Solutions to Worksheet #1

Algebra V - Galois theory

Fall 2024

Solution 1. *Here is the basic idea. We first observe that* 0 *and* 1 *are given and that we can easily construct* -1 *as follows. Let* $L(0,1)$ *be the line through* 0 *and* 1*, and let* $C(0;1)$ *be the circle centered at zero with radius* 1*. Then* -1 *can be obtained as a point in the intersection* $C(0;1) \cap L(0,1)$*. Now, once you have constructed* n *and* $-n$*,* $n \in \mathbb{N}$*, we can construct* $n+1$ *(resp.* $-(n+1)$*) as a point in the intersection of the line* $L(0,1)$ *and the circle* $C(n;1)$ *(resp.* $C(-n;1)$*) centered at* n *(resp.* $-n$*) with radius* 1*.*

Solution 2. *We first show that* F *is a field. Since we know how to construct the integers (problem 1) and we assume Claim 2, it suffices to show that if* α *and* β *are constructible, then* $\alpha+\beta$ *is also constructible. To do this, we can argue as follows. If* $\alpha \cdot \beta = 0$*, there is nothing to do. Otherwise, we can draw the line* $L = L(0;\beta)$ *through* 0 *and* β*, and the parallel line through* α*, say* $\tilde{L}$*, and then draw the circle* $C = C(\alpha; |\beta|)$ *centered at* α *with radius* $|\beta|$*. By construction,* $\alpha \pm \beta \in \tilde{L} \cap C$*. Note that we also already know* $\mathbb{Q} \subset F$*. So, to complete the exercise, we need to show that we can construct numbers of the form* $a/b + c/di$*, where* $a, b, c, d \in \mathbb{Z}$*. But this can be constructed as a point in the intersection of the line* $x = a/b$ *and the circle centered at* a/b *with radius* $|c/d|$*.*

Solution 3. *First, note that we know how to construct* $\sqrt{2} = |1+i|$ *as the (positive) intersection of* $L(0,1)$ *(the real axis = x-axis) and the circle* $C(0; |1+i|)$*. Now, let* $\alpha \in F \cap \mathbb{R}_{>0}$ *(*$0 = \sqrt{0}$ *is given). Then* $\sqrt{1+\alpha^2} = |\alpha + i|$ *and* $1+\alpha$ *are also in* F*. Thus, the intersection points of the line* $x = \sqrt{1+\alpha^2}$ *and the circle centered at* 0 *with radius* $1+\alpha$ *are also in* F*. Since these points have the form*

$$\sqrt{1+\alpha^2} + i\beta,$$

where $\beta^2 = 2\alpha$*, we conclude that* $\sqrt{2\alpha}$ *is in* F*. But then* $\sqrt{\alpha}$ *is in* F*.*

To extend this to all $\alpha \in F$*, it is handy to use polar coordinates and write* $\alpha = (r,\theta)$*. Then, to show you can construct* $\pm\sqrt{\alpha} = \pm(\sqrt{r}, \theta/2)$*, it suffices to show you can bisect the angle* θ*.*

Solution 4. *Note that if we can prove the claim, then we conclude* $\alpha = \sqrt[3]{2}$ *is not constructible because* $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ *(the minimal polynomial of* α *is* $x^3 - 2$*). The claim is proved, for instance, in Milne's notes (Theorem 1.37 + Corollary 1.38).*