Exercise 1. We apply the Euler-Maclaurin formula as hinted:

$$\sum_{2\le n\le x}\frac{1}{n\log n}=\int_{3/2}^{x}\frac{1}{t\log t}\ \mathrm{d}t-\int_{3/2}^{x}\frac{\log t+1}{t^2\log^2 t}\psi(t)\ \mathrm{d}t-\frac{1}{x\log x}\psi(x),$$

using $\psi(3/2)=0$. The integral

$$C=\int_2^\infty\frac{\log t+1}{t^2\log^2 t}\ \mathrm{d}t$$

converges and we can use this information to write

$$\begin{aligned}\left|\int_2^x\frac{\log t+1}{t^2\log^2 t}\psi(t)\ \mathrm{d}t\right|&\le\int_2^x\frac{\log t+1}{t^2\log^2 t}\ \mathrm{d}t\\&=-\int_x^\infty\frac{\log t+1}{t^2\log^2 t}\ \mathrm{d}t+C=C-\frac{1}{x\log x}.\end{aligned}$$

Next notice that

$$\int_2^x\frac{1}{t\log t}\ \mathrm{d}t=\log\log x-\log\log 2.$$

Also we have

$$|\psi(x)\frac{1}{x\log x}|=O(\frac{1}{x\log x}).$$

Summarizing, setting $C_1=C-\log\log 2$ we get

$$\sum_{2\le n\le x}\frac{1}{n\log n}=\log\log x+C_1+O\left(\frac{1}{x\log x}\right).$$

Exercise 2. We have

$$\mu(n)^2=\begin{cases}1 & n\text{ square free}\\0 & \text{otherwise}\end{cases}$$

Recall also from last Exercise Sheet that $\mu(n)^2=\sum_{d^2|n}\mu(d)$. We have

$$\begin{aligned}Q_2(x)&=\sum_{n\le x}\mu(n)^2\\&=\sum_{n\le x}\sum_{d^2|n}\mu(d)\\&=\sum_{d\le\sqrt{x}}\sum_{n\le x/d^2}\mu(d)\\&=\sum_{d\le\sqrt{x}}\mu(d)\lfloor x/d^2\rfloor\\&=x\sum_{d\le\sqrt{x}}\frac{\mu(d)}{d^2}+O(\sqrt{x}).\end{aligned}$$

The summation $\sum_{d=1}^\infty\frac{\mu(d)}{d^2}$ is absolutely convergent, in particular a number. Also we have by triangular inequality

$$|\sum_{\sqrt{x}<d}\frac{\mu(d)}{d^2}|\le\int_{\sqrt{x}}^\infty\frac{1}{t^2}\ \mathrm{d}t=\frac{1}{\sqrt{x}}.$$

Hence

$$Q_2(x)=x\left(\sum_{d=1}^\infty\frac{\mu(d)}{d^2}\right)+O(\sqrt{x}).$$

Exercise 3. Write $F(x) = \sum_{n \le x} \log n$. We apply the summation by parts formula

$$\sum_{1<n\le x} \frac{\log n}{n} = \frac{F(x)}{x} + \int_1^x \frac{F(t)}{t^2}\,\mathrm{d}t.$$

Recall that $\log 1 = 0$. We can estimate $F(x)$ using monotonicity:

$$\begin{aligned} F(x) &= \sum_{2\le n\le x} \log n \\ &= \int_1^x \log t\,\mathrm{d}t + O(\log x) \\ &= x\log x - x + 1 + O(\log x) = x\log x - x + O(\log x). \end{aligned}$$

Now we estimate the integral trying not to abuse of the O-notation.[1] Write $E(t) = F(t) - t\log t + t$. Then by the above $E(t) = O(\log t)$ as $t \to \infty$. In particular there exists $t_0 > 0$ and $C > 0$ such that for all $t > t_0$ it holds that

$$|E(t)| \le C\log t.$$

Also since $F(t) \le t\log t$ (this is true for all $t \ge 1$) we have that $|E(t)| \le t_0$ for all $t < t_0$. In particular changing $C > 0$ above we can write $|E(t)| \le C\log t$ for all $t > 0$. Now we compute:

$$\begin{aligned} \int_1^x \frac{F(t)}{t^2}\,\mathrm{d}t &= \int_1^x \frac{t\log t}{t^2}\,\mathrm{d}t - \int_1^x \frac{1}{t}\,\mathrm{d}t + \int_1^x \frac{E(t)}{t^2}\,\mathrm{d}t \\ &= \int_1^x \frac{\log t}{t}\,\mathrm{d}t - \log x + \int_1^x \frac{E(t)}{t^2}\,\mathrm{d}t. \end{aligned}$$

We estimate

$$\begin{aligned} \left|\int_1^x \frac{E(t)}{t^2}\,\mathrm{d}t\right| &\le C\int_1^x \frac{\log t}{t^2}\,\mathrm{d}t \\ &= -\frac{C\log x}{x} + \int_C^x \frac{1}{t^2}\,\mathrm{d}t \\ &= -\frac{C\log x}{x} - \frac{C}{x} + C = C + O(\log x/x) \end{aligned}$$

And we also have

$$\int_1^x \frac{\log t}{t}\,\mathrm{d}t = \frac{1}{2}\log^2 x.$$

Putting all together we have

$$\begin{aligned} \sum_{n\ge 1} \frac{\log n}{n} &= \frac{F(x)}{x} + \int_1^x \frac{F(t)}{t^2}\,\mathrm{d}t \\ &= \log x - 1 + O(\log x/x) + \frac{1}{2}\log^2 x - \log x + C + O(\log x/x) \\ &= \frac{1}{2}\log^2 x + (C-1) + O(\log x/x). \end{aligned}$$

Exercise 4. Let $k \in \mathbb{N}$ and $p_1, \dots, p_k$ the smallest k'th primes. Order them so that $p_1 < \dots < p_k$. We have

$$\log(p_1 \cdots p_k) = \sum_{i=1}^k \log(p_i).$$

[1] In future we will often do it.

On the other hand $\tau(p_1 \cdots p_k) = 2^k$. Recall from the first exercise sheet that $\log(p_i) \le \log 2^{2^{i-1}} = 2^{i-1}\log 2$. Then

$$\sum_{i=1}^{k} \log p_i \le \log 2 \sum_{i=0}^{k-1} 2^i = (2^k - 1)\log 2 < 2^k$$

since $\log 2 < 1$. We have seen that

$$\tau(p_1 \cdots p_k) > \log(p_1 \cdots p_k).$$

In order to deal with $A > 0$ we need to exploit powers. Let $n = (p_1 \cdots p_k)^r$, with $r > 0$ to be chosen later, then

$$(\log n)^A = r^A (\sum_{i=1}^{k} \log p_i)^A \quad \text{and} \quad \tau(n) = (r+1)^k.$$

In particular if we choose $k > 2A$ and $r > (\sum_{i=1}^{k} \log p_i)^A$ (ex. $r \ge 2^{kA}$), we see that

$$\tau(n) > (r+1)^{2A} > (\log n)^A.$$

Exercise 5. Fix $\epsilon > 0$. First notice that for any prime number p we have $p \ge 2$ and so

$$\tau(p^k) = k+1, \qquad p^{k\epsilon} \ge 2^{k\epsilon}.$$

Let M_ϵ be so that $M_\epsilon 2^{k\epsilon} \ge k+1$. This can be chosen independently of k since the real function

$$[0,\infty) \ni x \mapsto \frac{x+1}{2^{x\epsilon}}$$

is decreasing for $x > \frac{1}{\epsilon \log 2}$ (compute the derivative) and is continuous, hence it attains a *global* maximum. Call it M_ϵ. We have shown that for any prime number p we have

$$\frac{\tau(p^k)}{p^{k\epsilon}} \le \frac{k+1}{2^{k\epsilon}} \le M_\epsilon.$$

For big prime number p we can do way better, in particular for $p > e^{1/\epsilon}$ we have for any $k \ge 0$

$$\log(k+1) \le k\epsilon \log p,$$

i.e. after taking the exponential $\tau(p^k) \le p^{k\epsilon}$.

Now let $n \in \mathbb{N}$ and write $n = \prod_{p\in\mathcal{P}} p^{v_p(n)}$, where $v_p(n) = 0$ for all but finitely many $p \in \mathcal{P}$.[2] Then

$$\begin{aligned}\frac{\tau(n)}{n^\epsilon} &= \prod_{p\in\mathcal{P}} \frac{\tau(p^{v_p(n)})}{p^{v_p(n)\epsilon}} \\ &\le \prod_{p\le P_\epsilon} M_\epsilon \\ &= M_\epsilon^{\pi(P_\epsilon)},\end{aligned}$$

where $\pi(P_\epsilon)$ is the number of prime numbers smaller or equal than P_ϵ. Set $C_\epsilon = M_\epsilon^{\pi(P_\epsilon)}$.

Exercise 6. Let f as in the exercise. Let $\mathbb{1}_\mathcal{P}$ denote the characteristic function of $\mathcal{P} \subset \mathbb{N}$. The assumption reads

$$G(x) := \sum_{n\le x} f(n)\log(n)\mathbb{1}_\mathcal{P}(n) = (ax+b)\log x + cx + r(x), \tag{0.1}$$

[2] Recall we use $\mathcal{P}$ for the set of prime numbers.

for some uniformly bounded function $r(t)$. We apply next summation by parts to $F(x) \coloneqq \sum_{p\le x} f(p) = \sum_{n\le x} f(n)\mathbb{1}_{\mathcal{P}}(n)$:

$$\begin{aligned} F(x) &= f(2) + \sum_{2<n\le x} \frac{f(n)\log(n)}{\log(n)}\mathbb{1}_{\mathcal{P}}(n) \\ &= f(2) + \frac{G(x)}{\log x} - \frac{G(2)}{\log 2} + \int_2^x \frac{G(t)}{t\log^2 t}\,\mathrm{d}t \\ &= \frac{G(x)}{\log x} + \int_2^x \frac{G(t)}{t\log^2 t}\,\mathrm{d}t. \end{aligned}$$

Now we insert (0.1) into the integral

$$\int_2^x \frac{G(t)}{\log^2 t}\,\mathrm{d}t = \int_2^x \frac{(at+b)}{t\log t}\,\mathrm{d}t + c\int_2^x \frac{1}{\log^2 t}\,\mathrm{d}t + \int_2^x \frac{r(t)}{t\log^2 t}\,\mathrm{d}t$$

First

$$a\int_2^x \frac{1}{\log t}\,\mathrm{d}t = a\left(\frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{1}{\log^2 t}\,\mathrm{d}t\right).$$

Second

$$b\int_2^x \frac{1}{t\log t}\,\mathrm{d}t = b(\log\log x - \log\log 2).$$

Last

$$|\int_2^x \frac{r(t)}{t\log^2 t}\,\mathrm{d}t| \le C(\frac{1}{\log 2} - \frac{1}{\log x}),$$

for some constant $C \ge \sup_t |r(t)|$. In particular the integral

$$I \coloneqq \int_2^\infty \frac{r(t)}{t\log^2 t}\,\mathrm{d}t$$

converges and we have $\int_2^x \frac{r(t)}{t\log^2 t}\,\mathrm{d}t = I + O\left(\frac{1}{\log x}\right)$. Hence

$$F(x) = ax + b + (a+c)\frac{x}{\log x} + (a+c)\int_2^x \frac{1}{\log^2 t}\,\mathrm{d}t + (I - b\log\log 2 - a\frac{2}{\log 2}) + O(1/\log x),$$

which is in the desired form after setting $B = I - b\log\log 2 - a\frac{2}{\log 2}$.