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Problem Set 5 Solutions

Exercise 1. Consider the representation of the group $U(1) = \{e^{i\theta}, \theta \in [0, 2\pi[\} \subset \mathbb{C}$ in $V = \mathbb{C}^2$ given by the rotation matrix

$$\rho(e^{i\theta}) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Decompose V into a direct sum of two irreducible unitary complex representations of $U(1)$.

Solution 1. By multiplying $\rho(e^{i\theta})\rho(e^{i\phi})$ we check that the give map defines a representation of $U(1)$. The characteristic polynomial of the matrix is $\lambda^2 - 2\lambda\cos(\theta) + 1$, and the eigenvalues are $\lambda_{1,2} = \cos(\theta) \pm i\sin(\theta) = e^{\pm i\theta}$. It is easy to check that the eigenvectors for $e^{\pm i\theta}$ are

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

respectively. So we have $V = \mathbb{C}v_1 \oplus \mathbb{C}v_2$ as a complex representation of $U(1)$. Note that V is irreducible over the real numbers.

Exercise 2. Let A be an associative algebra over a field k. For a representation V of A, consider the vector space $\mathrm{End}_A(V)$ of endomorphisms of the representation V (linear maps $V \to V$ commuting with the action of A in V). Let V be the left regular representation, $V = A$. Show that $\mathrm{End}_A(A)$ is an associative algebra isomorphic to A^{op}, the algebra A with the opposite multiplication.

Solution 2. Let $\phi : A \to A$ be an endomorphism of the left regular representation of A. Then $b\phi(a) = \phi(ba)$ for any $a, b \in A$ by the definition of a homomorphism. In particular, $b\phi(1) = \phi(b)$ for any $b \in A$. Therefore, a homomorphism $\phi : A \to A$ is uniquely determined by its value $\phi(1)$. Define a map $F : \mathrm{End}_A(A) \to A$ by setting $F(\phi) = \phi(1) \in A$. We now check that the map is an isomorphism of vector spaces: $F(\phi) = \phi(1) = 0$ implies that $\phi(b) = b\phi(1) = 0$ for all $b \in A$, therefore F is injective. For any $a \in A$, the assignment $\phi(1) = a$ defines a homomorphism by setting $\phi(b) = \phi(b \cdot 1) = b\phi(1) = ba$, then $\phi(cb) = cba = c\phi(b)$ for any $c, b, a \in A$, which shows that ϕ is indeed an endomorphism of representations (here we use that ϕ is a homomorphism of the *left* regular representation). Therefore the map F is surjective. Now consider the multiplicative structure given by composition in $\mathrm{End}_A(A)$. Let $\phi_1(1) = a, \phi_2(1) = b$. We have

$$\phi_1 \circ \phi_2(x) = \phi_1(x\phi_2(1)) = \phi_1(xb) = xb\phi_1(1) = xba.$$

Therefore, $\phi_1 \circ \phi_2 \mapsto ba$ and we have $F : \mathrm{End}_A(A) \simeq A^{\mathrm{op}}$.

Exercise 3. Let $A = \mathrm{Mat}_d(k)$ for a field k. Prove that the algebra A is semisimple, meaning that any finite dimensional representation of A over k is isomorphic to a direct sum of irreducible representations.

Solution 3. Let $E_{ij} \in A$ denote the matrix with 1 in position (i, j) and zeros everywhere else. Notice that $\sum_{i=1}^d E_{ii} = \mathrm{Id}$. Let V be a finite dimensional representation of A and $v \in V$ any vector. Then $v = \mathrm{Id}\, v = \sum_{i=1}^d E_{ii}v$. This holds uniquely for any $v \in V$, and therefore we get a direct sum decomposition

$$V = \bigoplus_{i=1}^{d} E_{ii}V.$$

Define linear maps $\phi_i : E_{11}V \to E_{ii}V$ by $\phi_i(v) = E_{i1}v$. Then if $w \in E_{ii}V$, there exist a unique $v \in E_{11}V$ such that $w = E_{i1}v$, namely left-multiplying by E_{1i} gives $v = E_{1i}w$. So $\phi_i : E_{11}V \to E_{ii}V$ is a bijection.

Let $v \in E_{11}V$ be a nonzero vector, and let $L(v)$ be the linear span of the vectors $\{E_{11}v, E_{21}V, \ldots E_{d1}v\}$. Since we have a direct sum decomposition $V = \bigoplus_{i=1}^d E_{ii}V$ and each of the operators $\phi_i : E_{11}V \to E_{ii}V$ is a bijection, it follows that the obtained vectors are nonzero and linearly independent. This implies that they span a k -vector space $L(v)$ of dimension d. We will show next that this vector space is a representation of A isomorphic to the irreducible representation k^d. Let $f : L(v) \to k^d$ be given by $f(E_{i1}v = e_i$, where $\{e_i\}_{i=1}^d$ is a standard basis of k^d. Let $a \in A$, then $a = \sum_{jk} c_{jk} E_{jk}$. We have

$$aE_{i1}v = \sum_{jk} c_{cj} E_{jk} E_{i1} v = \sum_j c_{ji} E_{j1} v \in L(v).$$

This shows that $L(v)$ is a subrepresentation of V. Now we check that $f : L(v) \to k^d$ is an isomorphism.

$$f(aE_{i1}v) = f(\sum_j c_{ji} E_{j1} v) = \sum_j c_{ji} e_j,$$

on the other hand

$$af(E_{i1}v) = ae_i = \sum_{jk} c_{jk} E_{jk} e_i = \sum_j c_{ji} e_j.$$

By construction f is an isomorphism of vector spaces, and so we have that $L(v) \simeq k^d$ as representations of A.

Fix a basis $\{v_i\}_{i=1}^k$ in $E_{11}V$. Since $V = \bigoplus_{i=1}^n E_{ii}V$ is a direct sum decomposition and the components are isomorphic to $E_{11}V$ by maps ϕ_i, we have a unique decomposition

$$v = \sum_{j=1}^k \sum_{i=1}^d c_{ij} E_{i1} v_j.$$

Therefore we have the decomposition

$$V = L(v_1) \oplus L(v_2) \oplus \ldots \oplus L(v_k)$$

into a direct sum of irreducible representations of $A = \mathrm{Mat}_d(k)$ with each irreducible component isomorphic to k^d.

Exercise 4. Let A be a finite dimensional algebra, and $\mathrm{Rad}(A)$ the set of all elements of A that act by 0 in all irreducible representations of A.

(a) Show that $\mathrm{Rad}(A)$ is a two-sided ideal in A.

(b) Let $I \subset A$ be a two-sided nilpotent ideal, meaning that there exist $n \in \mathbb{N}$ such that $x^n = 0$ for all $x \in I$. Show that $I \subset \mathrm{Rad}(A)$.

Solution 4. (a) Let $a \in A$ and $x \in \mathrm{Rad}(A)$. Then by definition $\rho(x)V = 0$ for any irreducible representation V of A. We have $\rho(ax)V = \rho(a)\rho(x)V = \rho(a) \cdot 0 = 0$, so $\mathrm{Rad}(A)$ ia a left ideal.
Let $v \in V$. Then $\rho(xa)v = \rho(x)\rho(a)v = \rho(x)w = 0$, where $w \in V$, since $\rho(x)V = 0$. Therefore $\mathrm{Rad}(A)$ is also a right ideal, and so it is a two-sided ideal in A.

(b) Let $I \subset A$ be a nilpotent two-sided ideal and let V be an irreducible representation of A. Let $v \in V$, and consider the subspace $\rho(I)v \subset V$. It is a subrepresentation of V, because $aI \subset I$ for any $a \in A$. If $\rho(I)v \neq 0$, then $\rho(I)v = V$ since V is irreducible. But then there is $x \in I$ such that $\rho(x)v = v$, then $\rho(x)^n \neq 0$ for any natural n. Since $x \in I$ has the property $x^n = 0$ for some n, we arrive at a contradiction. Therefore all elements of the ideal I act by zero in V, and therefore $I \subset \mathrm{Rad}(A)$.

Exercise 5. Recall that the character of a finite dimensional representation V of an algebra A over a field k is defined as $\chi_V(a) = \mathrm{Tr}_V \rho(a)$. Show that if V is a finite dimensional representation of A, and $W \subset V$ a subrepresentation, then the character $\chi_V = \chi_W + \chi_{V/W}$.

Solution 5. Let $\{w_1, ..., w_m\}$ be a basis of W and complete it to a basis $B = \{w_1, ..., w_m, u_1, ..., u_n\}$ of V. Similar to exercise 3 in problem set 1, the matrix of $\rho(a)$ in B is in the following block form:

$$\left(\begin{array}{c|c} M & * \\ \hline 0 & N \end{array}\right)$$

where M is the matrix representing $\rho(a)|_W = \rho_W(a)$. Moreover, the matrix N coincides with the matrix of $\rho_{V/W}(a)$ in the basis $B' = \{u_1 + W, ..., u_n + W\}$ of V/W. Indeed, let $u \in \mathrm{Span}(u_1, ..., u_n)$ and write $u + W \in V/W$ as a vector $x \in k^n$ in the basis B'. We have

$$\left(\begin{array}{c|c} M & * \\ \hline 0 & N \end{array}\right) \left(\begin{array}{c} 0 \\ x \end{array}\right) = \left(\begin{array}{c} * \\ Nx \end{array}\right).$$

This shows that $\rho_{V/W}(a)(u + W)$ is represented by Nx in the basis B'. Since the trace of an operator in $\mathrm{End}(V)$ is independent of the choice of basis, we have

$$\chi_V(a) = \mathrm{Tr}(M) + \mathrm{Tr}(N) = \chi_W(a) + \chi_{V/W}(a)$$

for all $a \in A$.

Exercise 6. (a) Construct all possible representations of the cyclic group $C_3 = \langle t \mid t^3 = 1 \rangle$ in V, where V is a two-dimensional vector space over the field $\mathbb{F}_2$. Decompose the obtained representations into a direct sum of irreducibles.

(b) For the obtained irreducible representations, consider the intertwiners $\phi : V \to V$ that commute with the action of the group C_3. Show how the Schur's lemma fails in the case of the field $\mathbb{F}_2$, which is not algebraically closed.

Solution 6. (a) Let $\rho : C_3 \to GL(2, \mathbb{F}_2)$. We need to find all 2×2 matrices M over $\mathbb{F}_2$ such that $M^3 = \mathrm{Id}$. It is easy to see that matrices with exactly one or exactly two nonzero elements (1's), or all 4 nonzero elements do not satisfy this requirement. Two matrices M and M^2 with 3 nonzero entries satisfy:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad M^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad M^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that the representations $\rho(t) = M$ and $\rho'(t) = M^2$ are isomorphic by the change of basis performed by the matrix conjugation with the matrix S:

$$SMS^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = M^2.$$

So we will consider only the representation $\rho(t) = M$ in the 2-dimensional vector space V over $\mathbb{F}_2$. It is irreducible over $\mathbb{F}_2$. Indeed, the characteristic equation of the matrix M is $\lambda^2 + \lambda + 1$ which has no solutions in $\mathbb{F}_2$, so there are no nontrivial invariant subspaces in V. (It is also easy to see by explicitly applying the matrix M to vectors in V that no one-dimensional subspace of V is invariant under the action of C^3). Finally, up to isomorphism we have only one 2-dimensional representation of C_3 over $\mathbb{F}_2$ and it is irreducible.
Remark: Here we have an example where the consequence of the Schur's lemma over an algebraically closed field, namely that any irreducible representation of an abelian group is one-dimensional, fails over a non-algebraically closed field $\mathbb{F}_2$. Notice also that Maschke's theorem works in this case: the characteristic 2 does not divide the order of the group C_3. So we can be sure that any representation of C_3 over $\mathbb{F}_2$ is completely reducible.

(b) Note that the matrix M itself can serve as a homomorphism $M : V \to V$ that intertwines the action of C_3 in V, because $\rho(t) = M$ commutes with itself. A quick computation with 2×2 matrices over $\mathbb{F}_2$ shows that the only linear maps $V \to V$ commuting with M are the powers of M. Therefore, the ring of endomorphisms of V over C_3, or the ring of intertwiners, is generated by M and given by

$$\mathbb{F}_2[M]/\langle M^2 + M + 1 \rangle.$$

One can conclude, recalling some properties of polynomial rings over finite fields, that it is an extension of $\mathbb{F}_2$ isomorphic to $\mathbb{F}_4$.
Here we have another evidence that Schur's lemma for algebraically closed fields fails in general over non-algebraically closed fields: we obtained two nontrivial intertwiners M and M^2 of an irreducible representation over $\mathbb{F}_2$ that are not a scalar operators.