Rings and modules – Final

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Your Name ____

This examination booklet contains 8 problems on 28 sheets of paper including the front cover and the empty sheets.

Do all of your work in this booklet, if you need extra paper, ask the proctors to give you yellow paper, show all your computations and justify/explain your answers. Calculators, books, notes, electronic devices etc. are NOT allowed.

Problem	Possible score	Your score
1	10	
2	12	
3	10	
4	5	
5	18	
6	20	
7	15	
8	10	
Total	100	

By k we always denote an arbitrary field.

Question 1 [10]

Compute $\mathrm{Tor}_i^{\mathbb{Z}}((\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z}(4,6), \mathbb{Z}/m\mathbb{Z})$ for all $i \geq 0$ and all $m \in \mathbb{Z}$.

The functors $\mathrm{Tor}_i^{\mathbb{Z}}(\cdot, \mathbb{Z}/m\mathbb{Z})$ are the derived functors of the functor $\cdot \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$. Hence,

$\mathrm{Tor}_0^{\mathbb{Z}}((\mathbb{Z}\oplus\mathbb{Z})/\mathbb{Z}(4,6), \mathbb{Z}/m\mathbb{Z}) = ((\mathbb{Z}\oplus\mathbb{Z})/\mathbb{Z}(4,6))\otimes_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z} \cong (\mathbb{Z}/m\mathbb{Z}\oplus\mathbb{Z}/m\mathbb{Z})/\mathbb{Z}/m\mathbb{Z}(\overline{4}, \overline{6})$.

The other deriver functors are computed via projective resolutions. A free resolution of $(\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z}(4,6)$ is

$$0 \longrightarrow \mathbb{Z} \stackrel{n \mapsto (4n,6n)}{\longrightarrow} \mathbb{Z} \oplus \mathbb{Z}$$

Tensoring by $\mathbb{Z}/m\mathbb{Z}$ over $\mathbb{Z}$ gives a complex

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \stackrel{n \mapsto (4n,6n)}{\longrightarrow} \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}.$$

Thus,

$$\begin{aligned} \mathrm{Tor}_1^{\mathbb{Z}}((\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z}(4,6), \mathbb{Z}/m\mathbb{Z}) &= \ker(\mathbb{Z}/m\mathbb{Z} \stackrel{n \mapsto (4n,6n)}{\longrightarrow} \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}) \\ &= \{n \in \mathbb{Z} : 4n, 6n \in m\mathbb{Z}\}/m\mathbb{Z} = \begin{cases} \frac{m}{2}\mathbb{Z}/m\mathbb{Z} & \text{if } 2 \mid m; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Also $\mathrm{Tor}_i^{\mathbb{Z}}((\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z}(4,6), \mathbb{Z}/m\mathbb{Z}) = 0$ for $i \geq 2$.

Question 2 [12]

Let R be a commutative ring which is an integral domain. Let M be a finitely generated R-module. Let $Tor(M)$ be the torsion submodule of M.

(a) Show that $\mathrm{Tor}_1^R(R/rR, M)$ is the r-torsion submodule of M for all $r \in R$. [6]

(b) Show that if M is flat then $Tor(M) = 0$. [3]

(c) Show that if R is a PID, then M is flat if and only if $Tor(M) = 0$. [3]

(a) Since R is an integral domain, the sequence

$$0 \longrightarrow R \xrightarrow{r\cdot} R \longrightarrow R/rR \longrightarrow 0$$

is exact. Hence, the complex

$$0 \longrightarrow R \xrightarrow{r\cdot} R$$

is a free resolution of R/rR. Then

$$\mathrm{Tor}_1^R(R/rR, M) = \ker(R \otimes_R M \xrightarrow{r\cdot} R \otimes_R M)/\mathrm{im}(0 \otimes_R M \to R \otimes_R M) = \ker(M \xrightarrow{r\cdot} M).$$

(b) If M is flat, then $\mathrm{Tor}_1^R(N, M) = 0$ for every R-module N. Thus in particular the r-torsion submodule $Tor_r(M)$ of M is 0 for all $r \in R$. Then $Tor(M) = \bigcup_{r \in R} Tor_r(M) = 0$.

(c) If R is a PID, then $M \cong R^n \oplus Tor(M)$ by the structure theorem for finitely generated modules over a PID. Thus if $Tor(M) = 0$, then M is a free module and hence flat.

Question 3 [10]

Let e_1, e_2, e_3 be a basis of the free $\mathbb{Z}$-module $\mathbb{Z}^3$. Consider the submodule N generated by $7e_1 - 3e_2$, $3e_1 - e_2 + 4e_3$, $6e_1 - 2e_2 + 2e_3$. Compute the Smith normal form of the following matrix with entries in $\mathbb{Z}$:

$$A = \begin{pmatrix} 7 & 3 & 6 \\ -3 & -1 & -2 \\ 0 & 4 & 2 \end{pmatrix}$$

Then put the $\mathbb{Z}$-module $M = \mathbb{Z}^3/N$ into the form described by the structure theorem for finitely generated modules over a PID.

We put the given matrix into Smith normal form using row and column operations:

$$\begin{pmatrix} 7 & 3 & 6 \\ -3 & -1 & -2 \\ 0 & 4 & 2 \end{pmatrix} \to \begin{pmatrix} 3 & 7 & 6 \\ -1 & -3 & -2 \\ 4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & -3 & -2 \\ 3 & 7 & 6 \\ 4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & -2 \\ -3 & 7 & 6 \\ -4 & 0 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -3 & -2 \\ 0 & -2 & 0 \\ 0 & -12 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & -12 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Let $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the linear map given by A. The image of f is equal to the kernel of the natural projection $\mathbb{Z}^3 \to M$. By changing bases of the two copies of $\mathbb{Z}^3$ so that the matrix of M is in Smith normal form, we have found generators of the module M which exhibit it as

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}.$$

Since $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ by the Chinese reminder theorem, the canonical form of M according to the structure theorem for finitely generated modules over a PID is (up to reordering the summands)

$$M \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}.$$

Question 4 [5]

Let R be a commutative ring. Let T be a multiplicatively closed subset of R. Show that if R is noetherian, then $T^{-1}R$ is noetherian.

The ring $T^{-1}R$ is noetherian if and only if it is noetherian as $T^{-1}R$-module, if and only if all its submodules are finitely generated. The submodules of $T^{-1}R$ as $T^{-1}R$-module are the ideals of $T^{-1}R$. Let I be an ideal of $T^{-1}R$. We recall that $I = I^{ce}$ under the morphism $\iota : R \to T^{-1}R$ given by the universal property of localization. Since R is noetherian, the ideal I^c is finitely generated. Then $I^{ce} = \iota(I^c)T^{-1}R$ is finitely generated as an ideal of $T^{-1}R$.

Question 5 [18]

Let k be a field. For each of the following rings R compute the nilpotent radical of R, compute a minimal primary decomposition of (0), compute the prime ideals of height 0 of R, compute the Krull dimension of R.

(a) $R = k[x, y]/(x^2y, xy^2)$. [6]
(b) $R = k[x, y, z]/(x^4 + y^4 + zy)$. [6]
(c) $R = k[x]/(x^2 + 1)$ (the answer should depend on the field). [6]

(a) Let J be the radical of the ideal $I = (x^2y, xy^2)$ of $k[x, y]$. Then $xy \in J$, $I \subseteq (xy) \subseteq J$ and $\mathrm{rad}(xy) = (xy)$ (the last holds because if $xy \mid f^n$ for some $f \in k[x, y]$, then $xy \mid f$ using the fact that $k[x, y]$ is a UFD and x, y are irreducible elements of $k[x, y]$). So $J = (xy)$, and the nilpotent radical of R is the principal ideal $\overline{xy}R$. We observe that $I = (xy) \cap (x^2, y^2) = (x) \cap (y) \cap (x^2, y^2)$. The ideals (x) and (y) are prime and the ideal (x^2, y^2) is primary because its radical is the maximal ideal (x, y). This decomposition is minimal because $(y) \cap (x^2, y^2) = (x^2y, y^2) \nsubseteq (x)$, $(x) \cap (x^2, y^2) = (x^2, xy^2) \nsubseteq (y)$ and $(x) \cap (y) = (xy) \nsubseteq (x^2, y^2)$. Thus $(0) = (\overline{x}) \cap (\overline{y}) \cap (\overline{x}^2, \overline{y}^2)$ is a minimal primary decomposition of (0) in R. Any prime ideal of R contains the nilpotent radical of R. In particular it contains $\overline{x}$ or $\overline{y}$. Thus the prime ideals of height 0 in R are $(\overline{x})$ and $(\overline{y})$, and R has Krull dimension 1 as $(\overline{x}) \subseteq (\overline{x}, \overline{y})$ is a maximal chain of prime ideals.

(b) We observe that the polynomial $x^4 + y^4 + zy = x^4 + y(y^3 + z)$ is irreducible by the Eisenstein criterion applied $k[y, z][x]$ with the prime y. Then R is an integral domain, hence (0) is the nilpotent radical of R, it is the only prime ideal of height 0, and (0) is also a minimal primary decomposition of (0). The Krull dimension of R is computed by $\mathrm{trdeg}_k \mathrm{Frac}(R)$. We observe that $\mathrm{Frac}(R) = k(y, z)[x]/(x^4 + y^4 + zy)$ is an algebraic extension of $k(x, y)$ and hence has the same transcendence degree, which is 2. So $\dim R = 2$.

(c) If k has characteristic 2, then $x^2+1 = (x+1)^2$ in $k[x]$, so $\mathrm{rad}(x^2+1) = (x+1)$ and the nilpotent radical of R is $(\overline{x+1})$, which is a maximal ideal of R. Then (0) is a primary ideal of R and also the mimimal primary decomposition of (0). Every prime ideal of R contains the nilpotent radical, hence $(\overline{x+1})$ is the only prime ideal of R, it has height 0, and $\dim R = 0$.

If k has not characteristic 2, and does not contain a square root of -1, then $x^2 + 1$ is an irreducible polynomial in $k[x]$, and R is a field. Then the nilpotent radical of R is (0), it is also the minimal primary decomposition of (0) and the only prime ideal of height 0. The Krull dimension of R is 0.

If k has not characteristic 2 and contains a square root ζ of -1, then $x^2 + 1 = (x + \zeta)(x - \zeta)$ is a decomposition in irreducible factors in $k[x]$. We observe that the ideal $(x^2 + 1) = (x + \zeta) \cap (x - \zeta)$ of $k[x]$ equals its radical because it is an intersection of prime ideals. Thus the nilpotent radical of R is (0), a minimal primary decomposition of (0) is $(0) = (\overline{x + \zeta}) \cap (\overline{x - \zeta})$. The prime ideals of height 0 are $(\overline{x + \zeta})$ and $(\overline{x - \zeta})$, they are maximal ideals, so $\dim R = 0$.

QUESTION 6 [20]

(a) Let R be a commutative integral domain. Let $T \subseteq R$ be a multiplicatively closed subset. Prove that if R is integrally closed, then $T^{-1}R$ is integrally closed. [5]

(b) Show that the following rings are integral domains and compute their integral closure:

(i) $\mathbb{R}[x,y]/(xy-1)$ [5]

(ii) $\mathbb{Q}[x,y,z]/((x+y)^2 - yz^4)$ [10]

(a) Let α be an element of $Frac(T^{-1}R) = Frac(R)$ that satisfies an integral relation

$$\alpha^n + \sum_{i=0}^{n-1} \frac{a_i}{t_i}\alpha^i = 0$$

with $a_i \in R$ and $t_i \in T$ for all $i = 0, \dots, n-1$. Let $t = \prod_{i=0}^{n-1} t_i$. We multiply the equation by t^n to obtain

$$0 = t^n \left(\alpha^n + \sum_{i=0}^{n-1} \frac{a_i}{t_i}\alpha^i \right) = (t\alpha)^n + \sum_{i=0}^{n-1} b_i (t\alpha)^i$$

with $b_i = a_i t^{n-1-i} \prod_{\substack{0 \le j \le n-1 \\ j \ne i}} t_i \in R$. Since R is integrally closed, we have $t\alpha \in R$. Then $\alpha = \frac{t\alpha}{t} \in T^{-1}R$.

(b) (i) $\mathbb{R}[x,y]/(xy-1) \cong \mathbb{R}[x]_x$ is an integral domain because it is a localization of an integral domain. Since $\mathbb{R}[x]$ is a PID it is a UFD and hence integrally closed. Then $\mathbb{R}[x,y]/(xy-1) \cong \mathbb{R}[x]_x$ is integrally closed by previous exercise because it is a localization of $\mathbb{R}[x]$. Alternatively, $\mathbb{R}[x,y]/(xy-1)$ is a PID because it is the localization of a PID. Hence it is integrally closed.

(ii) Let $R = \mathbb{Q}[x,y,z]/((x+y)^2 - yz^4)$. We observe that $R = \mathbb{Q}[y,z][x]/((x+y)^2 - yz^4)$ is an integral domain because $\mathbb{Q}[y,z]$ is a UFD and the polynomial $(x+y)^2 - yz^4 = x^2 + 2yx + y(y-z^4)$ is irreducible by the Eisenstein criterion with the prime y. We observe that $t := \frac{\overline{x+y}}{\overline{z}^2} \in Frac(R)$ satisfies the integral relation $t^2 - \overline{y} = 0$ over R. Therefore the subring $S := \mathbb{Q}[\overline{x}, \overline{y}, \overline{z}, t] \subseteq Frac(R)$ is an integral extension of R. We observe that $S = \mathbb{Q}[\overline{z}, t]$ because $\overline{y} = t^2$ and $\overline{x} = \overline{z}^2 t - t^2$ in S. Since $R \subseteq S \subseteq Frac(R)$, we have $Frac(S) = Frac(R)$. Since $\operatorname{trdeg}_{\mathbb{Q}} Frac(R) \ge 2$, the two elements $t, \overline{y}$ are algebraically independent over $\mathbb{Q}$. Then S is isomorphic to a polynomial ring in two variables over $\mathbb{Q}$. In particular, S is a UFD, and hence integrally closed in $Frac(S) = Frac(R)$. So S is the integral closure of R in $Frac(R)$.

Question 7 [15]

Let R be a commutative ring containing a multiplicatively closed subset T. Endow the R-module $R \oplus R$ with the coordinatewise multiplication $(r_1, r_2) \cdot (r_1', r_2') := (r_1 r_1', r_2 r_2')$ for all $r_1, r_1', r_2, r_2' \in R$. Show that the subset $T \oplus T \subseteq R \oplus R$ is a multiplicatively closed subset of $R \oplus R$. Then prove, using the universal property of localisation, that there is an isomorphism of rings $(T \oplus T)^{-1}(R \oplus R) \cong (T^{-1}R) \oplus (T^{-1}R)$.

As a set $T \oplus T = \{(t_1, t_2) \in R \oplus R : t_1, t_2 \in T\}$. We observe that the neutral element $(1, 1)$ of the multiplication of $R \oplus R$ belongs to $T \oplus T$ because $1 \in T$. Moreover, if $(t_1, t_2), (t_1', t_2') \in T \oplus T$, then $(t_1, t_2) \cdot (t_1', t_2') = (t_1 t_1', t_2 t_2') \in T \oplus T$, because $t_1 t_1', t_2 t_2' \in T$, as T is a multiplicatively closed subset of R.

Let $\iota : R \to T^{-1}R$ be the morphism associated to the localization $T^{-1}R$. Let $\iota \oplus \iota : R \oplus R \to (T^{-1}R) \oplus (T^{-1}R)$ be the induced morphism (i.e., the morphism that sends (r_1, r_2) to $(\iota(r_1), \iota(r_2))$ for all $r_1, r_2 \in R$). We show that $((T^{-1}R) \oplus (T^{-1}R), \iota \oplus \iota)$ satisfies the universal property that defines $(T \oplus T)^{-1}(R \oplus R)$.

Assume that $f : R \oplus R \to S$ is a ring homomorphism such that $f(T \oplus T) \subseteq S^\times$. Any morphism $g : (T^{-1}R) \oplus (T^{-1}R) \to S$ such that $g \circ (\iota \oplus \iota) = f$ must satisfy

$$f((t_1, t_2)) g\left(\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right)\right) = g(\iota(t_1), \iota(t_2)) g\left(\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right)\right) = g((\iota(r_1), \iota(r_2)) = f((r_1, r_2))$$

for all $r_1, r_2 \in R$ and all $t_1, t_2 \in T$. The equalities take place in S. Since $f((t_1, t_2)) \in S^\times$ by assumption, we conclude that if g exists it is unique, because it must satisfy $g\left(\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right)\right) = f((t_1, t_2))^{-1} f((r_1, r_2))$.

Let us define g by $g\left(\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right)\right) := f((t_1, t_2))^{-1} f((r_1, r_2))$ for all $r_1, r_2 \in R$ and $t_1, t_2 \in T$. It remains to show that g is well defined and a ring homomorphism. If $\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right) = \left(\frac{r_1'}{t_1'}, \frac{r_2'}{t_2'}\right) \in (T^{-1}R) \oplus (T^{-1}R)$, then there exist $u_1, u_2' \in T$ such that $u_1(t_1' r_1 - t_1 r_1') = u_2(t_2 r_2' - t_2' r_2) = 0$. Then

$$\begin{aligned} f((t_1, t_2))^{-1} f((r_1, r_2)) &= f((r_1, r_2)) f((t_1' u_1, t_2' u_2)) f((t_1 t_1' u_1, t_2 t_2' u_2))^{-1} \\ &= f((r_1', r_2')) f((t_1 u_1, t_2 u_2)) f((t_1 t_1' u_1, t_2 t_2' u_2))^{-1} = f((t_1', t_2'))^{-1} f((r_1', r_2')). \end{aligned}$$

So g is well defined. Also $g\left(\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right)\left(\frac{r_1'}{t_1'}, \frac{r_2'}{t_2'}\right)\right) = g\left(\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right)\right) g\left(\left(\frac{r_1'}{t_1'}, \frac{r_2'}{t_2'}\right)\right)$ and $g\left(\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right) + \left(\frac{r_1'}{t_1'}, \frac{r_2'}{t_2'}\right)\right) = g\left(\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right)\right) + g\left(\left(\frac{r_1'}{t_1'}, \frac{r_2'}{t_2'}\right)\right)$ for all $\left(\frac{r_1}{t_1}, \frac{r_2}{t_2}\right), \left(\frac{r_1'}{t_1'}, \frac{r_2'}{t_2'}\right) \in (T^{-1}R) \oplus (T^{-1}R)$, because $f((t_1 t_1', t_2 t_2'))^{-1} f((r_1 r_1', r_2 r_2')) = f((t_1, t_2))^{-1} f((r_1, r_2)) f((t_1', t_2'))^{-1} f((r_1', r_2'))$ and $f((t_1 t_1', t_2 t_2'))^{-1} f((t_1' r_1 + t_1 r_1', t_2' r_2 + t_2 r_2')) = f((t_1, t_2))^{-1} f((r_1, r_2)) + f((t_1', t_2'))^{-1} f((r_1', r_2'))$ as f is a ring homomorphism.

Question 8 [10]

State and prove the Going-Up Theorem. You can use without proof all the preliminary lemmas and propositions we proved before the actual proof of the Going-Up Theorem.

Statement.

Let $S \to R$ be an integral extension.

(1) If $p \subseteq S$ is a prime ideal, then there is a prime ideal $q \subseteq R$, such that $q \cap S = p$.

Addendum: if there are prime ideal $p' \subsetneq p \subseteq S$ and $q' \subseteq R$, such that $q' \cap S = p'$, then we may choose q such that $q' \subseteq q$.

(2) Let $q \subsetneq q' \subseteq R$ be prime ideals. Then $q \cap S \neq q' \cap S$.

Proof.

(1) Choose $p \subseteq S$ prime. Then
  (a) $S_p \to R_p$ is an integral extension.
  (b) S_p is a local ring with maximal ideal $m := pS_p$.

Choose now, any maximal ideal n of R_p. Then $n \cap S_p$ is a maximal ideal because $S_p \to R_p$ is an integral extension, hence $n \cap S_p$ is necessarily m. Define q then to be the contraction of n along $R \to R_p$. We have the following commutative diagram:

$$\begin{array}{ccc} S & \overset{\iota}{\hookrightarrow} & R \\ \downarrow{\scriptstyle j_S} & & \downarrow{\scriptstyle j_R} \\ S_p & \overset{\iota_p}{\hookrightarrow} & R_p \end{array}$$

Using the notations of the diagram:

$$q \cap S = \iota^{-1} q = \iota^{-1} j_R^{-1} n = j_S^{-1} \iota_p^{-1} n = j_S^{-1} m = p,$$

where we used the correspondence between prime ideals under localization in the last step.

For the addendum, just note that $q' R_p$ is a proper ideal such that $j_R^{-1}(q' R_p) = q'$ (we are using that $q' \cap S = p' \subseteq p$ and hence $q' \cap (S \setminus p) = \emptyset$). Hence, we may pick n to contain $q' R_p$, and hence, $q := j_R^{-1}(n)$ contains q'.

(2) Assume the contrary, that is, $p := q \cap S = q' \cap S$. Perform then the same localization construction as in the diagram. As above, $qR_p \subsetneq q' R_p$ are proper prime ideals, as they avoid $S \setminus p$, and hence their contraction in R is q and q' respectively. Also, their contraction in S_p are prime ideals that contract to p. Hence, these two contractions are equal:

$$pS_p = S_p \cap q = S_p \cap q'.$$

Hence, using that pS_p is maximal and $S_p \to R_p$ is an integral extension, both qR_p and $q' R_p$ are maximal, which is a contradiction.