoluteness

Definition 195. *Let $\mathbf{M}, \mathbf{N}$ be non-empty classes, and $\varphi_{(x_1,\ldots,x_n)}$ a formula with free variables among $x_1, \ldots, x_n$.*

(1) If $\mathbf{M} \subseteq \mathbf{N}$,

$$\varphi \textit{ is absolute for } \mathbf{M}, \mathbf{N}$$

$$\Longleftrightarrow$$

$$\forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M} \left((\varphi)^{\mathbf{M}} \longleftrightarrow (\varphi)^{\mathbf{N}} \right).$$

(2) φ is absolute for $\mathbf{M}$ if φ is absolute for $\mathbf{M}, \mathbf{V}$. i.e.,

$$\forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M} \left((\varphi)^{\mathbf{M}} \longleftrightarrow \varphi \right).$$

Remark 196.

$$\left.\begin{array}{c} \mathbf{M} \subseteq \mathbf{N} \\ \varphi \text{ is absolute for } \mathbf{M} \\ \varphi \text{ is absolute for } \mathbf{N} \end{array}\right\} \Longrightarrow \varphi \text{ is absolute for } \mathbf{M}, \mathbf{N}.$$

Absolute formulas are closed under boolean operations.

Lemma 197. *Let $\mathbf{M} \subseteq \mathbf{N}$ be non-empty classes, and $\varphi_{(x_1,\ldots,x_n)}$ a formula with free variables among $x_1, \ldots, x_n$.*

(1)

$$\varphi \textit{ is absolute for } \mathbf{M}, \mathbf{N} \Longrightarrow \neg\varphi \textit{ is absolute for } \mathbf{M}, \mathbf{N}.$$

(2)

$$\left.\begin{array}{l}\varphi \textit{ is absolute for } \mathbf{M},\mathbf{N}\\ \psi \textit{ is absolute for } \mathbf{M},\mathbf{N}\end{array}\right\} \Longrightarrow \begin{cases}\big(\varphi \wedge \psi\big) \textit{ is absolute for } \mathbf{M},\mathbf{N}\\ \big(\varphi \vee \psi\big) \textit{ is absolute for } \mathbf{M},\mathbf{N}\\ \big(\varphi \rightarrow \psi\big) \textit{ is absolute for } \mathbf{M},\mathbf{N}\\ \big(\varphi \leftrightarrow \psi\big) \textit{ is absolute for } \mathbf{M},\mathbf{N}.\end{cases}$$

Proof of Lemma 197: Immediate from the definition of both relativization and absoluteness.

□ 197

Definition 198. *φ is a Δ_0^{0-rud}-formula if*

- $\varphi := x = y$
- $\varphi := x \in y$

or

- *ψ is a Δ_0^{0-rud}-formula and*
  - $\varphi := \neg\psi$

or

- *φ_0, φ_1 are Δ_0^{0-rud}-formulas and*
  - $\varphi := \big(\varphi_0 \wedge \varphi_1\big)$
  - $\varphi := \big(\varphi_0 \vee \varphi_1\big)$
  - $\varphi := \big(\varphi_0 \longrightarrow \varphi_1\big)$
  - $\varphi := \big(\varphi_0 \longleftrightarrow \varphi_1\big)$

or

- *ψ is a Δ_0^{0-rud}-formula and*
  - $\varphi := \exists x \, \big(x \in y \wedge \psi\big)$
  - $\varphi := \forall x \, \big(x \in y \longrightarrow \psi\big)$.

As usual, "$\exists x \, \big(x \in y \wedge \psi\big)$" is abbreviated as "$\exists x \in y \; \psi$", and "$\forall x \, \big(x \in y \longrightarrow \psi\big)$" is shortened to "$\forall x \in y \; \psi$".

Lemma 199. *Let* $\mathbf{M}$ *be any non-empty class, and* φ *any* Δ_0^{0-rud}*-formula.*

If $\mathbf{M}$ *is transitive, then* φ *is absolute for* $\mathbf{M}$.

Proof of Lemma 199: The proof is by induction on $ht(\varphi)$. The only case that matters is $\varphi := \exists x \left(x \in y \wedge \psi\right)$. Assuming that the free variables of ψ are among $x_1, \ldots, x_n$, one has

$$\begin{aligned} & \forall y \in \mathbf{M}\, \forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M} \ \left((\varphi)^{\mathbf{M}} \leftrightarrow \varphi\right) \\ \Longleftrightarrow\ & \forall y \in \mathbf{M}\, \forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M} \ \left(\left(\exists x \ (x \in y \wedge \psi)\right)^{\mathbf{M}} \leftrightarrow \exists x \left(x \in y \wedge \psi\right)\right) \\ \Longleftrightarrow\ & \forall y \in \mathbf{M}\, \forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M} \ \left(\exists x \in \mathbf{M} \left(x \in y \wedge (\psi)^{\mathbf{M}}\right) \leftrightarrow \exists x \left(x \in y \wedge \psi\right)\right). \end{aligned}$$

To show that we have

$$\forall y \in \mathbf{M}\, \forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M} \ \left(\exists x \in \mathbf{M} \left(x \in y \wedge (\psi)^{\mathbf{M}}\right) \leftrightarrow \exists x \ \left(x \in y \wedge \psi\right)\right)$$

notice that since the induction hypothesis yields $\left((\psi)^{\mathbf{M}} \leftrightarrow \psi\right)$, we have both

(1) $\exists x \in \mathbf{M} \left(x \in y \wedge (\psi)^{\mathbf{M}}\right) \Longrightarrow \exists x \in \mathbf{M} \left(x \in y \wedge \psi\right) \Longrightarrow \exists x \ \left(x \in y \wedge \psi\right)$

(2) and the transitivity of $\mathbf{M}$ gives us that x belongs to $\mathbf{M}$ (since $x \in y \in \mathbf{M}$) which leads to $\exists x \in \mathbf{M} \left(x \in y \wedge (\psi)^{\mathbf{M}}\right) \Longleftarrow \exists x \in \mathbf{M} \left(x \in y \wedge \psi\right) \Longleftarrow \exists x \ \left(x \in y \wedge \psi\right)$

□ 199

Lemma 200. *Let* $\mathbf{M} \subseteq \mathbf{N}$ *be non-empty classes,* $\varphi(x_1, \ldots, x_n)$ *and* $\psi(x_1, \ldots, x_n)$ *be any* $\mathcal{L}_{\mathrm{ST}}$*-formulas whose free variables are among* $x_1, \ldots, x_n$*, and* $\mathcal{T}$ *some* $\mathcal{L}_{\mathrm{ST}}$*-theory.*
If the following conditions are satisfied

- $\mathcal{T} \vdash_c \forall x_1 \ldots \forall x_n \left(\varphi \leftrightarrow \psi\right)$
- $\mathbf{ZF} \vdash_c (\mathcal{T})^{\mathbf{M}}$
- $\mathbf{ZF} \vdash_c (\mathcal{T})^{\mathbf{N}}$,

then

$\mathbf{ZF} \vdash_c$ "*φ is absolute for* $\mathbf{M}, \mathbf{N}$ " $\longleftrightarrow$ "*ψ is absolute for* $\mathbf{M}, \mathbf{N}$ ".

Proof of Lemma 200: Exercise.

□ 200

In particular, if φ is equivalent to some Δ_0^{0-rud}-formula, then φ is absolute for transitive models.

Proposition 201. *The following relations and functions are provably equivalent in* **ZF** *to* Δ_0^{0-rud}*-formulas, hence they are absolute for transitive models of* **ZF***.*

(1) $x = y$

(2) $x \in y$

(3) $x \subseteq y$

(4) $\varnothing$

(5) $\{x, y\}$

(6) $\{x\}$

(7) (x, y)

(8) $x \cup y$

(9) $x \cap y$

(10) $x \cup \{x\}$

(11) $x \smallsetminus y$

(12) $\bigcup x$

(13) $\bigcap x \quad (x \neq \varnothing)$

(14) *"x is transitive".*

Proof of Proposition 201:

(1) Immediate

(2) immediate

(3) $x \subseteq y$ iff $\forall z \in x \; z \in y$

(4) $x = \varnothing$ iff $\neg \exists z \in x$ or equivalently $\forall z \in x \; \neg z = z$

(5) $p = \{x, y\}$ iff $\big(\forall z \in p \; (p = x \vee p = y) \;\wedge\; (x \in p \wedge \; y \in p)\big)$

(6) $s = \{x\}$ iff $(x \in s \;\wedge\; \forall z \in s \;\; z = x)$

(7) $c = (x, y)$ iff $\Big(\forall z \in c \;\; \big(z = \{x\} \;\vee\; z = \{x, y\}\big) \;\wedge\; \big(\{x\} \in c \;\wedge\; \{x, y\} \in c\big)\Big)$

(8) $u = x \cup y$ iff $\Big(\forall z \in u \; (z \in x \;\vee\; z \in y) \;\wedge\; (x \subseteq u \;\wedge\; y \subseteq u)\Big)$

(9) $i = x \cap y$ iff $\Big(\forall z \in x \; (z \in y \;\longrightarrow\; z \in i) \;\wedge\; (i \subseteq x \;\wedge\; i \subseteq y)\Big)$

(10) $y = x \cup \{x\}$ iff $\Big((x \in y \;\wedge\; x \subseteq y) \;\wedge\; \forall z \in y \; (z \in x \;\vee\; z = x)\Big)$

(11) $d = x \smallsetminus y$ iff $\Big(d \subseteq x \;\wedge\; \forall z \in x \; (\neg z \in y \;\longleftrightarrow\; z \in d)\Big)$

(12) $u = \bigcup x$ iff $\big(\forall z \in u \; \exists y \in x \;\; z \in y \;\wedge\; \forall y \in x \;\; y \subseteq u\big)$

(13) $i = \bigcap x$ (assuming $\bigcap \varnothing = \varnothing$) iff

$$\Big(\big(\forall z \in i \;\; \forall y \in x \;\; z \in y \;\wedge\; \forall y \in x \;\; \forall z \in y (\forall y' \in x \;\; z \in y' \longrightarrow z \in i)\big) \;\wedge\; (x = \varnothing \longrightarrow i = \varnothing)\Big)$$

(14) "x is transitive" iff $\forall y \in x \ \ \forall z \in y \ \ z \in x$

□ 201

Definition 202. *Let* $\mathbf{M} \subseteq \mathbf{N}$ *be any non-empty class.*

(1) A class-relation $\mathbf{R} \subseteq \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{n}$ *is absolute for* $\mathbf{M}, \mathbf{N}$ *if there exists some formula* $\varphi(x_1, \ldots, x_n)$ *whose free variables are among* $x_1, \ldots, x_n$ *which is is absolute for* $\mathbf{M}, \mathbf{N}$ *and such that*

$$\forall x_1 \ldots \forall x_n \quad \big((x_1, \ldots, x_n) \in \mathbf{R} \longleftrightarrow \varphi(x_1, \ldots, x_n)\big).$$

(2) A class-function $\mathbf{F} : \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{n} \to \mathbf{V}$ *is absolute for* $\mathbf{M}, \mathbf{N}$ *if there exists some formula* $\varphi(x_1, \ldots, x_n, y)$ *whose free variables are among* $x_1, \ldots, x_n, y$ *which is is absolute for* $\mathbf{M}, \mathbf{N}$ *and such that*

$$\forall x_1 \ldots \forall x_n \forall y \quad \big(\mathbf{F}(x_1, \ldots, x_n) = y \longleftrightarrow \varphi(x_1, \ldots, x_n, y)\big).$$

Formally, φ *must also satisfy:*

- $\forall x_1 \ldots \forall x_n \exists! y \quad \varphi(x_1, \ldots, x_n, y)$
- $\forall x_1 \in \mathbf{M} \ldots \forall x_n \in \mathbf{M} \exists! y \in \mathbf{M} \quad \varphi(x_1, \ldots, x_n, y)$
- $\forall x_1 \in \mathbf{N} \ldots \forall x_n \in \mathbf{N} \exists! y \in \mathbf{N} \quad \varphi(x_1, \ldots, x_n, y)$

Lemma 203. *Let* $\mathbf{M} \subseteq \mathbf{N}$ *be non-empty classes,* $\varphi(x_1, \ldots, x_n)$ *any* $\mathcal{L}_{\text{ST}}$*-formula whose free variables are among* $x_1, \ldots, x_n$*, and* $\mathbf{F} : \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{n} \to \mathbf{V}$, $\mathbf{G_1} : \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{k} \to \mathbf{V}, \ldots, \mathbf{G_n} : \underbrace{\mathbf{V} \times \ldots \times \mathbf{V}}_{k}$ *any class-functions.*

(1) If "$\varphi, \mathbf{G_1}, \ldots, \mathbf{G_n}$ *are absolute for* $\mathbf{M}, \mathbf{N}$*",*
then "$\varphi\Big(\mathbf{G_1}(z_1, \ldots, z_k), \ldots, \mathbf{G_n}(z_1, \ldots, z_k)\Big)$ *is absolute for* $\mathbf{M}, \mathbf{N}$*".*

(2) If "$\mathbf{F}, \mathbf{G_1}, \ldots, \mathbf{G_n}$ *are absolute for* $\mathbf{M}, \mathbf{N}$*",*
then "$\mathbf{F}\Big(\mathbf{G_1}(z_1, \ldots, z_k), \ldots, \mathbf{G_n}(z_1, \ldots, z_k)\Big)$ *is absolute for* $\mathbf{M}, \mathbf{N}$*".*

Proof of Remark 203: Exercise.

□ 203

Proposition 204. *The following relations and functions are absolute for transitive models of* **ZF**.

(1) *"c is a couple"*

(2) $C = A \times B$

(3) *"R is a (binary) relation"*

(4) $d = dom(R)$ *(R a relation)*

(5) $r = ran(R)$ *(R a relation)*

(6) *"f is a function"*

(7) $y = f(x)$

(8) *"f is a 1-1 function"*

(9) *"α is an ordinal"*

(10) *"α is a limit ordinal"*

(11) *"α is a successor ordinal"*

(12) *"α is a finite ordinal"*

(13) $x = \omega$

(14) $x = 4$

Proof of Proposition 206:

(1) "c is a couple" iff $\exists x \in \bigcup c \; \exists y \in \bigcup c \;\; (x, y) = c$.

More precisely, "c is a couple" iff $\varphi\big(G_1(c), G_2(c), G_3(c)\big)$ where $G_1(c) = G_2(c) = \bigcup c$ which is absolute, $G_3(c) = c$, and $\varphi\big(x_1, x_2, x_3\big)$ is the following Δ_0^{0-rud}-formula:

$$\varphi\big(x_1, x_2, x_3\big) := \exists x \in x_1 \;\; \exists y \in x_2 \;\; \Big(\forall z \in c \;\; \big(z = \{x\} \vee z = \{x, y\}\big) \wedge \big(\{x\} \in c \wedge \{x, y\} \in c\big)\Big)$$

$$:= \exists x \in x_1 \;\; \exists y \in x_2 \left(\begin{array}{c} \forall z \in c \;\; \big((x \in z \;\wedge\; \forall z' \in z \;\; z' = x) \;\vee\; (\forall z' \in z \; (z' = x \vee z' = y)\big) \\ \wedge \\ \exists s \in c \; (x \in s \;\wedge\; \forall z \in s \;\; z = x) \\ \wedge \\ \exists p \in c \; \big(\forall z \in p \; (p = x \vee p = y) \;\wedge\; (x \in p \wedge \; y \in p)\big) \end{array} \right)$$

(2) $C = A \times B$ iff $\big(\forall x \in A \, \forall y \in B \;\; (x, y) \in C \;\wedge\; \forall z \in C \, \exists x \in A \, \exists y \in B \;\; (x, y) = z\big)$

(3) "R is a (binary) relation" iff $\forall x \in R$ "x is a couple"

(4) $d = dom(R)$ iff

$$\Big(\forall x \in d \, \exists y \in \bigcup\bigcup R \;\; (x, y) \in R \;\wedge\; \forall y \in \bigcup\bigcup R \; \big(\exists x \in \bigcup\bigcup R \;\; (x, y) \in R \longrightarrow x \in d\big)\Big)$$

(5) $r = ran(R)$ iff

$$\left(\forall y \in r\, \exists x \in \bigcup\bigcup R \;\; (x,y) \in R \;\wedge\; \forall y \in \bigcup\bigcup R \, \left(\exists x \in \bigcup\bigcup R \;\; (x,y) \in R \longrightarrow y \in r\right)\right)$$

(6) "f is a function" iff

$$\left(\begin{array}{c} \text{“}f \textit{ is a relation}\text{”} \\ \wedge \\ \forall x \in \bigcup\bigcup f \;\; \forall y \in \bigcup\bigcup f \;\; \forall z \in \bigcup\bigcup f \;\; \left(\left((x,y) \in f \;\wedge\; (x,z) \in f\right) \longrightarrow y = z\right) \end{array}\right)$$

(7) $y = f(x)$ iff "f is a function" $\wedge \;\; (x,y) \in f$

(8) "f is a 1-1 function" iff

$$\left(\begin{array}{c} \text{“}f \textit{ is a function}\text{”} \\ \wedge \\ \forall x \in dom(f) \; \forall x' \in dom(f) \; \forall y \in ran(f) \;\; \left(\left(f(x) = y \;\wedge\; f(x') = y\right) \longrightarrow x = x'\right) \end{array}\right)$$

(9) We make use of the following

Claim 205 (ZF). *Let A be any set.*

If A is both transitive and totally ordered by $\in$, then A is an ordinal.

Proof of Claim 205: It is enough to prove that every non-empty $B \subseteq A$ contains a $\in$-least element. By **Foundation**, we have $\exists y \left(y \in B \;\wedge\; \neg\exists z \left(z \in B \;\wedge\; z \in y\right)\right)$ which provides the requested $\in$-least element.

□ 205

$$\text{“}\alpha \text{ is an ordinal”} \text{ iff } \left(\begin{array}{c} \text{“}\alpha \textit{ is transitive}\text{”} \\ \wedge \\ \forall x \in \alpha \, \forall y \in \alpha \, \forall z \in \alpha \;\; \left((x \in y \wedge y \in z) \longrightarrow x \in z\right) \\ \wedge \\ \forall x \in \alpha \, \forall y \in \alpha \;\; (x \in y \vee y \in x \vee x = y) \end{array}\right)$$

(10) "α is a limit ordinal" iff $\left(\begin{array}{c} \neg\alpha = \varnothing \\ \wedge \\ \text{"}\alpha \textit{ is an ordinal}\text{"} \\ \wedge \\ \forall x \in \alpha\, \exists y \in \alpha \;\; x \in y \end{array}\right)$

(11) "α is a successor ordinal" iff $\left(\begin{array}{c} \neg\alpha = \varnothing \\ \wedge \\ \neg\text{"}\alpha \textit{ is a limit ordinal}\text{"} \end{array}\right)$

(12) "α is a finite ordinal" iff $\left(\begin{array}{c} \alpha = \varnothing \\ \vee \\ \forall x \in \alpha \cup \{\alpha\}\ \text{"}x \textit{ is a successor ordinal}\text{"} \end{array}\right)$

(13) $x = \omega$ iff $\left(\text{"}x \textit{ is a limit ordinal}\text{"} \;\wedge\; \forall y \in x\ \text{"}y \textit{ is a successor ordinal}\text{"}\right)$

(14) $x = 4$ iff $x = \Big\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \Big\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\Big\}\Big\}$.

□ 206

Proposition 206. *Let* **M** *be any transitive model of* **ZF**.

If $A \subseteq \mathbf{M}$ *is finite, then* $A \in \mathbf{M}$.

Proof of Proposition 206: If $A = \varnothing$, then the result comes from the fact $\varnothing$ is absolute. Otherwise, let $A = \{a_1, \ldots, a_k\}$. For each $1 \leqslant i \leqslant k$, $\{a_i\} \in \mathbf{M}$ holds by Proposition 201 (6), and Proposition 201 (8) yields $\{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_k\} \in \mathbf{M}$.

□ 206

Proposition 207. *The following relations and functions are absolute for transitive models of* "**ZF**".

(1) "x *is finite*"

(2) $x = A^n$

(3) $x = A^{<\omega}$

(4) "$<_A$ *well-orders* A"

(5) $x = \mathit{type}(A, <_A)$

Proof of Proposition 207:

(1) "*x is finite*" iff there exists some mapping $f : x \xrightarrow{1-1} \omega$ such that $ran(f) \in \omega$.

Notice that the following formula $\varphi(x, f)$ is absolute for transitive models of **ZF**:

$$\varphi(x, f) := \big(\text{"}f \textit{ is a 1-1 function}\text{"} \;\wedge\; dom(f) = x \;\wedge\; ran(f) \in \omega\big)$$

So, given $\mathbf{M}$ any transitive model of **ZF**, we only need to show

$$\Big(\big(\exists f \; \varphi(x, f)\big)^{\mathbf{M}} \longleftrightarrow \big(\exists f \; \varphi(x, f)\big)^{\mathbf{V}}\Big)$$

i.e.,

$$\Big(\big(\exists f \; \varphi(x, f)\big)^{\mathbf{M}} \longleftrightarrow \exists f \; \varphi(x, f)\Big).$$

The direction $(\longrightarrow)$ is obvious. For the other direction $(\longleftarrow)$, notice that if f exists in $\mathbf{V}$, then f is a finite set of couples of elements of $\mathbf{M}$. By absoluteness, $\mathbf{M}$ is closed under the operation

$$\begin{array}{rcccl} couple: & \mathbf{M} & , & \mathbf{M} & \mapsto & \mathbf{M} \\ & x & , & y & \to & (x, y) \end{array}$$

and by Proposition 206, $\mathbf{M}$ contains all its finite subsets, hence it contains f.

(2) The proof is by induction on n. For $n = 0$, we have $x = A^0$ iff $x = \{\varnothing\}$ which is absolute. For $n := n + 1$,

$$\begin{aligned} x = A^{n+1} \iff x &= \big\{s \cup \{(n, a)\} \mid s \in A^n \wedge a \in A\big\} \\ &= \big\{s \cup \{(n, a)\} \mid (s, a) \in A^n \times A\big\}. \end{aligned}$$

By Proposition 206 (2), $A^n \times A$ is absolute for transitive models of **ZF**. The class-function $\begin{array}{rccc} \mathbf{F}: & A^n \times A & \to & \mathbf{V} \\ & (s, a) & \mapsto & s \cup \{(n, a)\} \end{array}$ is absolute as well; hence $x = A^{n+1}$ is absolute.

(3) Define the functionnal $\mathbf{F}(A)$ by $\mathbf{F}(A) = \big\{f \mid f \textit{ is a function} \;\wedge\; dom(f) \in \omega \;\wedge\; ran(f) \subseteq A\big\}$. Since the notions involved are absolute for transitive models of **ZF**, it turns out that given any $A \in \mathbf{M}$, we have $\big(\mathbf{F}(A)\big)^{\mathbf{M}} = \mathbf{F}(A)$.

(4) We need to prove $\big(\text{"} <_A \textit{ well-orders } A\text{"}\big)^{\mathbf{M}} \longleftrightarrow \text{"} <_A \textit{ well-orders } A\text{"}$.

$(\longleftarrow)$ Notice that "$<_A$ *totally orders* A" is absolute since it corresponds to the following

Δ_0^{0-rud}-formula:

$$\left(\begin{array}{c} \forall x \in A\ \forall y \in A\ \Big((x,y) \in <_A \ \vee\ (y,x) \in <_A \ \vee\ x = y\Big) \\ \bigwedge \\ \forall x \in A\ \forall y \in A\ \forall z \in A\ \Big(\big((x,y) \in <_A \ \vee\ (y,z) \in <_A\big) \longrightarrow (x,z) \in <_A\Big) \end{array}\right)$$

For well-ordering, we must check that

$$\left(\begin{array}{c} \forall X\ \Big(\big(X \subseteq A\ \wedge\ X \neq \varnothing\big) \longrightarrow \exists y \in X\ \forall z \in X\ \ (z,y) \notin <_A\Big) \\ \longrightarrow \\ \forall X\ \Big(\big(X \subseteq A\ \wedge\ X \neq \varnothing\big) \longrightarrow \exists y \in X\ \forall z \in X\ \ (z,y) \notin <_A\Big)^{\mathbf{M}} \end{array}\right)$$

i.e.,

$$\forall X\ \varphi(X, A, <_A) \longrightarrow \Big(\forall X\ \varphi(X, A, <_A)\Big)^{\mathbf{M}}$$

$$\text{where } \varphi(X, A, <_A) := \Big(\big(X \subseteq A\ \wedge\ X \neq \varnothing\big) \longrightarrow \exists y \in X\ \forall z \in X\ \ (z,y) \notin <_A\Big).$$

Notice that $\varphi(X, A, <_A)$ is absolute form $\mathbf{M}$. Also, that a **universal** quantification over an absolute formula relativizes down from $\mathbf{V}$ to $\mathbf{M}$. i.e., in this particular case, given any non-empty $\big(X \subseteq A\big)^{\mathbf{M}}$, first we notice that $\big(X \subseteq A\big)^{\mathbf{M}}$ holds iff $X \cap \mathbf{M} \subseteq A$ and since $\mathbf{M}$ is transitive, we have $X \cap \mathbf{M} = X$, hence $X \cap \mathbf{M} \subseteq A$ iff $X \subseteq A$. But since in $\mathbf{V}$ $<_A$ well-orders A, there exists some minimal element $y \in X$, which also exists in $\mathbf{M}$ and is $<_A$-minimal.

($\longrightarrow$) Notice that $\mathbf{M} \models \exists \alpha\ \exists f\ \ \varphi(\alpha, f, A, <_A)$ where the formula $\varphi(\alpha, f, A, <_A)$ is

$$\big(\text{"}\alpha \textit{ is an ordinal}\text{"}\ \wedge\ \text{"}f \textit{ is an isomorphism between } \alpha \textit{ and } (A, <_A)\text{"}\big),$$

and that $\varphi(\alpha, f, A, <_A)$ is absolute for transitive models of **ZF**. Also, that an **existential** quantification over an absolute formula relativizes up from $\mathbf{M}$ to $\mathbf{V}$. i.e., in this particular case, since $\big(\text{"} <_A \textit{ well-orders } A\text{"}\big)^{\mathbf{M}}$ holds, there exists in $\mathbf{M}$ both an ordinal α and an isomorphism f, which remain an ordinal and an isomorphism in $\mathbf{V}$ (by absoluteness of $\varphi(\alpha, f, A, <_A)$), showing not only that $\big(\text{"} <_A \textit{ well-orders } A\text{"}\big)^{\mathbf{V}}$ but also that the order type of $(A, <_A)$ is unchanged when one goes from $\mathbf{M}$ to $\mathbf{V}$.

(5) See the last argument in case (4) right above.

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