December 10, 2024

Problem Set 12 Solutions

Exercise 1. Recall that an irreducible Specht module V_λ for S_n is determined by a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_p)$, such that $\sum_{i=1}^{p} \lambda_i = n$. It can be defined by $V(\lambda) = \mathbb{C}[S_n]c_\lambda$, where $c_\lambda = a_\lambda b_\lambda$ with

$$a_\lambda = \frac{1}{|P_\lambda|} \sum_{g \in P_\lambda} g; \quad b_\lambda = \frac{1}{|Q_\lambda|} \sum_{g \in Q_\lambda} (-1)^g g.$$

Here the subgroups $P_\lambda \in S_n$ and $Q_\lambda \in S_n$ are the stabilizers respectively of rows and columns of a Young tableau T_λ of shape λ.

(a) In class we showed that $c_\lambda^2 = x(\lambda)c_\lambda$, where $x(\lambda) \in \mathbb{Q}$ is a coefficient. Find $x(\lambda)$. *Hint:* Consider the action of c_λ in the *right* regular representation of S_n.

(b) Let $C = \sum_{i<j} (ij) \in \mathbb{C}[S_n]$ be the sum of all transpositions. Show that C acts on the Specht module V_λ by multiplication by the scalar $z(\lambda) = \sum_{j=1}^{p} \sum_{i=1}^{\lambda_j} (i-j)$. (The integer $z(\lambda)$ is called the *content* of the Young diagram of shape λ.)

Solution 1. (a) The element c_λ is proportional to an idempotent because $c_\lambda^2 = a_\lambda(b_\lambda a_\lambda)b_\lambda = x(\lambda)c_\lambda$. If κ is an eigenvalue of c_λ in any representation, then $\kappa^2 = x(\lambda)\kappa$, and $\kappa = x(\lambda)$ or $\kappa = 0$. To find the nonzero eigenvalue of c_λ, notice that since $P_\lambda \cap Q_\lambda = \{1\}$, the coefficient of 1 in c_λ is $\frac{1}{|P_\lambda||Q_\lambda|}$. Taking the trace of c_λ in the *right* regular representation, we get

$$\mathrm{tr}(\rho_\mathrm{r}(\mathrm{c}_\lambda)) = \frac{\mathrm{n!}}{|\mathrm{P}_\lambda||\mathrm{Q}_\lambda|}.$$

On the other hand, we have $\dim\mathbb{C}[S_n]c_\lambda = \dim V_\lambda$. Now we want to find the trace of the right regular action of c_λ on V_λ. It is given by $\rho_r(c_\lambda)(vc_\lambda) = vc_\lambda^2 = \kappa v c_\lambda$ for any $v \in \mathbb{C}[S_n]$. Therefore, We have

$$\mathrm{tr}(\rho_\mathrm{r}(\mathrm{c}_\lambda)) = \dim \mathrm{V}_\lambda \kappa.$$

Therefore,

$$\kappa = x(\lambda) = \frac{n!}{|P_\lambda||Q_\lambda|\dim(\mathrm{V}_\lambda)}.$$

Using the hook length formula for $\dim(\mathrm{V}_\lambda)$, we get

$$x(\lambda) = \frac{\prod_{i,j \in \lambda} h(i,j)}{\prod_j \lambda_j \prod_i \lambda_i^*},$$

where λ_j is the number of boxes in the jth row of the Young diagram of λ, λ_i^* – the number of boxes in the ith column, and $h(i,j)$ – the length of the hook starting at the square (i,j).

(b) First, notice that C is central: $gCg^{-1} = \sum_{i<j} g(i,j)g^{-1} = \sum_{i<j}(g(i), g(j)) = C$. Therefore, C acts on V_λ by a scalar, and we have $C \cdot c_\lambda = z(\lambda)c_\lambda$. Then $C \cdot c_\lambda = Ca_\lambda b_\lambda = a_\lambda C b_\lambda$. We have:

$$a_\lambda(i,j)b_\lambda = \begin{cases} a_\lambda b_\lambda = c_\lambda, & \text{if } (i,j) \in P_\lambda \\ -a_\lambda b_\lambda = -c_\lambda, & \text{if } (i,j) \in Q_\lambda \end{cases}$$

If $(i,j) \notin P_\lambda \cup Q_\lambda$, then i and j are in different rows and different columns of the Young diagram. Suppose i is in the longest of the two rows. Then there is a transposition in P_λ, (i,s), that moves i to the column of j, and $(i,j)(i,s)(i,j) = (j,s) \in Q_\lambda$. Then, similarly to the proof we did in class, we have

$$a_\lambda(i,j)b_\lambda = a_\lambda(i,s)(i,j)b_\lambda = a_\lambda(i,j)(j,s)b_\lambda = -a_\lambda(i,j)b_\lambda = 0.$$

So the value of $z(\lambda)$ is the number of all transpositions in P_λ minus the number of all transpositions in Q_λ. This number is

$$z(\lambda) = \sum_{j=1}^{p} \sum_{i=1}^{\lambda_j} (i-1) - \sum_{j=1}^{p} \lambda_j \cdot (j-1) = \sum_{j=1}^{p} \sum_{i=1}^{\lambda_j} (i-1) - \sum_{j=1}^{p} \sum_{i=1}^{\lambda_j} (j-1) = \sum_{j=1}^{p} \sum_{i=1}^{\lambda_j} (i-j).$$

Exercise 2. Let $G = SL(2, \mathbb{F}_q)$ be the group of 2×2 matrices of determinant 1 with coefficients in the field of q elements $\mathbb{F}_q$ ($q \geq 3$ a prime). Consider the 2-dimensional $\mathbb{F}_q$-vector space V with the basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

(a) Find the order of G.

(b) Show that G acts transitively on $V \setminus \{(0, 0)\}$.

(c) Find the stabilizer $N \subset G$ of e_1.

(d) Show that the representation (F, ρ) of G in the complex vector space F of functions $f : V \setminus \{(0, 0)\} \to \mathbb{C}$ given by

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(x, y) = f(dx - by, -cx + ay)$$

is isomorphic to the induced representation of G from the trivial representation of N.

(e) Use Frobenius reciprocity to deduce that ρ is not irreducible.

Solution 2. (a) The condition $ad - bc = 1$ means that if $a \neq 0$, then d is uniquely determined with any values of b and c, and if $a = 0$, then b and c have to be nonzero. In the first case, we have $(q-1)q^2$ possibilities, and in the second, $q(q-1)$, q choices for d and $(q-1)$ for b. Totally, the order of G is $(q-1)q^2 + q(q-1) = (q-1)(q^2+q) = q(q^2-1)$.

Another way to look at it is to first compute the order of the group of invertible matrices $GL(2, \mathbb{F}_q)$. To be invertible, the columns have to be linearly independent. So we have $q^2 - 1$ choices for the first column (it has to be nonzero) and $q^2 - q$ choices for the second column that cannot be a multiple of the first. Then the order of $GL(2, \mathbb{F}_q)$ is $(q^2 - 1)(q^2 - 2)$. Now consider the group homomorphism $\det : GL(2, \mathbb{F}_q) \to (\mathbb{F}_q)^*$. The kernel is by definition isomorphic to $SL(2, \mathbb{F}_q)$. Since the order of the image is $(q - 1)$, we get the order of $SL(2, \mathbb{F}_q)$ equal to $(q^2 - 1)(q^2 - q)/(q - 1) = q(q^2 - 1)$ as before.

(b) This amounts to checking that any vector (x, y) can be transformed into, say, $(0, 1)$ by the action of G. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

With the condition $ad - bc = 1$, if $y \neq 0$, then $a = y, b = -x, d = \frac{1+bc}{a}$. If $y = 0$, set $a = 0, b = -x$, $c = \frac{1}{x}$, and d is arbitrary. The inverse matrix then will map $(0, 1)$ to any given vector (x, y). Therefore, the action of G on $V \setminus \{0\}$ is transitive.

(c) By direct computation, the stabilizer of $e_1 = (1, 0)$ is

$$N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

where $b \in \mathbb{F}_q$ is arbitrary. Check by the orbit-stabilizer theorem, knowing from (b) that the orbit of any vector is $V \setminus \{0\}$:

$$|N| \cdot |\text{Orb}(e_1)| = |N| \cdot |V \setminus \{0\}| = q \cdot (q^2 - 1)) = q(q^2 - 1) = |G|.$$

The order of $V \setminus \{0\}$ is given by all 2-vectors except the zero vector.

(d) Let F denote the given representation of G. Since G acts transitively on $V \setminus \{0\}$, we can represent obtain any vector $(x, y)^T$ as the result of action of an element $g \in G$ on e_1: $(x, y)^T = g^{-1}e_1$. Let V_0 be the trivial representation of N and consider the map $\Phi : \text{Ind}_N^G V_0 \to F$ defined by

$$\Phi(f)(x, y) = \Phi(f)(g^{-1}e_1) = f(g).$$

Then it is a G- homomorphism. Let $t \in G$, then

$$\Phi(tf)(x, y) = \Phi(tf)(g^{-1}e_1) = tf(g) = f(gt) = \Phi(f)((gt)^{-1}e_1) = \Phi(f)(t^{-1}g^{-1}e_1) =$$

$$= t\Phi(f)(g^{-1}e_1) = t\Phi(f)(x, y).$$

Here we used that the action of $t \in G$ of a function $\Phi(f) \in F$ is by $t\Phi(f)(x, y) = \Phi(f)(t^{-1}(x, y))$.

If $\Phi(f)(V \setminus \{0\}) = 0$, then $f(G) = 0$, so the kernel of Φ is trivial. Characteristic functions of the elements in $V \setminus \{0\}$ form a basis of F. The representation $\text{Ind}_N^G V_0$ has dimension $|G|/|N|$, which equals to $|\text{Orb}(e_1)| = |V \setminus \{0\}|$ by (c). Therefore, Φ is an isomorphism of representations of G.

(e) Let W_0 be the trivial representation of G. Frobenius reciprocity shows that the multiplicity of the trivial representation of G in Ind_N^G is 1:

$$\mathrm{Hom}_G(W_0, \mathrm{Ind}_N^G V_0) = \mathrm{Hom}_N(\mathrm{Res}_N^G(W_0), V_0) = \mathrm{Hom}_N(V_0, V_0) = 1.$$

This is a general result: an induction from a trivial representation of a subgroup always contains the trivial representation of the group as a subrepresentation. In this particular case, the trivial subrepresentation corresponds to the subspace of constant functions on $V \setminus \{0\}$.

Exercise 3. Let $K \subset G$ be a subgroup, and $\mathbb{C}_\chi$ a one-dimensional representation of K with character $\chi : K \to \mathbb{C}^\star$. Consider the central idempotent corresponding to χ:

$$e_\chi = \frac{1}{|K|} \sum_{g \in K} \chi(g)^{-1} g \quad \in \mathbb{C}[K].$$

Show that the induced representation $\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}} \mathbb{C}_\chi$ is naturally isomorphic to $\mathbb{C}[G]e_\chi$, with the action of G in $\mathbb{C}[G]e_\chi$ by left multiplication.

Solution 3. Let $k \in K$, and $\{x_i\} \in G, i = 1, \ldots, |G/K|$ be the representatives of the left cosets G/K. Then

$$ke_\chi = \frac{1}{|K|} \sum_{g \in K} \chi(g)^{-1} kg = \frac{1}{|K|} \sum_{t \in K} \chi(k^{-1}t)^{-1} t = \chi(k) e_\chi.$$

Any element of G can be written uniquely as $g = x_i k$ for some $x_i \in G/K$ and some $k \in K$. Then

$$\mathbb{C}[x_i k e_\chi] = \mathbb{C}[\chi(k) x_i e_\chi] = \mathbb{C}[x_i e_\chi].$$

The elements $\{x_i e_\chi\}, i = 1, \ldots |G/K|$ form a basis in $\mathbb{C}[G]e_\chi$. Similarly, the elements $\{x_i \otimes_{\mathbb{C}[K]} \mathbb{C}_\chi\}, i = 1, \ldots |G/K|$ form a basis in

$$\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}} \mathbb{C}_\chi \simeq \mathbb{C}[\mathrm{G}] \otimes_{\mathbb{C}[\mathrm{K}]} \mathbb{C}_\chi.$$

Then map $\Phi : \mathbb{C}[G] \otimes_{\mathbb{C}[K]} \mathbb{C}_\chi \to \mathbb{C}[G]e_\chi$ given by $\Phi(g \otimes_{\mathbb{C}[K]} \mathbb{C}_\chi) = ge_\chi$ for any $g \in G$ sends the basis to the basis and commutes with the action of G: if $g_1 g = x_j k$ for some x_j and some $k \in K$, then

$$\Phi(g_1 g \otimes_{\mathbb{C}[K]} \mathbb{C}_\chi) = \Phi(x_j k \otimes_{\mathbb{C}[K]} \mathbb{C}_\chi) = \chi(k)\Phi(x_j \otimes_{\mathbb{C}[K]} \mathbb{C}_\chi) = \chi(k) x_j e_\chi =$$

$$= x_j k e_\chi = g_1 g e_\chi = g_1 \Phi(g \otimes_{\mathbb{C}[K]} \mathbb{C}_\chi).$$