Worksheet #6

Algebra V - Galois theory

October 18, 2024

Problem 1. *Let F be a finite field. Prove that $\mathrm{Aut}_F(\bar{F})$ is an Abelian group and that every element of this group has infinite order.*

Problem 2. *Let $G \neq \{1\}$ be a finite Abelian group.*

(i) Prove that there exists a Galois extension $L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \simeq G$.

(ii) Let $K/\mathbb{Q}$ be a finite extension. Prove that there exist infinitely many Galois extensions L/K such that $\mathrm{Gal}(L/K) \simeq G$.

Problem 3. *Let K be a field such that every finite extension L/K is cyclic. Show that there exists $\sigma \in \mathrm{Aut}_K(\bar{K})$ such that $K = \bar{K}^{\sigma} = \{x \in \bar{K}\,;\, \sigma(x) = x\}$.*

Problem 4. *Let L/K be a Galois extension with Galois group G. Define the so-called trace map $T : L \to L$ (cf. Problem 8, worksheet 3) by*

$$\ell \mapsto \sum_{g \in G} g(\ell)$$

(i) Prove that $\mathrm{im}(T) \subset K$ and T is additive.

(ii) Assume that $[L : K] = |G| = n$ and $G =< \sigma >$ is cyclic and prove that $\ker(T) = \mathrm{im}(\sigma - \mathrm{id}_L)$.

Problem 5. *Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic extension of $\mathbb{Q}$.*

Problem 6. *Let $f \in \mathbb{Q}[x]$ be a cubic polynomial such that $\mathrm{Gal}(SF_{\mathbb{Q}}(f)/\mathbb{Q}) = C_3$. Prove that all the roots of f are real.*

Problem 7. *Let L/K be a cyclic extension of degree three of fields of characteristic not equal to two. Write $G = \mathrm{Gal}(L/K) = \{1, \sigma, \sigma^2\}$ and for $\beta \in L$ define (cf. Problem 8, worksheet 3)*

$$N_{L/K}(\beta) = \prod_{g \in G} g(\beta).$$

Fix $\beta \in L^{\times} \backslash K$ not a square with $N_{L/K}(\beta) = 1$ and let α be a root of $x^2 - \beta$.

(i) Prove that the normal closure of $L(\alpha)$ over K is a Galois extension of K with Galois group A_4.

(ii) Let now $L = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ and $K = \mathbb{Q}$, where ζ_7 is a primitive 7-th root of unity. Then, L/K is a cubic cyclic extension. Check that $\beta = \zeta_7 + \zeta_7^{-1}$ has the properties outlined above.