Analytic number theory
Solutions to Exercise Sheet 1

Exercise 1. By taking logarithms of all quantities, we see that we are left with to compare

$$(\log x)^3, \epsilon \log x, \sqrt{\log x}, B\log x, x \text{ and } A\log\log x.$$

We clearly have

$$A\log\log x \ll \sqrt{\log x} \ll \epsilon\log x \ll B\log x \ll (\log x)^3 \ll x$$

for large enough x. and so

$$(\log x)^A \ll e^{\sqrt{\log x}} \ll x^{\epsilon} \ll x^B \ll e^{(\log x)^3} \ll e^x$$

Exercise 2. (a) We apply integration by parts and get

$$\begin{aligned}\int_2^x \frac{1}{\log t}\,\mathrm{d}t &= \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{t}{t\log^2 t}\,\mathrm{d}t \\ &= \frac{x}{\log(x)} + \int_2^x \frac{1}{\log^2 t}\,\mathrm{d}t + O(1).\end{aligned}$$

To estimate the integral notice that the order of magnitude of $\log(x)$ (as $x \to \infty$) is constant in the interval $[x^{\eta}, x]$, with $x \geq 2$ and $0 < \eta < 1$ a fixed number, since $\log(x^{\eta}) = \eta\log(x)$. This motivates the following manipulation (choose $\eta = 1/2$ for readability):

$$\begin{aligned}\int_2^x \frac{1}{\log^2 t}\,\mathrm{d}t &= \int_2^{x^{1/2}} \frac{1}{\log^2 t}\,\mathrm{d}t + \int_{x^{1/2}}^x \frac{1}{\log^2(t)}\,\mathrm{d}t \\ &\leq \frac{x^{1/2}}{\log^2 2} + 2\frac{x - x^{1/2}}{\log^2 x}.\end{aligned}$$

The first summand is $O(x/(\log x)^2)$ as $x \to \infty$ since $(\log x)^2 = O(x^{1/2})$ $(x \to \infty)$ and clearly the second summand is $O(x/\log^2(x))$ $(x \to \infty)$.

(b) Instead of estimatin the integral we continue applying integration by parts: we have that

$$\int_2^x \frac{1}{\log^2(t)}\,\mathrm{d}t = \frac{x}{\log^2 x} - \frac{2}{\log^2(2)} + 2\int_2^x \frac{1}{\log^3 t}\,\mathrm{d}t.$$

Note the recurrence. In particular one can show by induction that

$$\int_2^x \frac{1}{\log(t)}\,\mathrm{d}t = x\sum_{j=1}^{k} \frac{1}{\log^j x} + (k-1)!\int_2^x \frac{1}{\log^k t}\,\mathrm{d}t - 2\sum_{j=1}^{k} \frac{1}{\log^k 2}.$$

In the same way as before we can prove that

$$\int_2^x \frac{1}{\log^k t}\,\mathrm{d}t = O\left(\frac{x}{\log^k(x)}\right).$$

In particular we notice that for fixed k the term $\sum_{j=1}^{k} \frac{1}{\log^k 2}$ is $O(1)$.

Exercise 3. Recall that $\epsilon(n) = n$ for all $n \geq 1$. Also $\mu * \epsilon = e$, where

$$e(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{else.} \end{cases}$$

Denote by ν the arithmetic function

$$\nu(n) = \begin{cases} 1 & n = 1 \\ (-1)^r & n = p_1 \cdots p_r \text{ product of } r \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

We show that $\nu * \epsilon = e$. This will imply $\nu = \mu$ We have

$$1 = \sum_{d|1} \nu(d) = \nu(1).$$

Let now $n = \prod_{i=1} p_i^{l_i}$ for pairwise distinct prime numbers $p_1, \ldots, p_r$ and integers $l_i \geq 1$, $i = 1, \ldots, r$. Then

$$\begin{aligned} \nu * \epsilon(n) &= \sum_{d|n} \nu(d) \\ &= \sum_{j_1=0}^{l_1} \cdots \sum_{j_r=0}^{l_r} \nu(p_1^{j_1} \cdots p_r^{j_r}) \\ &= \sum_{j_1=0}^{1} \cdots \sum_{j_r=0}^{1} \nu(p_1^{j_1} \cdots p_r^{j_r}) \\ &= \sum_{j_1=0}^{1} \cdots \sum_{j_{r-1}=0}^{1} \left((-1)^{j_1+\cdots+j_{r-1}+1} + (-1)^{j_1+\cdots+j_{r-1}}\right) = 0. \end{aligned}$$

Hence $\nu = \mu$.

Exercise 4. We equivalently show that

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}. \tag{0.1}$$

If we denote by id the arithmetic function $id(n) = n$, then we have that the right hand side is $\mu * id(n)$. The arithmetic functions φ and $\mu * id$ are multiplicative and hence to show (0.1) it suffices to show the equality at prime powers p^k. We have

$$\begin{aligned} \varphi(p^k) &= |(\mathbb{Z}/p^k\mathbb{Z})^\times| \\ &= |\{1 \leq a \leq p^k,\ (a, p^k) = 1\}| \\ &= |\{1 \leq a \leq p^k,\ (a, p) = 1\}| = (p-1)p^{k-1}. \end{aligned}$$

On the other hand

$$\begin{aligned} \mu * id(p^k) &= \sum_{d|p^k} \mu(d) \frac{p^k}{d} \\ &= \mu(1)p^k + \mu(p)p^{k-1} \\ &= p^k - p^{k-1} = p^{k-1}(p-1). \end{aligned}$$

This concludes the proof.

Exercise 5. We proceed as hinted: the first prime number is 2 and the assertion is trivially verified. Suppose that the assertion holds for the first n prime numbers. Euclid's argument shows that any of $p_1, \ldots, p_n$ do not divide $p_1 \cdots p_n + 1$. In particular we infer that $p_{n+1} \leq p_1 \cdots p_n + 1$, since the latter is divided by some prime number. By induction hypothesis we have then

$$p_{n+1} \leq \prod_{i=1}^{n} 2^{2^{i-1}} + 1 = 2^{\sum_{i=0}^{n-1} 2^i} + 1 = 2^{2^n - 1} + 1 < 2^{2^n}.$$

This concludes the induction proof.

Using $p_n < e^{e^{n-1}}$, and the monotonicity of log we get that $\log(\log(p_n)) < n - 1$. Let $x > 0$ and let $n = \pi(x)$. In particular $p_{n+1} > x$. By what we said before we have

$$\pi(x) = n > \log(\log(p_{n+1})) > \log\log x$$

and so $\pi(x) > \log\log x$.

Exercise 6. For this exercise recall that $\tau(n) = \sum_{d|n} 1$ is multiplicative.

(a) The arithmetic function $\mathbb{N} \ni n \mapsto \tau(n^2)$ is then also multiplicative. The right hand side corresponds to $\mu * \tau^2(n)$. This arithmetic function is also multiplicative as both μ and τ^2 are. In particular it suffices to check the equality at prime powers p^k:

$$\begin{aligned} \tau(p^{2k}) &= 2k + 1 \\ \mu * \tau^2(p^k) &= \sum_{l=0}^{k} \mu(p^l)\tau^2(p^{k-l}) \\ &= \tau^2(p^k) - \tau^2(p^{k-1}) = (k+1)^2 - k^2 = 2k + 1. \end{aligned}$$

(b) The arithmetic function $\mu(n)$ and

$$\theta(n) = \sum_{d^2|n} \mu(d)$$

are both multiplicative. As before it suffices to check the equality at prime powers p^k. Clearly $\theta(1) = 1$. Moreover

$$\theta(p) = \sum_{d^2|p} \mu(d) = \mu(1) = 1 = \mu(p)^2$$

and for $k \geq 2$

$$\theta(p^k) = \sum_{d^2|p^k} \mu(d) = \sum_{0 \leq i \leq k/2} \mu(p^i) = \mu(1) + \mu(p) = 0.$$

(c) We would like to get rid of the coprime conditions in the sum. For this we use the fact that $\mu * \epsilon = e$, or in formulas that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n \geq 2 \end{cases}.$$

In particular we have

$$\sum_{\substack{1\le a\le n\\(a,n)=1}} e^{2\pi i a/n} = \sum_{a=1}^{n}\sum_{d|(a,n)} \mu(d)e^{2\pi i a/n}$$

$$= \sum_{d|n}\mu(d)\sum_{a=1}^{n/d} e^{2\pi i(ad)/n}$$

$$= \sum_{d|n}\mu(d)\sum_{a=1}^{n/d} e^{2\pi i a/(n/d)}.$$

The inner sum is 0 unless $n/d = 1$, in that case it is 1. Hence the sum above is $\mu(n)$.

Exercise 7. Suppose f^{-1} is completely multiplicative. Let p be a prime and $l \ge 2$ and integer. Then

$$0 = \sum_{d|p^l} f(d)f^{-1}\left(\frac{p^l}{d}\right)$$

$$= \sum_{d=0}^{2} f(p^d)f^{-1}(p^{l-d})$$

$$= \sum_{d=0}^{l} f(p^d)f^{-1}(p)^{l-d}$$

$$= f^{-1}(p)\sum_{d=0}^{l-1} f(p^d)f^{-1}(p)^{l-1-d} + f(p^l) = f^{-1}(p)(f*f^{-1})(p^{l-1}) + f(p^l)$$

Since $f*f^{-1}(p^{l-1}) = 0$ we see that $f(p^l) = 0$.

Conversely suppose that $f(p^l) = 0$ for all $l \ge 2$ and all prime p. We already know that f^{-1} is multiplicative from the class. Hence to show that f^{-1} is completely multiplicative it suffices to show that $f^{-1}(p^k) = f^{-1}(p)f^{-1}(p^{k-1})$ for all $k \ge 2$. We have

$$0 = f*f^{-1}(p^k)$$

$$= \sum_{d=0}^{k} f(p^d)f^{-1}(p^{k-d})$$

$$= f^{-1}(p^k) + f(p)f^{-1}(p^{k-1})).$$

We also have

$$0 = f*f^{-1}(p) = f(p) + f^{-1}(p).$$

Substituting the latter equation into the former we get $f^{-1}(p^k) = f^{-1}(p)f^{-1}(p^{k-1})$.