Rings and modules (MATH-311) — Final exam

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Numéro
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Paper & pen: This booklet contains 6 exercises, on 28 pages, for a total of 100 points. Please use the space with the square grid for your answers. **Do not** write on the margins. Write all your solutions under the corresponding exercise, except if you run out of space at a given exercise. In that case, continue with your solution at the empty space left after your solution for another exercise. In this case, mark clearly where the continuation of your solution is. If even this way the booklet is not enough, then ask for additional papers from the proctors. Write your name and the exercise number clearly on the top right corner of the additional paper. At the end of your exam put the additional papers into the exam booklet under the supervision of a proctor, and sign on to the number of additional papers on the proctor's form. We provide scratch paper. You are not allowed to use your own scratch paper. Please write with a pen, NOT with a pencil.

Duration of the exam: It is not allowed to read the inside of the booklet before the exam starts. The length of the exam is 180 minutes. If you did not leave until the final 20 minutes, then please stay seated until the end of the exam, even if you finish your exam during these 20 minutes. The exams are collected by the proctors at the end of the exam, during which please remain seated.

Cheat sheet: You can use a cheat sheet, that is, two sides of an A4 paper handwritten by yourself. At the end, we collect the cheat sheets.

CAMIPRO & coats: Please prepare your CAMIPRO card on your table. Your bag and coat should be placed close to the walls of the room, NOT in the vicinity of your seat.

Results of the course: You can use all results seen during the lectures or in the exercise sessions (that is, all results in the lectures notes or on the exercise sheets), except if the given question asks exactly that result or a special case of it. If you are using such a result, please state explicitly what you are using, and why the assumptions are satisfied.

Separate points can be solved separately: You get maximum credit for solving any point of an exercise assuming the statements of the previous points, even if you did not solve (all of) those previous points.

Assumptions: all rings are with identity.

Question:	1	2	3	4	5	6	Total
Points:	12	16	14	12	20	26	100
Score:							

Exercise 1 [**12 pts**]

Let R be a commutative ring. We defined during the course the subset

$$V(I) = \{ \, p \in \operatorname{Spec} R \mid I \subseteq p \, \} \subseteq \operatorname{Spec} R,$$

for any ideal $I \subseteq R$.

Show that the $V(I)$'s form a topology on $\operatorname{Spec} R$ as I runs through all ideals of R.

Exercise 2 [**16 pts**]

(1) State the definition of the transcendence basis of a field extension $K \subseteq L$.

[*You do not have to define the notions of being algebraically independent or being algebraic over a field.*]

(2) Show that if $S \hookrightarrow R$ is an integral ring extension such that both S and R are domains, then S is a field if and only if R is a field.

(3) Show that if F is a field and R is domain, which is also a finitely generated F-algebra with $\operatorname{trdeg}_F \operatorname{Frac}(R) > 0$, then R is not a field.

[*You can use without proof:*

- *the existence of Noether normalizations*
- *integral extensions of domains induce algebraic extensions after passing to the fraction fields.*
- *if $K \subseteq L$ is a field extension, then the number of elements in a finite transcendence basis of L over K is independent of the choice of a finite transcendence basis, and hence we may define the transcendence degree* $\operatorname{trdeg}_K L$ *of L over K as this number.*]

Exercise 3 [**14 pts**]

Let R be a commutative ring and let $T \subseteq R$ be a multiplicatively closed subset. Let $\iota : R \to T^{-1}R$ be the localization homomorphism, and let us mean extension and contraction of ideals with respect to ι. In the following exercise you can use without proof that extension and contraction gives a bijection

$$\operatorname{Spec} T^{-1}R \longleftrightarrow \left\{ p \in \operatorname{Spec} R \;\middle|\; p \cap T = \emptyset \right\}$$

and that for any ideal $I \subseteq R$ we have

$$I^e = \left\{ \frac{r}{t} \in T^{-1}R \;\middle|\; r \in I, t \in T \right\}.$$

(1) Show that if $p \in \operatorname{Spec} T^{-1}R$, then $\operatorname{ht} p = \operatorname{ht} p^c$.

(2) Show that if $p \in \operatorname{Spec} R$, then R_p is a local ring, with maximal ideal p^e.

(3) Show that if $p \in \operatorname{Spec} R$, then $\operatorname{ht} p = \dim R_p$.

Exercise 4 [**12 pts**]

Let R be a ring, and consider the following commutative diagram of R-modules, where the rows are exact:

$$\begin{array}{ccccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\ A' & \xrightarrow{f_1'} & B' & \xrightarrow{f_2'} & C' & \xrightarrow{f_3'} & D' \end{array}$$

Show that if a and c are surjective, and d is injective, then b is also surjective.

Exercise 5 [**20 pts**]

Consider the following two $R = \mathbb{R}[x, y]$-modules, where $\mathbb{R}$ denotes the field of real number:

$$M = \mathbb{R}[x,y] \big/ \big(x^2, x^2 + y^2\big), \qquad \text{and} \qquad N = \mathbb{R}[x,y] \big/ \big(x^2, x^2 + y^2 + 1\big)$$

(1) Show that $\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} N = 4$.

(2) Show that $\operatorname{length}_R M = 4$.

(3) Show that $\operatorname{length}_R N = 2$.

Exercise 6 [**26 pts**]

Let F be an algebraically closed field (which is then in particular infinite), and let $R = F[x_1, \dots, x_n]$ for some integer $n \geq 1$.

(1) Show that over an arbitrary ring S two simple modules M and N are isomorphic if and only if $\mathrm{Ann}_S(M) = \mathrm{Ann}_S(N)$.

(2) Show that if M is a non-zero simple R-module then $M \cong R/(x_1 - c_1, \dots, x_n - c_n)$.

(3) Show that if M is finite length R-module, then $\dim_F M = \mathrm{length}_R M$.

(4) Compute the $F[x,y]$-module $\mathrm{Ext}^1_{F[x,y]}\left(F[x,y]/(x,y), F[x,y]/(x,y)\right)$. Give your result in the form of a direct sum of non-zero quotients of $F[x,y]$.

(5) Give non-trivial Yoneda extensions of $F[x,y]$-modules:

$$0 \longrightarrow F[x,y]/(x,y) \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} F[x,y]/(x,y) \longrightarrow 0,$$

indexed by some index set I, such that $M_i \ncong M_j$ as $F[x,y]$-modules for any $i \neq j \in I$. In particular, you should explain why your examples are non-trivial extensions. You get maximal credit if I is infinite, and your arguments are mathematically correct.