Algebra IV - Rings and modules (MATH-311) — Final exam

25 January 2024, 9 h 15 – 12 h 15

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Numéro

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SCIPER : 42

Signature : ______________________

Paper & pen: This booklet contains 6 exercises, on 32 pages, for a total of 100 points. Please use the space with the square grid for your answers. **Do not** write on the margins. Write all your solutions under the corresponding exercise, except if you run out of space at a given exercise. In that case, continue with your solution at the empty space left after your solution for another exercise. In this case, mark clearly where the continuation of your solution is. If even this way the booklet is not enough, then ask for additional papers from the proctors. Write your name and the exercise number clearly on the top right corner of the additional paper. At the end of your exam put the additional papers into the exam booklet under the supervision of a proctor, and sign on to the number of additional papers on the proctor's form. We provide scratch paper. You are not allowed to use your own scratch paper. Please write with a pen, NOT with a pencil.

Duration of the exam: It is not allowed to read the inside of the booklet before the exam starts. The length of the exam is 180 minutes. If you did not leave until the final 20 minutes, then please stay seated until the end of the exam, even if you finish your exam during these 20 minutes. The exams are collected by the proctors at the end of the exam, during which please remain seated.

Cheat sheet: You can use a cheat sheet, that is, two sides of an A4 paper handwritten by yourself. At the end, we collect the cheat sheets.

CAMIPRO & coats: Please prepare your CAMIPRO card on your table. Your bag and coat should be placed close to the walls of the room, NOT in the vicinity of your seat.

Results of the course: You can use all results seen during the lectures or in the exercise sessions (that is, all results in the lectures notes or on the exercise sheets), except if the given question asks exactly that result or a special case of it. If you are using such a result, please state explicitly what you are using, and why the assumptions are satisfied.

Separate points can be solved separately: You get maximum credit for solving any point of an exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Assumptions: all rings are with identity.

Question:	1	2	3	4	5	6	Total
Points:	16	16	18	18	18	14	100
Score:							

Exercise 1 $\left[\right.$ **16 pts** $\left.\right]$

Consider the proof of Hilbert's basis theorem, where we proved that if a commutative ring R is Noetherian, then so is $R[x]$. We fixed an arbitrary ideal $I \subseteq R[x]$ and we wanted to show that I is finitely generated. We chose a finite generator set of the ideal

$$I_{\mathrm{in}} = \left\{ a_n \in R \;\middle|\; \sum_{i=0}^{n} a_i x^i \in I \text{ such that } a_n \neq 0 \right\}$$

of R. We chose then $f_1, \dots, f_r \in I$ such that their leading coefficients yielded exactly the above generator set, and such that their degrees were the same, say d. Then we also chose a generator set $f_{r+1}, \dots, f_s$ for the R-module

$$\left\{ f \in I \;\middle|\; \deg f < d \right\}$$

Show that $f_1, \dots, f_s$ generate I as an ideal of $R[x]$.

Exercise 2 $\left[\ \textbf{16 pts}\ \right]$

Let

$$0 \longrightarrow E^\bullet \xrightarrow{\alpha^\bullet} F^\bullet \xrightarrow{\beta^\bullet} G^\bullet \longrightarrow 0$$

be a short exact sequence of co-chain complexes, that is, $\alpha^\bullet$ and $\beta^\bullet$ are co-chain morphisms such that

$$0 \longrightarrow E^i \xrightarrow{\alpha^i} F^i \xrightarrow{\beta^i} G^i \longrightarrow 0$$

is exact for every $i \in \mathbb{Z}$. Let $e^i : E^i \to E^{i+1}$, $f^i : F^i \to F^{i+1}$ and $g^i : G^i \to G^{i+1}$ be the structure homomorphisms of $E^\bullet$, $F^\bullet$ and $G^\bullet$, respectively.

Show that the following sequence is exact for any $i \in \mathbb{Z}$:

$$H^i(E^\bullet) \xrightarrow{H^i(\alpha^\bullet)} H^i(F^\bullet) \xrightarrow{H^i(\beta^\bullet)} H^i(G^\bullet) \tag{1}$$

[*Remark: you can use without proof that H^i is functorial, that is, that $H^i(\beta^\bullet \circ \alpha^\bullet) = H^i(\beta^\bullet) \circ H^i(\alpha^\bullet)$.*]

Exercise 3 $\left[\ \textbf{18 pts}\ \right]$

(1) Show that if $0 \neq f \in \mathbb{R}[x]$ is an irreducible element, then either

- $f = x - c$ for some $c \in \mathbb{R}$, or
- $f = (x-c)^2 + a$ for some $a, c \in \mathbb{R}$ with $a > 0$.

(2) Show that up to isomorphisms of rings, there are three (non-isomorphic) possibilities for $\mathbb{R}[x]_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{R}[x]$.

Exercise 4 [**18 pts**]

Consider the following situation:

- *k is an algebraically closed field,*
- *$a, b \in k$ are arbitrary elements,*
- *$R = k[x, y]$,*
- *M is the R-module $R/(x, y)$,*
- *L is the R-module $R/(x-a, y-b)$,*
- *Q is an arbitrary finite length R-module.*

(1) Compute $\operatorname{Tor}_i^R(L, M)$ for an arbitrary $i \in \mathbb{Z}$. Your answer should depend on the value of $(a, b) \in k^2$.

(2) Show that $\operatorname{Tor}_i^R(Q, M)$ is of finite length over R.

Exercise 5 $\left[\ \textbf{18 pts}\ \right]$

Consider the following situation:

- *k is a field,*
- *$R = k[x,y]\big/(xy)$,*
- *for $a, b \in k$ such that $(a,b) \neq (0,0)$, let $\eta_{a,b} : k[t] \to R$ be the k-algebra homomorphism given by $t \mapsto ax + by$ (where $ax + by$ denotes the element $ax + by + (xy) \in R$).*

Without proof you can admit that $\eta_{a,b}$ is injective (for example, it is easy to see that the elements $(ax+by)^i \in R$ are k-linearly independent).

Show that $\eta_{a,b}$ yields an integral extension if and only if $a \neq 0$ and $b \neq 0$.

Exercise 6 [14 pts]

Consider $R = \mathrm{Mat}(3, \mathbb{C})$ *and the subring*

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in R \;\middle|\; a, b, c, d, e, f \in \mathbb{C} \right\} \subseteq R$$

You do not have to show that S *is a subring of* R. *Let* $M = \mathbb{C}^3$ *be the* 3*-dimensional complex vectorspace, the elements of which we regard as column vectors. Then multiplication on the left turns* M *into a (left) module both over* S *and* R. *That is, the module structures are given by the multiplication* $A \cdot v = Av$ *for every* $A \in R$ *or* $A \in S$ *and for every* $v \in M$ *(you do not have to verify this).*

Note that scalar multiples of the identity matrix give $\mathbb{C}$ *as a subring of both* S *and* R. *Hence every* R *and* S *module is also automatically a* $\mathbb{C}$*-vectorspace.*

(1) Show that every non-zero S-submodule $N \subseteq M$ contains the S-submodule

$$M_1 = \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \;\middle|\; a \in \mathbb{C} \right\}$$

(You do not have to verify that M_1 is an S-submodule)

(2) Conclude that M is not a semi-simple S-module.

(3) Show that M is a simple R-module.