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Problem Set 7 Solutions

Exercise 1. Let D_{2n} be the dihedral group, the group of symmetries of a regular n-gon. This group has $2n$ elements.

(a) Describe all irreducible complex representations of D_n. Start with the 1-dimensional representations, then consider the complexifications of the symmetries of a regular n-gon, and use the sum of the squares formula to complete the classification. Consider cases of odd and even n.

(b) Use the character table to find the decompositions of the tensor products $V_i \otimes V_j$ into a direct sum of irreducible representations. (It is enough to consider the case where $\mathrm{dimV_i} > 1, \mathrm{dimV_j} > 1$).

Solution 1. (a) $D_n = \{r, s : r^n = 1, s^2 = 1, srs = r^{-1}\}$. If n is odd, we have $s^2 = 1$ and $(rs)^2 = 1$, while also $r^n = 1$. There are two one-dimensional representations: ρ_{11} where both r and s act by 1, and ρ_{12}, where r acts by 1 and s acts by -1. The two-dimensional representations are given by

$$\rho_{2,k}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho_{2,k}(r) = \begin{pmatrix} e^{\frac{2\pi i k}{n}} & 0 \\ 0 & e^{-\frac{2\pi i k}{n}} \end{pmatrix},$$

where $k = 1, \ldots \frac{n-1}{2}$ (we can consider k going up to $n-1$, but the representation with the rotation angle ϕ is isomorphic to the representation with the rotation by $-\phi$). Check by the "sum of squares" formula, that we have listed all irreducible representations: $2n = 1+1+\frac{n-1}{2}\cdot 4$. Another way to see that all the remaining representations have to be two dimensional is to recall that this group has $\frac{n+3}{2}$ conjugacy classes.
For even n, we have $s^2 = 1, (rs)^2 = 1$, and now $r^n = 1$ where n is even. In this case, there are 4 one-dimensional representations: both r and s can act by ± 1. The two-dimensional representations are given by the same formulas as above, where now $k = 1, \ldots \frac{n-2}{2}$, because the representation with the rotation by π splits into a direct sum of two one-dimensional representations. Check by the "sum of squares" formula: $2n = 1+1+1+1+\frac{n-2}{2}\cdot 4$, or recall that the number of conjugacy classes is $\frac{n+6}{2}$ for even n.

(b) Suppose that n is odd. The character of a two-dimensional representation of D_n $\chi_{2,k}(s) = 0$, $\chi_{2,k}(r) = 2\cos(\frac{2\pi k}{n})$. The conjugacy classes are 1, $\{sr^j\}_{0\le j\le n-1}$, and $\frac{n-1}{2}$ classes containing $\{r^m, r^{-m}\}$ for $m = 1 \ldots \frac{n-1}{2}$. The characters are given by the following formulas:

$$\chi_{2,k}(1) = 2, \quad \chi_{2,k}(sr^j) = 0, \quad \chi_{2,k}(r^m) = 2\cos\left(\frac{2\pi k m}{n}\right).$$

Then the character of the tensor product

$$\chi_{V_{2,k}\otimes V_{2,l}}(r^m) = \chi_{2,k}(r^m)\chi_{2,l}(r^m) = 4\cos\left(\frac{2\pi k m}{n}\right)\cos\left(\frac{2\pi l m}{n}\right) =$$

$$= 2\left(\cos\left(\frac{2\pi m(k-l)}{n}\right) + \cos\left(\frac{2\pi m(k+l)}{n}\right)\right).$$

The remaining characters are

$$\chi_{(V_{2,k}\otimes V_{2,l})}(1) = 4, \qquad \chi_{(V_{2,k}\otimes V_{2,l})}(sr^j) = 0$$

In this case, it is easier to see how the obtained character decomposes in the basis of irreducible characters, rather than compute inner products. We immediately observe that if $k \neq l$, and $k + l \neq n$, we have

$$\chi_{V_{2,k}\otimes V_{2,l}}(g) = \chi_{V_{2,k+l}}(g) + \chi_{V_{2,|k-l|}}(g)$$

for all $g \in D_n$. Therefore, in this case

$$V_{2,k}\otimes V_{2,l} = V_{2,k+l} \oplus V_{2,|k-l|}.$$

If $k = l$, we get

$$V_{2,k} \otimes V_{2,l} = V_{1,1} \oplus V_{1,-1} \oplus V_{2,k+l}.$$

Finally, if $k + l = n$, we get

$$V_{2,k} \otimes V_{2,l} = V_{1,1} \oplus V_{1,-1} \oplus V_{2,|k-l|}.$$

The case of even n can be solved similarly.

In fact, if we define $V_{2,0} = V_{2,n} = V_{1,1} \oplus V_{1,-1}$, a reducible representation for any n, and for n even, let $V_{2,n/2} = V_{-1,1} \oplus V_{-1,-1}$ be the sum of the two one dimensional representations where the action of r is by -1. Then, the formula

$$\chi_{V_{2,k} \otimes V_{2,l}}(g) = \chi_{V_{2,k+l}}(g) + \chi_{V_{2,|k-l|}}(g)$$

holds without restrictions, implying that

$$V_{2,k} \otimes V_{2,l} = V_{2,k+l} \oplus V_{2,|k-l|}$$

since two representations have the same characters iff they're isomorphic. From here all cases are obtained, although you have to take into account that some of the summands in the right hand side might be reducible.

Exercise 2. Use results in representation theory of finite groups over $\mathbb{C}$ to show that every group of order p^2, where p is a prime, is abelian.

Solution 2. The sum of squares formula implies

$$|G| = p^2 = \sum_{i=1}^{r} (\mathrm{dim}(\mathrm{V_i}))^2 = 1 + \sum_{\mathrm{i=2}}^{\mathrm{r}} (\mathrm{dim}(\mathrm{V_i}))^2.$$

The dimensions of irreducible representations divide the order of the group, therefore in this case $\mathrm{dimV_i} = \mathrm{p}$ or $\mathrm{dimV_i} = 1$. The former cannot hold because there is a trivial representation of dimension 1. Therefore, all irreducible representations of G are 1-dimensional. Then the number of conjugacy classes in G is equal to the order of G, and therefore the group is abelian.

Exercise 3. Let G be a group of invertible upper triangular 2×2 matrices with coefficients in $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$.

(a) Find the conjugacy classes of G.

(b) Find a normal subgroup $H \subset G$ such that G/H is abelian.

(c) Use (a), (b) and the "sum of squares" formula to find the dimensions of the irreducible complex representations of G.

(d) Use the orthogonality relations to compute the table of characters of G.

Solution 3. (a) Let

$$g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

be an element of G, where $a, b, c \in \mathbb{F}_3 = \{0, 1, -1\}$, $a \neq 0, c \neq 0$. Then $|G| = 12$. Conjugation with another upper triangular matrix does not change the values of a and c. Therefore, we have at least 4 conjugacy classes, with (a, c) given by $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$. Next, the identity matrix and its negative are central, and therefore constitute their own conjugacy classes. It is easy to see that all other elements with fixed (a, c) are conjugate. Finally, we have 6 conjugacy classes:

$$c_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 1 & \neq 0 \\ 0 & 1 \end{pmatrix}, \quad c_4 = \begin{pmatrix} -1 & \neq 0 \\ 0 & -1 \end{pmatrix}$$

$$c_5 = \begin{pmatrix} 1 & * \\ 0 & -1 \end{pmatrix}, \quad c_6 = \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix}$$

(b) The abelian quotient is likely to contain the diagonal matrices. Indeed, the subgroup

$$H = \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \right\} \subset G$$

is normal in G of order 3, and the quotient

$$G/H \simeq \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \qquad a, c \neq 0 \right\}$$

is abelian. It is easy to see that $G/H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(c) Each irreducible representation of G/H can be lifted to an irreducible representation of G. The abelian group $G/H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ has 4 inequivalent irreducible 1-dimensional representations, where the matrices with $(a, c) = (1, -1)$ and $(a, c) = (-1, 1)$ act each by 1 or -1. We know that there are 6 conjugacy classes, so that we must have 6 inequivalent irreducible representations. The "sum of squares" formula gives

$$12 = 1 + 1 + 1 + 1 + d_5^2 + d_6^2.$$

The only solution is $d_5 = d_6 = 2$.

(d) Using (c), we can fill out the first 4 rows of the character table, and we know the dimensions of the remaining two representations:

	c_1	c_2	c_3	c_4	c_5	c_6
$\lvert C_g \rvert$	1	1	2	2	3	3
χ_{11}	1	1	1	1	1	1
χ_{12}	1	-1	1	-1	-1	1
χ_{13}	1	-1	1	-1	1	-1
χ_{14}	1	1	1	1	-1	-1
χ_{21}	2					
χ_{22}	2					

Using the identity $\sum_V \chi_V(g)\overline{\chi_V(g)} = |Z_g| = |G|/|C_g|$, we immediately obtain that $\chi_{21}(c_5, c_6) = \chi_{22}(c_5, c_6) = 0$. The element $s \in c_2$ is central, with $s^2 = 1$, and therefore $\chi_{21}(s)$ and $\chi_{22}(s)$ can only be equal to ± 2. Using the result in 4(b), we know that V_{21} and $V_{21} \otimes V_{12}$ are both irreducible, and they have different value of character on c_2. Therefore, we can assume $\chi_{21}(c_2) = 2$ and $\chi_{22}(c_2) = -2$. By orthogonality $(\chi_{11}, \chi_{21}) = 0$, $(\chi_{12}, \chi_{22}) = 0$, we have $\chi_{21}(c_3) = \chi_{21}(c_4) = -1$. Then using $\chi_{22} = \chi_{21}\chi_{12}$, we obtain $\chi_{22}(c_3) = -1$, $\chi_{22}(c_4) = 1$. Here is the complete character table:

	c_1	c_2	c_3	c_4	c_5	c_6
$\lvert C_g \rvert$	1	1	2	2	3	3
χ_{11}	1	1	1	1	1	1
χ_{12}	1	-1	1	-1	-1	1
χ_{13}	1	-1	1	-1	1	-1
χ_{14}	1	1	1	1	-1	-1
χ_{21}	2	2	-1	-1	0	0
χ_{22}	2	-2	-1	1	0	0