Rings and Modules – Final exam

30.01.2018, 8:15-11:15

Your Name ______________________________

This examination booklet contains 6 problems on 20 pages of paper including the front cover and the empty pages.

First sign the booklet above! Do all of your work in this booklet, if you need extra paper, ask the proctors to give you yellow paper, show all relevant computations and justify/explain your answers. The exercises do not require any involved computations or elaborate discussions – try being to the point. Calculators, books, notes, electronic devices etc. are NOT allowed. In particular, please mute the ringer and leave the phone in you bag. You might unstaple the booklet, we are prepared to staple it back. However, it is your responsibility to put the papers in the right order.

Problem	Possible score	Your score
1	13	
2	15	
3	20	
4	20	
5	20	
6	12	
Total	100	

Question 1 [13]

Let R be an integral domain, and let K be its fraction field. Prove the following statements:

(1) if $f \in R$ is a non-zero element, then $\mathrm{Ext}^1_R(R/(f), K) = 0$, [7]
(2) more generally if $f_1, \dots, f_n$ is a sequence of elements such that for every $1 \le i \le n$ the multiplication by f_i is injective on $R/(f_1, \dots, f_{i-1})$ then

$$\mathrm{Ext}^1_R(R/(f_1, \dots, f_n), K) = 0.$$

[6]

Proof. (1) Since f is a non-zero and R is an integral domain the sequence (not exact at the right spot):

$$P^\bullet \colon 0 \to R \xrightarrow{f\cdot} R \to 0$$

is the projective resolution of $R/(f)$. Consequently, we compute $\mathrm{Ext}^1(R/(f), K)$ by taking $\mathrm{Hom}(-, K)$ functor of the $P^\bullet$ and computing cohomology at the first spot – note that the arrows get reversed. Identifying $\mathrm{Hom}(R, K)$ with K (by an isomorphism given by evaluation at $1 \in R$) we see that $\mathrm{Ext}^1(R/(f), K)$ is isomorphic to the cokernel of multiplication by f on K. This is clearly zero, because f is invertible in K, that is, $x = f \cdot f^{-1}x \in K$. In terms of homomorphisms every $\phi \in \mathrm{Hom}(R, K)$ is f-divisible – consider post multiplication by $f^{-1} \in K$.

(2) We reason by induction. The case $n = 1$ is (1). Since $f_n \in R/(f_1, \dots, f_{n-1})$ is a non-zero divisor we obtain a short exact sequence:

$$0 \to R/(f_1, \dots, f_{n-1}) \xrightarrow{f_n\cdot} R/(f_1, \dots, f_{n-1}) \to R/(f_1, \dots, f_{n-1}, f_n) \to 0.$$

By considering corresponding long exact sequence of Ext groups (note that the arrows get reversed) we obtain:

$$\dots \to \mathrm{Hom}(R/(f_1, \dots, f_{n-1}), K) \to \mathrm{Ext}^1(R/(f_1, \dots, f_n), K) \to \mathrm{Ext}^1(R/(f_1, \dots, f_{n-1}), K) \to \dots$$

The right term in the sequence is zero by induction. The left one is zero because $R/(f_1, \dots, f_{n-1})$ is f_1-torsion and K is f_1-divisible (every element is divisible by f_1). Consequently, we see that

$$\mathrm{Ext}^1(R/(f_1, \dots, f_n), K) = 0,$$

because it fits between to zeroes in a long exact sequence.

□

Question 2 [15]

Let k be a field. We set

$$R = k[x, y, z, u]/(xyzu - 1).$$

Recalling the proof of Noether's normalization find an integral extension $S \subset R$ such that S is a polynomial ring in n variables. What is n equal to? [8]

Next, state the going up theorem and use it for the proof of the fact that Krull dimension $\dim R = n$ (you are required to use going up for this). [7]

Proof. As in the proof of Noether's normalization, we change variables by setting $x' = x - u$, $\overline{y} = y - u$ and $\overline{z} = z - u$. It is clear that R is then equal to $K[\overline{x}, \overline{y}, \overline{z}, u]/(\overline{x} + u)(\overline{y} + u)(\overline{z} + u) - 1$. However, the polynomial $(\overline{x} + u)(\overline{y} + u)(\overline{z} + u) - 1$ is monic in u and therefore the R is integral over $K[\overline{x}, \overline{y}, \overline{z}] = K[x - u, y - u, z - u]$. The variables $\overline{x}, \overline{y}, \overline{z}$ are algebraically independent (the only generating relation contains u) and therefore $n = 3$.

For the rest take a look in the lecture notes. □

Question 3 [20]

Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Formulate Nakayama's lemma for R, and then show that every finitely generated flat module F over R is free by proving the following statements. [5]

(1) A homomorphism $M \to N$ of finitely generated R-modules is surjective if and only if the induced map of $R/\mathfrak{m}$ vector spaces $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective, [5]

(2) Let $f_1, \ldots, f_n$ be elements of F such that their images $\overline{f_1}, \ldots, \overline{f_n}$ under the natural map $F \to F/\mathfrak{m}F$ form a basis. Prove that the map $g\colon R^n \to F$ defined by the associations $e_i \mapsto f_i$ is surjective. [5]

(3) Prove that the kernel $K = \{x \in R^n : g(x) = 0\}$ is zero, and hence g is an isomorphism, by considering the exact sequence:

$$0 \to K \to R^n \to F \to 0$$

and the associated long exact sequence of $\mathrm{Tor}^i_R(-, R/\mathfrak{m})$ modules. You may use Nakayama's lemma and the fact that an R-module M is flat if and only if $\mathrm{Tor}^1_R(M, P) = 0$ for every R-module P. [5]

Proof. For the statement of Nakayama's lemma, see the lecture.

(1) We consider the homomorphism $f\colon M \to N$. If f is surjective then $f \otimes R/\mathfrak{m}\colon M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective because

$$f(z \bmod \mathfrak{m}M) = x \bmod \mathfrak{m}N$$

where $f(z) = x$. For the other direction, we consider the cokernel Q of the morphism f. We need to prove that $Q = 0$. By Nakayama's lemma is suffices to show that $Q/\mathfrak{m}Q = 0$, note that the module Q is a quotient of N and is therefore finitely generated. However, using right exactness of the tensor product $\otimes R/\mathfrak{m}$ and identification $P \otimes R/\mathfrak{m} = P/\mathfrak{m}P$ we see that the sequence

$$M/\mathfrak{m}M \to N/\mathfrak{m}N \to Q/\mathfrak{m}Q \to 0$$

is exact which gives the desired claim because the first map is surjective by assumption.

(2) By the choice of elements f_i we see that $g \bmod \mathfrak{m}\colon (R/\mathfrak{m})^n \to F/\mathfrak{m}F$ is surjective and hence (1) applies.

(3) We take a long exact sequence of $\mathrm{Tor}(-, R/\mathfrak{m})$ groups associated with

$$0 \to K \to R^n \to F \to 0.$$

We obtain (note that we use the identification $P \otimes R/\mathfrak{m} = P/\mathfrak{m}P$ again):

$$\ldots \to \mathrm{Tor}^1(F, R/\mathfrak{m}) \to K/\mathfrak{m}K \to R^n/\mathfrak{m}R^n \to F/\mathfrak{m}F \to 0.$$

Since F is flat, by the result from the exercise session we see that $\mathrm{Tor}^1(F, R/\mathfrak{m}) = 0$. Since $R^n/\mathfrak{m}R^n \to F/\mathfrak{m}F$ is an isomorphism (by construction), we therefore see that $K/\mathfrak{m}K = 0$ and hence $K = 0$ by Nakayama's lemma. This implies that g yields a desired isomorphism.

□

1. Question 4 [20]

Let A be a noetherian ring, and let $I = \sqrt{0}$ be its nilpotent radical.

(1) Recall the definition of the Krull dimension, and then prove that the Krull dimensions of the rings A and A/I are equal. In particular, for every ideal J in $k[x_1, \ldots, x_n]$ the dimension of $k[x_1, \ldots, x_n]/J$ equals the dimension of $k[x_1, \ldots, x_n]/\sqrt{J}$. [4]

Let k be a field. For each of the following rings R: compute the nilpotent radical of R, compute the prime ideals of height zero of R, compute the Krull dimension of R.

(2) $R = k[x, y, z]/(xz^3, yz^2)$. [8]
(3) $R = k[x, y, z]/(x^6 + y^6 + z^6)$ (this depends on the characteristic of k). [8]

For (2) you may use the fact that a prime ideal $\mathfrak{p}$ contains an intersection of ideals $I \cap J$ if and only if it contains either I or J.

Recall that the characteristic of the field k is either zero or the smallest prime number p such that $p = 0$ in k. Note that if characteristic is equal to $p > 0$ then $a^p + b^p = (a + b)^p$ for every $a, b \in k$.

Proof. (1) If $\mathfrak{p} \subset R$ is a prime ideal then the height $\mathrm{ht}(\mathfrak{p})$ is defined as the length of the longest strictly descending chain of prime ideals starting with $\mathfrak{p}$. The dimension is the supremum of heights of all prime ideals. For the statement concerning dimensions, we observe that the nil radical is an intersection of all prime ideals, and therefore it is contained in every prime ideal. We conclude using the correspondence between prime ideals in A/I and prime ideals of A containing I. We finish by observing that $\sqrt{0} \in k[x_1, \ldots, x_n]$ is exactly $\sqrt{J}/J$.

(2) We easily observe, by taking third powers, that xz and yz are in the nil radical. Consequently the ideal $(xz, yz) \subset \sqrt{(xz^3, yz^2)}$. However the ideal (xz, yz) is an intersection of prime ideals (z) and (x, y), and is therefore radical ($f^n \in \mathfrak{p}_i \cap \ldots \cap \mathfrak{p}_j$ implies that $f^n \in \mathfrak{p}_j$ for every j, but $\mathfrak{p}_j$ is prime so $f \in \mathfrak{p}_j$). We computed the nil radical. The minimal primes are exactly (z) and (x, y) because $xz \in \mathfrak{p}$ and $yz \in \mathfrak{p}$ implies that either $z \in \mathfrak{p}$ or $x, y \in \mathfrak{p}$. To compute the dimension we see that the longest chain of prime ideals needs to contain at least one minimal one – see the hint, and therefore we need to compute longest chains containing (z) or (x, y). Via standard correspondence those are equivalent to longest chains in $k[x, y, z]/(z) = k[x, y]$ and $k[x, y, z]/(x, y) = k[z]$ respectively. The dimension of $k[x, y]$ is larger and equal to two, and hence this is the dimension of our ring.

(3) First assume that characteristic of k is neither 2 nor 3. Then the polynomial y^6+z^6 have no irreducible factors of degree greater then one (over an algebraic closure $y+iz$ is such factor), and hence by Eisenstein's criterion $x^6+y^6+z^6$ is irreducible and consequently $(x^6+y^6+z^6)$ is prime, and hence R is an integral domain. Since R is an integral domain the minimal prime ideal is zero which is also the nil radical. By the theorem from the lecture, the dimension is equal to the transcendence degree of $\mathrm{Frac}(R)$ which is clearly equal to two – there is a single relations involving all variables and therefore y and z are

algebraically independent. The case of characteristic 2 and 3 is very similar except from relations $x^6 + y^6 + z^6 = (x^3 + y^3 + z^3)^2$ in characteristic two and $x^6 + y^6 + z^6 = (x^2 + y^2 + z^2)^3$ in characteristic three. The nil radicals are given by $(x^3 + y^3 + z^3)$ and $(x^2 + y^2 + z^2)$ respectively. Dimensions can be computed using (1) and the same arguments as before.

$\square$

Question 5 [20]

Prove the following statements.

(1) Let $A \subset B$ be an inclusion of commutative integral domains, and let C be the integral closure of A inside B. Let S be a multiplicately closed set in A. Prove that $S^{-1}C$ is the integral closure of $S^{-1}A$ inside $S^{-1}B$. [6]

(2) Deduce that if A is an integrally closed domain then $S^{-1}A$ is also integrally closed. [4]

(3) Prove that the ring

$$R = k[x, y, z]/\left((x+1)^4 - z(y+z^{2019})^4\right).$$

is a domain. Compute its integral closure, and then find an element $u \in R$ such that the localization R_u is integrally closed. [10]

Proof. (1) First we prove that $S^{-1}A \subset S^{-1}C$ is integral, that is, $S^{-1}C$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$. However, this is clear by the localization statement in the lecture. To see the other inclusion, we take an element b/s, for $b \in B$ and $s \in S$, integral over $S^{-1}A$. By definition b/s satisfies a relation:

$$\left(\frac{b}{s}\right)^n + \sum_{0 \le i \le n-1} \frac{a_i}{s_i}\left(\frac{b}{s}\right)^i = 0$$

Multiplying by $s^n(s_1 \cdots s_{n-1})^n$ we see that

$$(s_1 \cdots s_{n-1}b)^n + \sum_{0 \le i \le n-1} A_i\,(s_1 \cdots s_{n-1}b)^i = 0$$

where $A_i \in A$ (the denominators are cleared). Consequently, since C is the integral closure, we see that $s_1 \cdots s_{n-1}b$ is in C and hence b/s is in $S^{-1}C$.

(2) Clear by the previous item, because $C = A$ by assumptions.

(3) First, we change variables $v = x+1$ and $u = y + z^{2019}$. Then $k[x, y, z] = k[u, v, z]$ and $(x+1)^4 - z(y+z^{2019})^4 = v^4 - zu^4$. In this coordinated, we clearly see that $(v/u)^4 = z$ in the fraction field of $k[u, v, z]/(v^4 - zu^4)$. We therefore consider the ring $S = k[v/u, u]$ generated by v/u and u. It is clearly an integral extension of R, since $u = v/u \cdot u$ and $z = (v/u)^4$. It is also isomorphic to a polynomial ring and hence is the integral closure, because otherwise the transcendent degree of $\mathrm{Frac}(R)$ would be smaller than two, which is not the case. In coordinates u, v, z we see that $R_u = (k[u, v, z]/(v^4 - zu^4))_u = k[u, 1/u, v, z]/((v/u)^4 - z) = k[u, 1/u, v]$. The last ring is a localization of the polynomial ring and is hence integrally closed. This means we need to localize in u to get an integrally closed ring. To finished the proof we just express everything in terms of old variables:

$$S = k[v/u, u] = k[(x+1)/(y+z^{2019}), y + z^{2019}] \qquad u = y + z^{2019}.$$

□

Question 6 [12]

State and prove Hilbert's Basis Theorem.

Proof. See the lecture notes. □