(Partial) Mock Exam Solutions

An A4 two side sheet of personal notes is allowed.
Points are only given as an indication of the length and/or the difficulty of each exercise.

Last name:
First name:
Section:

/73 points

Problem 1: (38 points)

2 pt **Question 1.1:** One the following ordinal equalities is true. Which one?

- [ ] $3 \cdot \omega = \omega \cdot 3$
- [x] $\omega \cdot 2 + \omega^2 = \omega \cdot 3 + 2 + \omega^2$
- [ ] $\omega^2 + \omega^3 = \omega^2 + \omega^2$
- [ ] $4 \cdot (3 + \omega) = (3 + \omega) \cdot 4$

2 pt **Question 1.2:** Let $\oplus$ and $\otimes$ denote the cardinal operations. One of the following cardinal equalities is provably false in ZFC. Which one?

- [ ] $\aleph_0 \otimes \aleph_0 = \aleph_0$
- [x] $\aleph_\omega \oplus \aleph_1 = \aleph_{\omega+1}$
- [ ] $\aleph_2 \otimes 2^{(2^{\aleph_0})} = 2^{(2^{\aleph_0})}$
- [ ] $8 \oplus \aleph_1 = \aleph_1 \otimes 8$

2 pt **Question 1.3:** Assuming ZFC is consistent. Let $\mathrm{cof}(\alpha)$ denote the cofinality of α. One of the following ordinal equality is provably true in ZFC. Which one?

- [ ] $\mathrm{cof}(\omega^\omega) = \omega^\omega$
- [ ] $\mathrm{cof}(\omega_2) = \omega$
- [ ] $\mathrm{cof}(\omega_1 + \omega^\omega) = \omega_1$
- [x] $\mathrm{cof}(\aleph_\omega) = \omega$

4 pt **Question 1.4:** Working in ZFC and assuming ZFC is consistent, we consider the structure $(V_\omega, \in)$. Among the following axioms, check those that **do hold** in this structure.

- [x] Foundation axiom
- [x] Pairing axiom
- [ ] Axiom of infinity
- [x] Power Set axiom

3 pt **Question 1.5:** Check the correct equalities among the following.

- [x] $\text{rk}(\,\omega\,) = \omega$
- [x] $\text{rk}(\,{}^{\omega}\omega\,) = \omega + 1$
- [ ] $\text{rk}(\,{}^{\omega}\omega\,) = \omega$
- [x] $\text{rk}(\,\{0, 2\}\,) = 3$
- [ ] $\text{rk}(\,\{0, 2\}\,) = 2$
- [x] $\text{rk}(\,\{\omega_1\}\,) = \omega_1 + 1$

3 pt **Question 1.6:** Assuming ZFC is consistent. Which of the following statements is provable in ZFC?

- [x] there exists a surjection $f : \mathbb{R} \to \mathcal{P}(\aleph_0)$
- [ ] there exists a injection $f : \omega_2 \to \mathbb{R}$
- [x] there exists an injection $f : \omega_1 \to \mathcal{P}(\mathbb{R})$
- [x] there exists a bijection $f : \mathbb{R} \to \mathcal{P}(\aleph_0)$
- [ ] there exists a surjection $f : \omega_2 \to \mathbb{R}$
- [x] there exists a surjection $f : \omega_2 \to \omega_1$

3 pt **Question 1.7:** Let $\mathbf{L}$ be the class of constructible sets. Assuming ZF is consistent, which of the following statements is provable in ZF?

- [x] (Power Set Axiom)$^{\mathbf{L}}$
- [ ] (*"there exists some well-ordering of* $\mathbb{R}$ *of order type* ω_2*"*)$^{\mathbf{L}}$
- [x] (AC)$^{\mathbf{L}}$
- [x] (Axiom of Extensionality)$^{\mathbf{L}}$
- [x] $(2^{(2^{\aleph_0})} = \aleph_2)^{\mathbf{L}}$
- [ ] (*"*$\mathbb{R}$ *is a countable union of countable sets"*)$^{\mathbf{L}}$

3 pt **Question 1.8:** Let $\mathbf{M}$ be a transitive class model of ZFC. Which of the following formulas are absolute for $\mathbf{M}$?

- [x] "α is an ordinal"

- ☒ "α is an infinite ordinal"
- ☒ "α is a limit ordinal"
- ☐ "α is some uncountable ordinal"
- ☐ "α is some countable ordinal"
- ☒ "$f : x \to y$ is some injection"

4 pt **Question 1.9:** Assuming ZFC is consistent. Among the following statements, check those which are consistent with ZFC:

- ☒ There exists some regular cardinal κ such that $\aleph_0 \leqslant \kappa < 2^{\aleph_0}$.
- ☐ There exists no singular cardinal κ such that $\aleph_0 < \kappa < 2^{(\aleph_{\aleph_0})}$.
- ☒ There exists some singular cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$.
- ☒ There exists some singular cardinal κ and some regular cardinal λ such that $\aleph_0 < \kappa \leqslant \lambda < 2^{\aleph_0}$.
- ☒ There exists some singular regular κ and some singular cardinal λ such that $\aleph_0 < \kappa \leqslant \lambda < 2^{\aleph_0}$.
- ☒ There exists exactly one singular cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$.
- ☒ There exists no regular cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$.
- ☒ There exist exactly two singular cardinals κ,λ such that $\aleph_0 < \kappa < \lambda < 2^{\aleph_0}$.

5 pt **Question 1.10:** Assuming ZF is consistent. Among the following statements, check those which hold in all models of ZF:
(We recall that $A \overset{1-1}{\lesssim} B$ stands for *"there exists an injection from A to B"*.)

- ☒ $\mathbb{R} \overset{1-1}{\lesssim} {}^{\omega}\mathbb{R}$.
- ☒ ${}^{\omega}2 \overset{1-1}{\lesssim} \mathbb{R}$.
- ☒ ${}^{\omega}\mathbb{R} \overset{1-1}{\lesssim} \mathbb{R}$.
- ☒ $\mathbb{R} \overset{1-1}{\lesssim} {}^{2}({}^{\omega}2)$.
- ☒ $\mathbb{R} \overset{1-1}{\lesssim} {}^{\mathbb{R}}2$.
- ☒ ${}^{2}({}^{\omega}2) \overset{1-1}{\lesssim} \mathbb{R}$.
- ☐ ${}^{\mathbb{R}}2 \overset{1-1}{\lesssim} \mathbb{R}$.
- ☐ $\mathbb{R} \overset{1-1}{\lesssim} {}^{({}^{\omega}2)}2$.
- ☒ $\mathbb{R} \overset{1-1}{\lesssim} {}^{\omega}2$.
- ☒ ${}^{({}^{\omega}2)}2 \overset{1-1}{\lesssim} \mathbb{R}$.

2 pt **Question 1.11:** Assuming ZF is consistent. Among the following statements, check those which hold in all models of ZF+ "$\mathbb{R}$ *is a countable union of countable sets*":

- ☒ there is a partition $\mathcal{R}$ of $\mathbb{R}$ such that $\mathbb{R} \overset{1-1}{\lesssim} \mathcal{R}$
- ☒ there is a partition $\mathcal{R}$ of $\mathbb{R}$ such that $\mathbb{R} \overset{1-1}{\not\lesssim} \mathcal{R}$
- ☒ there is a partition $\mathcal{R}$ of $\mathbb{R}$ such that $\mathcal{R} \overset{1-1}{\not\lesssim} \mathbb{R}$
- ☒ there is a partition $\mathcal{R}$ of $\mathbb{R}$ such that both $\mathbb{R} \overset{1-1}{\lesssim} \mathcal{R}$ and $\mathcal{R} \overset{1-1}{\not\lesssim} \mathbb{R}$ hold.

2 pt **Question 1.12:** Assuming ZF is consistent. Let M be any countable transitive model of "ZFC", $(\mathbb{P}, \leqslant, \mathbb{1})$ any partial order over M, $\mathcal{G}$ any subgroup of the group of automorphisms of $\mathbb{P}$, and $\mathcal{F}$ a normal filter on $\mathcal{G}$

For all subgroups $\mathcal{H}$, $\mathcal{K}$ of $\mathcal{G}$ and all $\pi \in \mathcal{G}$ which of the following assertion holds?

- ☒ $\mathcal{G} \in \mathcal{F}$
- ☐ if $\mathcal{H} \in \mathcal{F}$ and $\mathcal{K} \subseteq \mathcal{H}$, then $\mathcal{K} \in \mathcal{F}$
- ☐ if $\mathcal{H} \in \mathcal{F}$ and $\mathcal{K} \in \mathcal{F}$, then $\mathcal{H} \cup \mathcal{K} \in \mathcal{F}$
- ☒ if $\mathcal{H} \in \mathcal{F}$, then $\pi^{-1} \circ \mathcal{H} \circ \pi \in \mathcal{F}$

3 pt **Question 1.13:** Assuming **ZFA** is consistent. Let **M** be any transitive model of **ZFA** with $\mathbb{A}$ as set of atoms, $\mathcal{G}$ any subgroup of the group of permutations of $\mathbb{A}$, $\mathcal{F}$ any normal filter on $\mathcal{G}$, $\mathbf{HS}_{\mathcal{F}}$ the class of all hereditarily symmetric sets, and $\mathbf{M}^{\mathbf{HS}_{\mathcal{F}}} = \mathbf{M} \cap \mathbf{HS}_{\mathcal{F}}$ the induced permutation model. Among the following statements, check those which hold:

- ☒ $\mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$ is transitive.
- ☒ $\mathcal{P}^{\infty}(\emptyset) \subseteq \mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$;
- ☒ $\mathbb{A} \subseteq \mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$.

☒ $\mathbb{A} \in \mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$.

☒ $\mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$ satisfies **ZFA**.

☐ $\mathbf{M}^{\mathbf{HS}_{\mathcal{F}}}$ satisfies **AC**.

/38 points

Problem 2: (6 points)

2 pt **Question 2.1:** Write a first order formula in the language of set theory which formalizes the expression

$$A = \bigcup(B \cap C).$$

Solution:

$$\forall x \Big(x \in A \leftrightarrow \exists y \big(x \in y \wedge (y \in B \wedge y \in C) \big) \Big)$$

4 pt **Question 2.2:** In the language of set theory, write a first order formula with one free variable which describes the class of transitive $\in$-well-founded sets.

Solution:

$$\phi(x) : \Big(\forall y \, \forall z \, \big((y \in x \wedge z \in y) \to z \in x \big) \wedge \exists y \big(y \in x \ \wedge \ \forall z \big(z \in y \to z \notin x \big) \big) \Big)$$

/6 points

Problem 3: (29 points)

We let $\mathbb{P}$ denote the following forcing notion:

$$\{p : F \to \omega \mid F \text{ is any finite subset of } \omega\}$$

partially ordered by $p \leqslant q$ if and only if p extends q, i.e. $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and the restriction of p to $\mathrm{dom}(q)$ equals q.
We assume that M is a countable transitive model of "ZFC" that contains $\mathbb{P}$.

4 pt **Question 3.1:** Among the following sets, check the ones which are $\mathbb{P}$-conditions.

- ☒ $p_0 = 0$
- ☒ $p_1 = \{(0,0)\}$
- ☐ $p_2 = \{\{0\}\}$
- ☐ $p_3 = \omega$
- ☐ $p_4 = \{(0,0),(0,3)\}$
- ☒ $p_5 = \{(0,0),(3,1)\}$
- ☐ $p_6 = \{(0,p_0),(3,p_3)\}$
- ☐ $p_6 = \{(p_0,0),(p_3,3)\}$
- ☐ $p_7 = \{p_0,p_3\}$

3 pt **Question 3.2:** Assume that σ and τ are $\mathbb{P}$-names and p and q are $\mathbb{P}$-conditions. Among the following sets, check the ones which are $\mathbb{P}$-names.

- ☒ $\tau_0 = 0$
- ☒ $\tau_1 = \{(0,0)\}$
- ☐ $\tau_2 = \{0,\tau_1\}$
- ☒ $\tau_3 = \{(\sigma,p),(\tau,q)\}$
- ☐ $\tau_4 = \{(p,\sigma),(q,\tau)\}$
- ☐ $\tau_5 = \{(p,p),(q,q)\}$

4 pt **Question 3.3:** Let G be $\mathbb{P}$-generic over M with $p = \{(0,0),(1,2),(5,7)\} \in G$.

1. Does $q_0 = \{(0,0),(5,7)\} \in G$ hold?

   ☒ yes ☐ no ☐ it depends

2. Does $q_1 = \{(1,0)\} \in G$ hold?

   ☐ yes ☒ no ☐ it depends

3. Does $q_2 = \{(2,1)\} \in G$ hold?

   ☐ yes ☐ no ☒ it depends

4. Does $q_3 = \{(0,0),(1,2),(2,3),(3,4),(5,7)\} \in G$ hold?

   ☐ yes ☐ no ☒ it depends

6 pt **Question 3.4:** We now consider the following $\mathbb{P}$-conditions:

$$p_0 = \{(0,0)\}, \quad p_1 = \{(0,0),(1,1)\}, \quad \text{and} \quad p_2 = \{(0,0),(1,2)\}.$$

We consider the following $\mathbb{P}$-names

$$\tau_0 = \{(\varnothing, p_0)\}, \quad \tau_1 = \{(\varnothing, p_1)\}, \quad \tau_2 = \{(\tau_0,p_0),(\varnothing, p_1)\}, \quad \tau_3 = \{(\tau_2, p_2)\}.$$

Let G be any $\mathbb{P}$-generic filter over M with $p_1 \in G$.
Compute $(\tau_0)_G$, $(\tau_1)_G$, $(\tau_2)_G$, and $(\tau_3)_G$.

Solution:

- $(\tau_0)_G = \{\varnothing\}$
- $(\tau_1)_G = \{\varnothing\}$
- $(\tau_2)_G = \{\varnothing,\{\varnothing\}\}$
- $(\tau_3)_G = \varnothing$

Does $p_1 \Vdash \tau_0 \in \tau_1$ hold?

☐ yes ☒ no ☐ it depends

Does $p_1 \Vdash (\tau_1 \in \tau_2 \wedge \tau_3 \in \tau_2)$ hold?

☒ yes ☐ no ☐ it depends

8 pt **Question 3.5:** Let G be any $\mathbb{P}$-generic filter over M and

$$\sigma = \{(\check{n}, p) \mid p \in \mathbb{P} \wedge n \in \omega \wedge p(n) = 1\}.$$

Does $(\sigma)_G$ belong to M?

☐ yes ☒ no ☐ it depends

4 pt **Question 3.6:** Show that for all $n \in \omega$ the set

$$D_n = \{p \in \mathbb{P} \mid \exists k \in \omega\ (k > n \wedge p(k) > k)\}$$

is dense in $\mathbb{P}$ and that D_n belongs to M.

Solution:
Fix $n \in \omega$ and let $q \in \mathbb{P}$, since $\mathrm{Dom}(q) \subseteq \omega$ is finite, there exists $k > n$ with $k \notin \mathrm{Dom}(q)$. Therefore $p = q \cup \{(k, k+1)\}$ satisfies $p \leqslant q$ and $p \in D_n$. So D_n is dense. Moreover since $\mathbb{P}$ belongs to M, M is transitive and a model of ZFC, by comprehension $D_n \in M$.

/29 points