Algebra IV - Rings and modules (MATH-311) — Final exam

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Paper & pen: This booklet contains 6 exercises, on 36 pages, for a total of 100 points. Please use the space with the square grid for your answers. **Do not** write on the margins. Write all your solutions under the corresponding exercise, except if you run out of space at a given exercise. In that case, continue with your solution at the empty space left after your solution for another exercise. In this case, mark clearly where the continuation of your solution is. If even this way the booklet is not enough, then ask for additional papers from the proctors. Write your name and the exercise number clearly on the top right corner of each additional sheet. At the end of your exam put the additional papers into the exam booklet under the supervision of a proctor, and sign on to the number of additional papers on the proctor's form. We provide scratch paper. You are not allowed to use your own scratch paper. Please write with a pen, NOT with a pencil.

Duration of the exam: It is not allowed to read the inside of the booklet before the exam starts. The length of the exam is 180 minutes. If you did not leave until the final 20 minutes, then please stay seated until the end of the exam, even if you finish your exam during these 20 minutes. The exams are collected by the proctors at the end of the exam, during which please remain seated.

Cheat sheet: You can use a cheat sheet, that is, two sides of an A4 paper handwritten by yourself. At the end, we collect the cheat sheets.

CAMIPRO & coats: Please prepare your CAMIPRO card on your table. Your bag and coat should be placed close to the walls of the room, NOT in the vicinity of your seat.

Results of the course: You can use all results seen during the lectures or in the exercise sessions (that is, all results in the lectures notes or on the exercise sheets), except if the given question asks exactly that result or a special case of it. If you are using such a result, please state explicitly what you are using, and why the assumptions are satisfied.

Separate points can be solved separately: You get maximum credit for solving any point of an exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Assumptions: All rings are commutative and with identity.

Question:	1	2	3	4	5	6	Total
Points:	18	18	12	14	18	20	100
Score:							

Exercise 1 [18 pts]

Let R be a ring and let A, C be R-modules. In this exercise you can freely use that a short exact sequence of chain/cochain complexes gives rise to a long exact sequence in homology/cohomology. Let $P_\bullet \xrightarrow{d_\bullet^A} A$ be a projective resolution of A and let $R_\bullet \xrightarrow{d_\bullet^C} C$ be a projective resolution of C. Let now

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules. Recall that the Horseshoe lemma explicitly constructs a projective resolution $Q_\bullet \xrightarrow{d_\bullet^B} B$ of B such that $Q_n = P_n \oplus R_n$ for every $n \geqslant 0$ and the natural morphisms $P_n \to Q_n$ and $Q_n \to R_n$ given respectively by the first inclusion and the second projection yield morphisms of chain complexes.

(1) Construct explicitly the zero'th differential $d_0^B : P_0 \oplus R_0 \to B$ and show that it is surjective.

(2) Suppose now that we have constructed morphisms $d_i^B : Q_i \to Q_{i-1}$ such that $(Q_\bullet, d_\bullet^Q)$ is a complex and such that the natural maps $(P_\bullet, d_\bullet^A) \to (Q_\bullet, d_\bullet^Q)$ and $(Q_\bullet, d_\bullet^Q) \to (R_\bullet, d_\bullet^C)$ explained before are morphisms of complexes. Show that $(Q_\bullet, d_\bullet^Q)$ is automatically a (projective) resolution of B.

(3) Using the Horseshoe lemma, prove that for any R-module N we have a long exact sequence involving the modules $\mathrm{Ext}^i(A, N)$, $\mathrm{Ext}^i(B, N)$ and $\mathrm{Ext}^i(C, N)$ with $i \geqslant 0$.

Solution:

(1) (**6 pts total**) To construct a morphism $P_0 \oplus R_0 \to B$, we need to find morphisms $P_0 \to B$ and $R_0 \to B$. (**1 pt**) For the first map, simply consider the composition $f : P_0 \to A \to B$. (**1 pt**) For the second map, since R_0 is projective and $B \to C$ is a surjection, there exists a lift $g : R_0 \to B$ of d_0^C. (**2 pts**)

Hence, let $d_0^B := f \oplus g : P_0 \oplus R_0 \to B$, and let us hsow that it is surjective. Let $b \in B$, and let $c \in C$ be the image of b in C. Then there exists $r \in R_0$ such that $d_0^C(r) = c$. In other words, $d_0^B(0, r) - b$ is in the kernel of $B \to C$, i.e. in A. But then, there exists $p \in P_0$ such that $d_A^0(p) = d_0^B(0, r) - b$. Thus,

$$d_0^B(-p, r) = -d_0^B(p, 0) + d_B^0(0, r) = b - d_0^B(0, r) + d_0^B(0, r) = b.$$

In other words, d_0^B is surjective. (**2 pts**)

(2) (**6 pts total**) Since each P_i and R_i are projective modules, so are their direct sum Q_i (**1 pt**). Hence, we only have to show that the complex made of $Q_\bullet$ and B at the very end is exact. (**1 pt**) Let $Q'_\bullet$ denote this complex, and let $P'_\bullet$ (resp. $R'_\bullet$) denote the analogous complexes with A and C. Then we obtain a cochain of complexes

$$0 \to P'_\bullet \to Q'_\bullet \to R'_\bullet \to 0.$$

Let us show that it is exact (i.e. it is termwise exact). At the first degree, this is exactly the assumption that

$$0 \to A \to B \to C \to 0$$

is exact. For all the other degrees, the sequence is

$$0 \to P_i \to P_i \oplus Q_i \to Q_i \to 0$$

where the maps are the ones explained in the exercise. It is immediate to see that these sequences are exact.

Thus, we can apply the long exact sequence is homology to obtain a long exact sequence

$$\cdots \to H^i(P'_\bullet) \to H^i(Q'_\bullet) \to H^i(R'_\bullet) \to \dots$$

Since both $P'_\bullet$ and $R'_\bullet$ are exact by assumption, each groups $H^i(P'_\bullet)$ and $H^i(R'_\bullet)$ vanish by assumption. Hence, so do each $H^i(Q'_\bullet)$, i.e. $Q_\bullet$ is a projective resolution. (**4 pts**)

(3) (**6 pts total**) Let $P_\bullet$, $Q_\bullet$ and $R_\bullet$ denote projective resolutions as in the Horseshoe lemma. Applying $\mathrm{Hom}(-, N)$ gives us a cochain complex

$$0 \to \mathrm{Hom}(R_\bullet, N) \to \mathrm{Hom}(Q_\bullet, N) \to \mathrm{Hom}(P_\bullet, N) \to 0.$$

If we knew that this was exact, then we would automatically conclude by the associated long exact sequence in cohomology. (**4 pts**) Thus, we are left to show that it is exact.

In other words, we have to show that this sequence is termwise exact. Since $\mathrm{Hom}(-, N)$ is left exact, it is enough to show that for all $i \geqslant 0$, the map $\mathrm{Hom}(Q_i, N) \to \mathrm{Hom}(P_i, N)$ is surjective. This is immediate, as for any map $f\colon P_i \oplus N$, we can extend it to $f \oplus 0\colon P_i \oplus R_i \to N$. (**2 pts**)

Exercise 2 [**18 pts**]

Let R be a ring. Throughout this exercise, you can use without proof that if $R \subset S$ is integral extension of domains, then R is a field if and only if S is a field.

(1) Let F be any field. Use Noether normalization to show that if $F[x_1, \cdots, x_n]/I$ is a field, then it is an algebraic extension of F.

(2) Let now K be an algebraically closed field. State the weak Hilbert Nullstellensatz and prove it using the previous point.

(3) Use the weak Nullstellensatz to prove the full Nullstellensatz: for any ideal $I \subseteq K[x_1, \dots, x_n]$, we have an equality $I(V(I)) = \sqrt{I}$. (Hint: for any ring R and any $g \in R$, the localization R_g is zero if and only if g is nilpotent).

Solution:

(1) (**5 pts total**) Let $R := F[x_1, \dots, x_n]/I$. By Noether normalization, there exists an integral extension $F[t_1, \dots, t_r] \hookrightarrow R$ (the left term is really a polynomial algebra). (**2 pts**) Thus, if R is a field, then we know that $F[t_1, \dots, t_r]$ is a field too. This forces $r = 0$ (otherwise t_1 does not have an inverse), so we have an integral extension of field $F \hookrightarrow R$. This is the same as saying that the extension R/F is algebraic. (**3 pts**)

(2) (**5 pts total**) The Weak Nullstellensatz states that if m is a maximal ideal of $K[x_1, \dots, x_n]$, then we have that $m = (x_1 - a_1, \dots, x_n - a_n)$. (**1 pt**)

Now, let us show it, so let $m \subseteq K[x_1, \dots, x_n]$ be a maximal ideal. Then $K[x_1, \dots, x_n]/m$ is a field, so by the previous point, it is an algebraic extension of K. Since K is algebraically closed, we deduce that the natural morphism $\theta\colon K \to K[x_1, \dots, x_n]/m$ is an isomorphism. (**2 pts**) Let $a_i = \theta^{-1}(x_i)$, and let us show that $m = x_1 - a_1, \dots, x_n - a_n)$. For a fixed i, we have that $x_i - a_i \in K[x_1, \dots, x_n]$ is sent to zero via $K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]/m \cong K$, so each $x_i - a_i$ are in m. Thus, $m \supseteq (x_1 - a_1, \dots, x_n - a_n)$. Since this latter ideal is maximal (the quotient of $K[x_1, \dots, x_n]$ by it is a field), we deduce that $m = (x_1 - a_1, \dots, x_n - a_n)$. (**2 pts**)

(3) (**8 pts total**) If $f \in \sqrt{I}$, say $f^n \in I$, the $f^n(a) = 0$ for all $a \in V(I)$ by definition. But then, this shows that $f(a) = 0$ for all $a \in V(I)$, so $f \in I(V(I))$. (**1 pt**)

Let us show the converse now, so let $f \in I(V(I))$, and let $R = K[x_1, \dots, x_n]$. Our goal is to show that $f \in \sqrt{I}$, or equivalently that f is nilpotent in R/I. To do so, we will actually show that the localization $(R/I)_f$ is zero. (**2 pts**) Since $(R/I)_f \cong (R/I)[t]/(tf - 1) \cong R[t]/(tf - 1, I)$, we want to show that $(tf - 1, I) = R[t]$. (**2 pts**) If this was not the case, then $(tf - 1, I)$ would belong to a maximal ideal, so there would exist $a_1, \dots, a_n, b \in K$ such that for all $h \in (tf - 1, I)$, $h(a_1, \dots, a_n, b) = 0$. (**1 pt**)

Combining the following two cases gives rise to a contradiction.

- If $(a_1, \dots, a_n) \in V(I)$, then evaluatiing $tf - 1$ at $(a_1, \dots, a_n, b)$ gives $-1 \neq 0$.
- If $(a_1, \dots, a_n) \notin V(I)$ (say $h \in I \subseteq R$ such that $h(a_1, \dots, a_n) \neq 0$), then the same element $h \in (tf - 1, I) \subseteq R[t]$ sends $(a_1, \dots, a_n, b)$ to $h(a_1, \dots, a_n) \neq 0$. (**2 pts**)

Exercise 3 [**12 pts**]

(1) Recall the definition of an Artinian ring.

(2) Prove that every Artinian ring has Krull dimension zero.

(3) Compute the Krull dimension of $\mathbb{Z}[x]/(6, x^2)$.

(4) Compute the length of $\mathbb{Z}[x]/(6, x^2)$ as a $\mathbb{Z}[x]$-module.

Solution:

(1) (**2 pts total**) A ring is said to be Artinian if any descending chain of ideals

$$\dots I_j \subseteq I_{j-1} \subseteq \dots \subseteq I_0,$$

stabilizes (i.e. $I_j = I_{j+1}$ for $j \gg 0$).

(2) (**3 pts total**) We have to show that any prime ideal is maximal. By quotienting, this is equivalent to showing that any Artinian domain is a field.

Let R be an Artinian domain, and let $r \neq 0$ be an element. Considering the descending chain

$$\dots \subseteq (r^j) \subseteq (r^{j-1}) \subseteq \dots \subseteq (r),$$

we obtain in particular that $(r^j) \subseteq (r^{j+1})$ for some $j > 0$. Hence, we can write $r^j = r^{j+1}s$ for some $s \in R$, or equivalently

$$r^j(1 - rs) = 0.$$

Since R is a domain, we deduce that $rs = 1$, so R is indeed a field.

(3) (**2 pts total**) Note that

$$\mathbb{Z}[x]/(6, x^2) \cong (\mathbb{Z}/6\mathbb{Z})[x]/(x^2),$$

which is a free $\mathbb{Z}/6\mathbb{Z}$–module of rank 2. In particular, this is a finite ring, so it is certainly Artinian. We conclude by the previous point that its Krull dimension is zero.

(4) (**5 pts total**) Consider the canonical short exact sequence

$$0 \to (6, x)/(6, x^2) \to \mathbb{Z}[x]/(6, x^2) \to \mathbb{Z}[x]/(6, x) \to 0.$$

(**1 pt**)

Let us show that the natural map $\theta\colon \mathbb{Z}[x] \to (6, x)/(6, x^2)$ sending 1 to x gives an isomorphism

$$\mathbb{Z}[x]/(6, x) \cong (6, x)/(6, x^2).$$

The morphism θ is surely surjective, so we only have to show that $\ker(\theta) = (6, x)$. The inclusion $(6, x) \subseteq \ker(\theta)$ is immediate. For the other inclusion, let $f \in \ker(\theta)$. Then we can write

$$xf = 6a + x^2 b$$

for some $a, b \in \mathbb{Z}[x]$. We can rewrite this as

$$6a = x(f - bx),$$

so given that 6 and x are coprime as $\mathbb{Z}[x]$ is a UFD, we deduce that 6 divides $f - bx$. In other words, $f \in (6, x)$. (**2 pts**)

Hence, we have a short exact sequence

$$0 \to \mathbb{Z}[x]/(6, x) \to \mathbb{Z}[x]/(6, x^2) \to \mathbb{Z}[x]/(6, x) \to 0,$$

so

$$\text{length}(\mathbb{Z}[x]/(6, x^2)) = 2\text{length}(\mathbb{Z}[x]/(6, x)).$$

By the Chinese Remainder theorem, the natural map of $\mathbb{Z}[x]$–modules

$$\mathbb{Z}[x]/(6, x) \to \mathbb{Z}[x]/(2, x) \oplus \mathbb{Z}[x]/(3, x)$$

is an isomorphism. Furthermore, the modules $\mathbb{Z}[x]/(2, x)$ and $\mathbb{Z}[x]/(3, x)$ have cardinality 2 and 3, which are prime numbers. Hence, they must automatically be simple, so we deduce that $\text{length}(\mathbb{Z}[x]/(6, x)) = 2$. In other words,

$$\text{length}(\mathbb{Z}[x]/(6, x^2)) = 4.$$

(**2 pts**)

Exercise 4 [14 pts]

Let R be a ring.

(1) Let $I \subset R$ be an ideal, and let M be a finitely generated R-module. Show that if $IM = M$, then there exists $x \in I$ such that $(1 + x)M = 0$. *Hint: Use adjugate matrices.*

(2) Deduce Nakayama's lemma for local rings: if R is local with maximal ideal m and M is a finitely generated R-module such that $mM = M$, then $M = 0$.

Solution:

(1) (**10 pts total**) Let $m_1, \dots, m_n \in M$ be generators of M. Then for all i, we can write $m_i = \sum a_{ij} m_j$, with $a_{ij} \in I$. Let $m = (m_1, \dots, m_n)^T$, and $A = (a_{ij}) \in I^{n \times n}$. Then

$$Am = m,$$

or equivalently $(A - Id)m = 0$ (**3 pts**). We then obtain that

$$\det(A - Id)m = \mathrm{adj}(A - Id)(A - Id)m = 0,$$

so it is enough to show that $\det(A - Id) \in 1 + I$. (**4 pts**)

By definition,

$$\det(A - Id) = \sum_{\sigma \in S^n} \mathrm{sgn}(\sigma) \prod_i (A - Id)_{i\sigma(i)}.$$

If $\sigma \neq id$, then one of the $(A - Id)_{i\sigma(i)}$ terms is in I, and hence so is the whole product. On the other hand, if $\sigma = id$, then each term $(A - Id)_{ii} = a_{ii} - 1$ is in $1 + I$, and hence so it their product. Thus, we conclude that $\det(A - Id) \in 1 + I$. (**3 pts**)

(2) (**4 points total**) By the previous point, we deduce that for some $x \in 1 + m$ (say $x = 1 + r$ with $r \in m$), then $xM = 0$. Hence, it is enough to show that x is a unit. (**2 pts**) If not, then $x \in m$ since m is the only maximal ideal, so $1 + r \in m$. Since $r \in m$, we deduce that $1 \in m$, which is a contradiction. (**2 pts**)

Exercise 5 [**18 pts**]

Let $R \subset S$ be an integral extension of Noetherian rings.

(1) State the Going-up theorem and explain why the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ induced by the contraction of ideals is surjective.

(2) Let $p \subset R$ be a prime ideal. Show that there is a one-to-one correspondence

$$\{ \text{ prime ideals } q \subset S \mid q^c = p\} \longleftrightarrow \mathrm{Spec}((S/p^e)_p).$$

Remark: The notation on the right–hand side of the bijection denotes the localization of the R–module S/p^e at p, which we see as a ring itself.

(3) Deduce that there are only finitely many prime ideals $q \subset S$ such that $q^c = p$ for a given $p \subset R$. *(Hint: you can use without proof that a Noetherian ring has only finitely many minimal prime ideals).*

Solution:

(1) (**3 pts total**) The Going-up theorem states that for any prime ideal p in R, there exists a prime ideal q in S such that $q \cap R = p$. (**2 pts**)

Since the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is precisely the map given by sending a prime q in S to $q^c = q \cap R$, saying that the contraction of ideals is surjective is precisely the Going-up theorem. (**1 pt**)

(2) (**7 pts total**) If a prime ideal $q \subseteq S$ satisfies that $q^c = p$, then surely q contains p^e. Thus, by the correspondence theorem, prime ideals of S contracting to p correspond to prime ideals of S/p^e satisfying some condition. (**2 pts**)

Let us show that this condition is exactly that $q \cap R \backslash p = \varnothing$. If this is the case, then we can conclude by the correspondence theorem for localization. (**3 pts**)

If q is a prime ideal containing p^e such that $q \cap R \backslash p = \varnothing$, then surely the inclusion $p \subseteq q \cap R$ must be an equality. Conversely, if $q \cap R = p$, then $q \cap (R \backslash p) = \varnothing$, so we are done. (**2 pts**)

(3) (**8 pts total**) By the previous point, we have to show that $(S/p^e)_p$ only has finitely many prime ideals. Since it is Noetherian, it is enough to show by the hint that any prime ideal in this ring is minimal, or in other words that its Krull dimension is zero. (**3 pts**)

Since we have an integral extension of rings $R \subseteq S$, the induced extension $R/p \subseteq S/p^e$ is also integral (note that $p^e \cap R = p$, because if q is a prime such that $q \cap R = p$, then we have inclusions $p \subseteq p^e \cap R \subseteq q \cap R = p$). (**1 pt**)

Since localization preserves injections (it is an exact functor), we obtain an extension $(R/p)_p \subseteq (S/p^e)_p$. Let us show that it is integral. Since the subset of integral elements is a $(R/p)_p$–module and that $(S/p^e)_p$ is general as a R_p–module by elements of the form $\frac{t}{1}$ with $t \in S/p^e$, then it is enough to show that each element $\frac{t}{1}$ is integral over $(R/p)_p$. However, if t satisfies a certain monic equation over R/p, then $\frac{t}{1}$ satisfies the same monic equation, so we deduce that, indeed, the extension $(R/p)_p \subseteq (S/p^e)_p$ is integral. (**2 pts**)

Now, note that by definition, $(R/p)_p = \mathrm{Frac}(R/p)$, so it is a field. In particular, it has Krull dimension zero. Since the Krull dimension is invariant under integral extension, we deduce that $(S/p^e)_p$ also has Krull dimension zero. (**2 pts**)

Alternative proof (found by some students!): As already mentioned, it is enough to show that every prime is minimal, or in other words there no inclusion $q_1' \subsetneq q_2'$ of primes in $(S/p^e)_p$. Assume by contradiction that there is such an inclusion, and let $q_i \subseteq S$ be the prime corresponding to q_i' (so that we still have that $q_1 \subsetneq q_2$ by the two correspondence theorems). Since $q_1 \cap R = q_2 \cap R$, this contradicts the addendum of the Going-up theorem.

Exercise 6 [**20 pts**]

Let R be a Noetherian ring and let M be a finitely generated module over R. In this exercise you can use without proof that localization is exact. We say that M is locally free if for every prime ideal $p \subset R$, the localization M_p of M at $R \setminus p$ is a free R_p-module.

(1) Show that if for some prime $p \subset R$ we have $M_p = 0$ then there exists $f \in R \setminus p$ such that $M_f = 0$.

(2) Show that if M_p is a free R_p-module for some prime ideal p, then there is $f \in R \setminus p$ such that M_f is a free A_f-module.

(3) Show that if M is locally free, then there exist $f_1, \cdots f_s \in R$ such that $(f_1, \cdots, f_s) = R$ and M_{f_i} is free over A_{f_i} for all $1 \leqslant i \leqslant s$.

Solution:

(1) (**6 pts total**) Let $m_1, \ldots, m_n$ be generators of M. By definition, for all i, we have that

$$\frac{m_i}{1} = 0 \in M_p,$$

or equivalently there exists $a_i \in R\backslash p$ such that $a_i m_i = 0$. (**3 pts**) Let $f = a_1 \ldots a_n \in R\backslash p$. Then by definition, $f m_i = 0$ for all i, so

$$\frac{m_i}{1} = 0 \in M_f$$

for all i. (**3 pts**)

(2) (**8 pts total**) Let $\frac{m_1}{s_1}, \ldots, \frac{m_n}{s_n}$ denote free generators of M_p over R_p. Since each $s_i \in R\backslash p$, they become unit in R_p. In other words, the elements $\frac{m_1}{1}, \ldots, \frac{m_n}{1}$ also denote free generators of M_p over R_p. (**2 pts**)

Now, consider the morphism $\theta \colon R^n \to M$ defined by sending e_i to M_i. Then we have an exact sequence

$$0 \to \ker(\theta) \to R^n \xrightarrow{\theta} M \to \operatorname{coker}(\theta) \to 0.$$

By the choice of the elements m_i, the localization of θ at p is an isomorphism. Since localization is exact, we deduce that $\ker(\theta)_p = \operatorname{coker}(\theta)_p = 0$. (**3 pts**) By the previous point, there exists $a, b \in R\backslash p$ such that $\ker(\theta)_a = 0$ and $\operatorname{coker}(\theta)_b = 0$. But then, if we write $f = ab$, then

$$\ker(\theta)_f = \operatorname{coker}(\theta)_f = 0,$$

so again by exactness of localization, we deduce that $\theta_f \colon R^n_f \to M_f$ is an isomorphism. (**3 pts**)

(3) (**6 pts total**) By the previous point, we deduce that for all prime p in R, there exists $f \notin p$ such that M_f is free as an R_f-module. But then, the ideal $(f_p)_{p \subseteq R}$ is not contained in any prime ideal, so

$$R = (f_p)_{p \subseteq R}.$$

(**3 pts**) In particular, we can write $1 = a_1 f_{p_1} + \cdots + a_n f_{p_n}$ for some primes p_i and elements $a_i \in R$. In other words, we deduce that $R = (f_{p_1}, \ldots, f_{p_n})$. (**3 pts**)