Algebra IV - Rings and modules (MATH-311) — Final exam

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Numéro
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Paper & pen: This booklet contains 6 exercises, on 32 pages, for a total of 100 points. Please use the space with the square grid for your answers. **Do not** write on the margins. Write all your solutions under the corresponding exercise, except if you run out of space at a given exercise. In that case, continue with your solution at the empty space left after your solution for another exercise. In this case, mark clearly where the continuation of your solution is. If even this way the booklet is not enough, then ask for additional papers from the proctors. Write your name and the exercise number clearly on the top right corner of each additional sheet. At the end of your exam put the additional papers into the exam booklet under the supervision of a proctor, and sign on to the number of additional papers on the proctor's form. We provide scratch paper. You are not allowed to use your own scratch paper. Please write with a pen, NOT with a pencil.

Duration of the exam: It is not allowed to read the inside of the booklet before the exam starts. The length of the exam is 180 minutes. If you did not leave until the final 20 minutes, then please stay seated until the end of the exam, even if you finish your exam during these 20 minutes. The exams are collected by the proctors at the end of the exam, during which please remain seated.

Cheat sheet: You can use a cheat sheet, that is, two sides of an A4 paper handwritten by yourself. At the end, we collect the cheat sheets.

CAMIPRO & coats: Please prepare your CAMIPRO card on your table. Your bag and coat should be placed close to the walls of the room, NOT in the vicinity of your seat.

Results of the course: You can use all results seen during the lectures or in the exercise sessions (that is, all results in the lectures notes or on the exercise sheets), except if the given question asks exactly that result or a special case of it. If you are using such a result, please state explicitly what you are using, and why the assumptions are satisfied.

Separate points can be solved separately: You get maximum credit for solving any point of an exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Assumptions: All rings are commutative and with identity.

Question:	1	2	3	4	5	6	Total
Points:	18	18	12	14	18	20	100
Score:							

Exercise 1 [**18 pts**]

Let R be a ring and let A, C be R-modules. In this exercise you can freely use that a short exact sequence of chain/cochain complexes gives rise to a long exact sequence in homology/cohomology. Let $P_\bullet \xrightarrow{d_\bullet^A} A$ be a projective resolution of A and let $R_\bullet \xrightarrow{d_\bullet^C} C$ be a projective resolution of C. Let now

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules. Recall that the Horseshoe lemma explicitly constructs a projective resolution $Q_\bullet \xrightarrow{d_\bullet^B} B$ of B such that $Q_n = P_n \oplus R_n$ for every $n \geqslant 0$ and the natural morphisms $P_n \to Q_n$ and $Q_n \to R_n$ given respectively by the first inclusion and the second projection yield morphisms of chain complexes.

(1) Construct explicitly the zero'th differential $d_0^B \colon P_0 \oplus R_0 \to B$ and show that it is surjective.

(2) Suppose now that we have constructed morphisms $d_i^B \colon Q_i \to Q_{i-1}$ such that $(Q_\bullet, d_\bullet^Q)$ is a complex and such that the natural maps $(P_\bullet, d_\bullet^A) \to (Q_\bullet, d_\bullet^Q)$ and $(Q_\bullet, d_\bullet^Q) \to (R_\bullet, d_\bullet^C)$ explained before are morphisms of complexes. Show that $(Q_\bullet, d_\bullet^Q)$ is automatically a (projective) resolution of B.

(3) Using the Horseshoe lemma, prove that for any R-module N we have a long exact sequence involving the modules $\mathrm{Ext}^i(A, N)$, $\mathrm{Ext}^i(B, N)$ and $\mathrm{Ext}^i(C, N)$ with $i \geqslant 0$.

Exercise 2 [**18 pts**]

Let R be a ring. Throughout this exercise, you can use without proof that if $R \subset S$ is integral extension of domains, then R is a field if and only if S is a field.

(1) Let F be any field. Use Noether normalization to show that if $F[x_1, \cdots, x_n]/I$ is a field, then it is an algebraic extension of F.

(2) Let now K be an algebraically closed field. State the weak Hilbert Nullstellensatz and prove it using the previous point.

(3) Use the weak Nullstellensatz to prove the full Nullstellensatz: for any ideal $I \subseteq K[x_1, \ldots, x_n]$, we have an equality $I(V(I)) = \sqrt{I}$. (Hint: for any ring R and any $g \in R$, the localization R_g is zero if and only if g is nilpotent).

Exercise 3 [**12 pts**]

(1) Recall the definition of an Artinian ring.

(2) Prove that every Artinian ring has Krull dimension zero.

(3) Compute the Krull dimension of $\mathbb{Z}[x]/(6, x^2)$.

(4) Compute the length of $\mathbb{Z}[x]/(6, x^2)$ as a $\mathbb{Z}[x]$-module.

Exercise 4 [**14 pts**]

Let R be a ring.

(1) Let $I \subset R$ be an ideal, and let M be a finitely generated R-module. Show that if $IM = M$, then there exists $x \in I$ such that $(1 + x)M = 0$. *Hint: Use adjugate matrices.*

(2) Deduce Nakayama's lemma for local rings: if R is local with maximal ideal m and M is a finitely generated R-module such that $mM = M$, then $M = 0$.

Exercise 5 [**18 pts**]

Let $R \subset S$ be an integral extension of Noetherian rings.

(1) State the Going-up theorem and explain why the map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ induced by the contraction of ideals is surjective.

(2) Let $p \subset R$ be a prime ideal. Show that there is a one-to-one correspondence

$$\{ \text{ prime ideals } q \subset S \mid q^c = p\} \longleftrightarrow \operatorname{Spec}((S/p^e)_p).$$

Remark: The notation on the right–hand side of the bijection denotes the localization of the R–module S/p^e at p, which we see as a ring itself.

(3) Deduce that there are only finitely many prime ideals $q \subset S$ such that $q^c = p$ for a given $p \subset R$. *(Hint: you can use without proof that a Noetherian ring has only finitely many minimal prime ideals).*

Exercise 6 [**20 pts**]

Let R be a Noetherian ring and let M be a finitely generated module over R. In this exercise you can use without proof that localization is exact. We say that M is locally free if for every prime ideal $p \subset R$, the localization M_p of M at $R \setminus p$ is a free R_p-module.

(1) Show that if for some prime $p \subset R$ we have $M_p = 0$ then there exists $f \in R \setminus p$ such that $M_f = 0$.

(2) Show that if M_p is a free R_p-module for some prime ideal p, then there is $f \in R \setminus p$ such that M_f is a free A_f-module.

(3) Show that if M is locally free, then there exist $f_1, \cdots f_s \in R$ such that $(f_1, \cdots, f_s) = R$ and M_{f_i} is free over A_{f_i} for all $1 \leqslant i \leqslant s$.