EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 13 - Solutions	

Exercise 1. Let R be a Noetherian ring. Show that R has only finitely many minimal prime ideals.

Hint: Reformulate this statement into a topological one by using spectra

Proof. Recall that given a radical ideal I, $V(I)$ is irreducible if and only if I is prime. Hence, the minimal primes of R correspond via Nullstellensatz (for general rings, so the much easier statement) to irreducible components of $\operatorname{Spec}(R)$. Since this is a Noetherian topological space, it has finitely many irreducible components, so we are done. □

Exercise 2. Let F be a field and let R be a ring, let $I = (f) \subseteq F[x]$ be a principal ideal, and let $\phi : F[x] \to R$ be a ring morphism. If we speak of extensions and contractions of ideals in this exercise, they are always understood to be with respect to ϕ. Let g be a generator of the ideal $I^{ec} \subseteq F[x]$, and note that g is uniquely defined up to multiplication by a unit. Give a formula for g in terms of the prime factors of f when ϕ is

(1) the localization $F[x] \to F[x]_x$.
(2) the localization $F[x] \to F[x]_{(x)}$ (i.e. localization at the prime ideal $(x) \subseteq F[x]$).

Additionally, characterize in both casers when $I^{ec} = I$, in terms of the prime factors of f.

Proof. If $f = 0$ we have $g = 0$ in both cases, so suppose $f \neq 0$. Write $f = x^n f_0$ where $f_0 \in F[x] \setminus \{0\}$ is such that x doesn't divide f_0 and $n \in \mathbb{Z}_{\geq 0}$.

(1) Using point (2) of Proposition 9.3.8 of the printed course notes we have
$$\begin{aligned} I^{ec} = \bigcup_{m\geq 0}(I : x^m) &= \{r \in F[x] \text{ such that } \exists m \geq 0 : \ x^m r \in I\} \\ &= \{r \in F[x] \text{ such that } \exists m \geq 0 : \ x^n f_0 \mid x^m r\} \\ &= \{r \in F[x] \text{ such that } f_0 \mid r\} = (f_0). \end{aligned}$$
Hence $g = f_0$, and thus $I^{ec} = I$ if and only if $f = 0$ or x doesn't divide f, i.e. $f(0) \neq 0$.

(2) Using point (2) of Proposition 9.3.8 of the printed course notes we have
$$\begin{aligned} I^{ec} = \bigcup_{h\notin (x)}(I : h) &= \{r \in F[x] \text{ such that } \exists h \notin (x) : \ hr \in I\} \\ &= \{r \in F[x] \text{ such that } \exists h \notin (x) : \ x^n f_0 \mid hr\} \\ &= \{r \in F[x] \text{ such that } x^n \mid r\} \end{aligned}$$
where for the last equality we used that as $x^n | hr$ and $h \notin (x)$ we have $x^n | r$ and if $x^n | r$ then we can take $h = f_0$ to obtain $x^n f_0 | f_0 r$. Hence $I^{ec} = (x^n)$, i.e. $g = x^n$. In particular, we have $I^{ec} = I$ if and only if f is of the form $f = \lambda x^n$ for $\lambda \in F$ and $n \geq 0$.

□

Exercise 3. If $S \subseteq R$ is a ring extension and $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of S resp. R, respectively, we say that $\mathfrak{q}$ lies above $\mathfrak{p}$ if and only if $\mathfrak{q}^c = \mathfrak{p}$. Show the following:

(1) Let R be a UFD. Then an ideal $\mathfrak{p} \subseteq R$ is a prime ideal of height 1 if and only there exists an irreducible element $f \in R$ such that $\mathfrak{p} = (f)$.

(2) If $S \subseteq R$ is an integral extension and $\mathfrak{p} \subseteq S$ is a prime ideal, then all prime ideals lying over $\mathfrak{p}$ have height at most that of $\mathfrak{p}$, with equality for at least one of them.
[*Hint:* Localize at $\mathfrak{p}$.]

(3) If $S \subseteq R$ is an integral extension of domains, then all primes of R lying over height 1 primes of S are of height 1.

(4) The ideal $\mathfrak{p} = (x^2 + y^2 + 1) \subseteq \mathbb{C}[x^2, y^2]$ is a height 1 prime, and there is a single prime in $\mathbb{C}[x, y]$ lying over it.

Proof. (1) Let f be an irreducible element of R. Then if $ab \in (f)$ for some $a, b \in R$ we have that f divides ab, and thus f must appear in the irreducible factor decomposition of either a or b. That is, either $a \in (f)$ or $b \in (f)$, and thus (f) is prime.
Now suppose that $\mathfrak{p} \subseteq R$ is a prime of height 1. In particular $\mathfrak{p} \neq (0)$, so let $r \in \mathfrak{p}$ be non-zero. As $\mathfrak{p}$ is prime, there must be an irreducible factor f of r such that $f \in \mathfrak{p}$. But then $(0) \subsetneq (f) \subseteq \mathfrak{p}$, so as $\mathfrak{p}$ is of height 1 and $(0), (f)$ are prime, we must have $\mathfrak{p} = (f)$.
Finally, if $f \in R$ is irreducible and by contradiction we have a chain $(0) \subsetneq \mathfrak{q} \subsetneq (f)$ with $\mathfrak{q}$ a prime ideal. Take some non-zero $s_0 \in \mathfrak{q}$. Then f divides s_0, so there is $s_1 \in R$ with $s_0 = fs_1$. As $f \notin \mathfrak{q}$, this implies $s_1 \in \mathfrak{q}$. Repeating this argument, we obtain a sequence of elements $(s_i)_i$ of $\mathfrak{q}$ such that $s_i = fs_{i+1}$, and thus f^i divides s_0 for every $i \geq 0$. This is a contradiction, so (f) must have height 1.

(2) Let $\mathfrak{q}$ be a prime of R lying over $\mathfrak{p}$. Let $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n = \mathfrak{q}$ be a strictly increasing chain of prime ideals of R. Then by point (2) of the Going-Up Theorem (Proposition 9.4.2 of the printed course notes) $\mathfrak{q}_0 \cap S \subsetneq \cdots \subsetneq \mathfrak{q}_n \cap S = \mathfrak{p}$ is a strictly increasing chain of prime ideals of S, and thus $n \leq \operatorname{ht} \mathfrak{p}$. Thus we conclude $\operatorname{ht} \mathfrak{q} \leq \operatorname{ht} \mathfrak{p}$.
To construct a prime ideal where we have equality, as in the proof of Proposition 9.4.2 denote $R_{\mathfrak{p}} := (S \setminus \mathfrak{p})^{-1} R$, and observe that $S_{\mathfrak{p}} \to R_{\mathfrak{p}}$ is integral. Hence by Corollary 9.4.4 in the printed course notes we have $\dim R_{\mathfrak{p}} = \dim S_{\mathfrak{p}}$, and by point (2) of Exercise 5 on Sheet 12 we have $\dim S_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p}$. Therefore, there exists a maximal ideal $\mathfrak{n}$ of $R_{\mathfrak{p}}$ such that $\operatorname{ht} \mathfrak{n} = \operatorname{ht} \mathfrak{p}$. Just as in the proof of Proposition 9.4.2, if $\mathfrak{q}$ denotes the contraction of $\mathfrak{n}$ under $R \to R_{\mathfrak{p}}$, then $\mathfrak{q}$ lies over $\mathfrak{p}$. But then by point (1) of Exercise 5 on Sheet 12 we have $\operatorname{ht} \mathfrak{n} = \operatorname{ht} \mathfrak{q}$ and thus $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$ with same height as $\mathfrak{p}$.

(3) Let $\mathfrak{p} \subseteq S$ be a prime of height 1 and let $\mathfrak{q} \subseteq R$ be a prime lying over S. By the previous point, we have $\operatorname{ht} \mathfrak{q} \leq 1$. If by contradiction $\operatorname{ht} \mathfrak{q} = 0$, then as R is a domain we must have $\mathfrak{q} = 0$, and thus also $\mathfrak{p} = 0$, which contradicts $\operatorname{ht} \mathfrak{p} = 1$. Hence $\operatorname{ht} \mathfrak{q} = 1$.

(4) As $\mathbb{C}[x^2, y^2] \cong \mathbb{C}[u, v]$, it is a UFD. Notice also that $\mathbb{C}[x^2, y^2] \subseteq \mathbb{C}[x, y]$ is an integral extension, as x, y are integral over $\mathbb{C}[x^2, y^2]$.
First of all, notice that $x^2 + y^2 + 1$ is an irreducible element of $\mathbb{C}[x^2, y^2]$, and thus by point (1) it is a prime of height 1. Let $\mathfrak{q} \subseteq \mathbb{C}[x, y]$ be a prime lying over $\mathfrak{p}$, which exists by Going-Up. But now notice that $x^2 + y^2 + 1$ is also irreducible in $\mathbb{C}[x, y]$, by seeing it as an element of $\mathbb{C}[x][y]$ and applying Eisenstein's criterion with the prime element $x + i$. Thus $(x^2 + y^2 + 1) \cdot \mathbb{C}[x, y]$ is a prime contained inside $\mathfrak{q}$, and as the latter is of height 1, we must have $\mathfrak{q} = (x^2 + y^2 + 1) \cdot \mathbb{C}[x, y]$. This is clearly a prime of height 1, and it lays over $\mathfrak{p}$: indeed, if $f \in \mathbb{C}[x, y]$ is such that $(x^2 + y^2 + 1) f \in \mathbb{C}[x^2, y^2]$, then f can't contain a monomial of the form $x^i y^j$ with at least one of i, j being odd,

because if we take such i, j with $i + j$ minimal then $x^i y^j$ also appears in $x^2 + y^2 + 1$, contradiction. So $\mathfrak{q} = (x^2 + y^2 + 1) \cdot \mathbb{C}[x, y]$ is the only prime of height 1 lying over $\mathfrak{p}$. □

Exercise 4. Let R be a ring which is the quotient of a polynomial ring over an algebraically closed field F by a radical ideal. This naturally determines an algebraic set X whose coordinate ring is R. Noether normalisation says there is a subring $S \subseteq R$ such that $S \cong F[t_1, ..., t_r]$ and R is an integral extension of S. Give a geometric interpretation of Noether normalisation. That is, the inclusion $S \to R$ corresponds to a morphism f of algebraic sets. Prove that the fibres of f are finite, i.e. the preimage of any point in F^r under f consists of a finite set of points in X.

Proof. Recall that if for two algebraic sets $X \subseteq F^m$ and $Y \subseteq F^n$ we have an F-algebra morphism $\lambda : A(Y) \to A(X)$ then this determines a morphism of algebraic sets $f : X \to Y$ such that $\lambda = \lambda_f$. Following the hint and using the same notations as in the solution to Exercise 5, let $\overline{\mathfrak{m}}_P$ be a maximal ideal of $A(X)$ (where $P = (a_1, \dots, a_m) \in X$). Let $h_1, \dots, h_n \in F[x_1, \dots, x_m]$ be such that $\lambda(y_j + I(Y)) = h_j + I(X)$ for all j. Let $\phi : F[y_1, \dots, y_n] \to F[x_1, \dots, x_m]$ be the F-algebra morphism defined by mapping y_j to h_j, and let $\pi_X : F[x_1, \dots, x_m] \to A(X)$ and $\pi_Y : F[y_1, \dots, y_m] \to A(Y)$ be the projection maps. Then by Exercise 4 we have $\pi_X \circ \phi = \lambda \circ \pi_Y$. Therefore

$$\pi_Y^{-1}(\lambda^{-1}(\overline{\mathfrak{m}}_P)) = \phi^{-1}(\pi_X^{-1}(\overline{\mathfrak{m}}_P)) = \phi^{-1}(\mathfrak{m}_P).$$

Now by construction we have $\phi(y_j - f(P)_j) = h_j - h_j(P)$ and thus evaluating $\phi(y_j - f(P)_j)$ at P gives 0. Hence $y_j - f(P)_j \in \phi^{-1}(\mathfrak{m}_P)$ for all j, and thus $\mathfrak{n}_{f(P)} := (y_1 - f(P)_1, \dots, y_n - f(P)_n) \subseteq \phi^{-1}(\mathfrak{m}_P)$. As $\mathfrak{n}_{f(P)}$ is maximal and $1 \notin \phi^{-1}(\mathfrak{m}_P)$, we thus have

$$\mathfrak{n}_{f(P)} = \phi^{-1}(\mathfrak{m}_P) = \pi_Y^{-1}(\lambda^{-1}(\overline{\mathfrak{m}}_P)).$$

Applying π_Y on both sides this gives

$$\overline{\mathfrak{n}}_{f(P)} = \lambda^{-1}(\overline{\mathfrak{m}}_P).$$

This expresses how one can obtain $f : X \to Y$ from $\lambda : A(Y) \to A(X)$ in terms of maximal ideals.

Now we are ready to tackle the Exercise. Let $\lambda : S \hookrightarrow R$ be the inclusion. By Exercise 7 of sheet 7, the algebraic sets determined by S and R can be identified with MaxSpec(S) resp. MaxSpec(R), and by the paragraph above λ determines a morphism of algebraic sets $f : \text{MaxSpec}(R) \to \text{MaxSpec}(S) \cong F^r$ given by $\mathfrak{m} \mapsto \lambda^{-1}\mathfrak{m} = \mathfrak{m} \cap S$. So to show that f has finite fibers, we need to show that for every maximal ideal $\mathfrak{n} \subseteq S$, there exist at most finitely many maximal ideals $\mathfrak{m}$ of R such that $\mathfrak{m} \cap S = \mathfrak{n}$. Any such $\mathfrak{m}$ contains $\mathfrak{n}^e = R \cdot \mathfrak{n}$, so we may suppose that the latter is non-trivial, and then the maximal ideals $\mathfrak{m} \subseteq R$ with $\mathfrak{m} \cap S = \mathfrak{n}$ are in one-to-one correspondence with the maximal ideals of $R/\mathfrak{n}^e$. Note that λ gives rise to a map $\overline{\lambda} : S/\mathfrak{n} \to R/\mathfrak{n}^e$. Furthermore, we have $S/\mathfrak{n} \cong F$ (by sending a scalar α to its class $\alpha + \mathfrak{n}$), and as the target ring is non-trivial we must have $\ker \overline{\lambda} = 0$. Hence, under the identification $S/\mathfrak{n} \cong F$, we have that $\overline{\lambda}$ is just the natural inclusion of F into $R/\mathfrak{n}^e$ (as R is a quotient of a polynomial ring over F, $R/\mathfrak{n}^e$ is too, and thus there is a natural inclusion $F \to R/\mathfrak{n}^e$). On the other hand, as λ is an integral extension, $\overline{\lambda}$ is too. Indeed, if $r + \mathfrak{n}^e \in R/\mathfrak{n}^e$ then there is a monic polynomial $T^d + s_{d-1}T^{d-1} + \cdots + s_0 \in S[T]$

annihilating r, and thus $T^d + (s_{d-1} + \mathfrak{n})T^{d-1} + \cdots + (s_0 + \mathfrak{n}) \in ({S}/{\mathfrak{n}})[T]$ is a monic polynomial annihilating $r + \mathfrak{n}^e$. In conclusion, ${R}/{\mathfrak{n}^e}$ is a finitely generated F-algebra which is integral over F. Let $g_1, \dots, g_l$ be generators of ${R}/{\mathfrak{n}^e}$, i.e. ${R}/{\mathfrak{n}^e} = F[g_1, \dots, g_l]$. Now let $N \in \mathbb{Z}_{>0}$ be such that for every g_i there exists a monic polynomial in $F[T]$ annihilating it. Then every power of g_i can be written as an F-linear combination of $1, g_i, \dots, g_i^{N-1}$. Hence every element of ${R}/{\mathfrak{n}^e} = F[g_1, \dots, g_l]$ can be written as an F-linear combination of $\{g_1^{c_1} \cdots g_l^{c_l} \mid c_1, \dots, c_l \in \{0, \dots, N-1\}\}$. In partiular, ${R}/{\mathfrak{n}^e}$ is finite as an F-vector space, so in particular Artinian. So by Exercise 7 on Sheet 9, ${R}/{\mathfrak{n}^e}$ has only finitely many maximal ideals, and hence the fiber $f^{-1}(\mathfrak{n})$ is finite.

Alternative approach, following the proof of Noether Normalization S is itself a polynomial ring, so it is the co-ordinate ring of the algebraic set F^r. Thus by the previous Question, the inclusion $S \to R$ corresponds to a morphism $f : X \to F^r$.
To show that the fibres (i.e. the set of pre-images of a point) are finite, use the notation of the proof of Noether normalisation for an infinite field as in the lecture notes. That is, we use induction on the number of variables n such that R is a quotient of a polynomial ring in n variables to prove that there exists a poynomial ring $S \subset R$ over which R is integral and such that the induced morphism of algebraic sets has finite fibers. Hence, we only need to modify the proof in the lecture notes slightly. For $n = 1$ the statement is clear since the algebraic set X in this case is the finite set of roots of the polynomial f. Let X' be the algebraic set determined by the ring R' as a quotient of $F[x_1 - c_1x_n, ..., x_{n-1} - c_{n-1}x_n]$ (notation as in the lecture notes). If we show that the fibres of $X \to X'$ are finite then we are done by induction. Suppose $P = (p_1, ..., p_{n-1}) \in X' \subset F^{n-1}$. Then we wish to show that the set $\Lambda = \{x \in F : (p_1 - c_1x, ..., p_{n-1} - c_{n-1}x, x) \in X\}$ is finite. In the proof of Noether normalisation, we found a polynomial $g'(y_1 - c_1y_n, ..., y_{n-1} - c_{n-1}y_n, y_n)$ which is satisfied everywhere on X but which is monic as a polynomial in y_n. But this then implies there can be only finitely many possible values of x in Λ, as these are the solutions of this polynomial for certain values of y_i for $i = 1, ..., n-1$.

□

Exercise 5. Let F be an algebraically closed field. Calculate the Krull dimension of the ring

$$F[w, x, y, z]\big/(x^2 - wy, y^2 - xz, wz - xy).$$

Proof. We saw already in Exercise 5 of sheet 7 (the same proof works over any algebraically closed field) that the $R = F[w, x, y, z]\big/(x^2 - wy, y^2 - xz, wz - xy)$ is the coordinate ring of the algebraic set $Z = \{(u^3, u^2v, uv^2, v^3) \mid u, v \in F\}$. In fact, define $\Phi : F[w, x, y, z] \to F[u^3, u^2v, uv^2, v^3]$ by $w \mapsto u^3, x \mapsto u^2v, y \mapsto uv^2, z \mapsto v^3$ (as in the solution to Exercise 3). The kernel is precisely the set of all polynomials $f \in F[w, x, y, z]$ that vanish on the set Z, i.e., the kernel of Φ is the ideal $I(Z) = (x^2 - wy, y^2 - xz, wz - xy)$. Thus R is isomorphic to the image of Φ, which is $F[u^3, u^2v, uv^2, v^3]$. There is an obvious inclusion of rings $F[u^3, u^2v, uv^2, v^3] \subset F[u, v]$ and the latter is obviously integral over the former. Therefore the dimension of $R \cong F[u^3, u^2v, uv^2, v^3]$ is the same as the dimension of the polynomial ring $F[u, v]$. As we have seen repeatedly in this course the dimension of a polynomial ring in two variables is two. So $\dim R = 2$. □

Exercise 6. Let F be an algebraically closed field. Calculate a primary decomposition for the ideals

(1) $(x^4 - 2x^3 - 4x^2 + 2x + 3) \subseteq F[x]$,
(2) $(x^2, xy^2) \subseteq F[x, y]$,
(3) $(x^2, xy, xz, yz) \subseteq F[x, y, z]$.

Proof. (1) Factorizing the polynomial, we get:

$$x^4 - 2x^3 - 4x^2 + 2x + 3 = (x - 3)(x - 1)(x + 1)^2$$

Therefore the ideal is the intersection of the primary factors $(x-3)$, $(x-1)$ and $(x+1)^2$. These are primary because their radicals are maximal.

(2) A primary decomposition is

$$(x^2, xy^2) = (x^2, y^2) \cap (x)$$

The first factor is primary as it has a radical which is a maximal ideal, while the second is prime. The above equation holds because if $p \in (x^2, y^2) \cap (x)$, then $p = x^2 a + y^2 b$ and $x \mid p$, so $b = xc$ for some c and $p = x^2 a + xy^2 c$. Hence $p \in (x^2, xy^2)$.

(3) It may help to first calculate the irreducible components of $V(I)$ where $I = (x^2, xy, xz, yz)$. If (a, b, c) is a point of F^3 where a^2, ab, ac, bc all vanish, the first thing we can deduce from $a^2 = 0$ is that $a = 0$. Hence $ab = ac = 0$ gives us no new information, and $bc = 0$ implies that at least one of b and c is zero. Hence $V(I) = V((x, y)) \cup V((x, z))$ is the decomposition into irreducible components of $V(I)$, and hence as a first guess, we may try if (x, y) and (x, z) themselves appear in the minimal primary decomposition. As

$$(x, y) \cap (x, z) = (x, yz)$$

we need at least another ideal. The point is that, as you may see later in your studies, the primary decomposition is somewhat related to the order of vanishing of elements in the ideal. Here, all elements vanish at order 2 at the origin (and no other point has this property). This suggests that we should try $(x, y, z)^2$ as the corresponding primary ideal (this is (x, y, z)-primary as its radical is (x, y, z) and hence maximal).

So let us try to show that $I = (x, yz) \cap (x, y, z)^2$. Let $p \in (x, yz) \cap (x, y, z)^2$, then on the one hand we can write p as $p = x\alpha + yz\beta(y, z)$, where we can suppose that β only depends on y, z as we can put everything with an x into α. On the other hand, as p is a combination of $x^2, y^2, z^2, xy, yz, zx$, we can write it as $p = x^2 a + xyb + xzc + yzd(y, z) + y^2 e(y, z) + z^2 f(y, z)$, where we can suppose that d, e, f only depend on y, z as we can put everything with xy resp. xz into b resp. c. Hence by evaluating at $x = 0$ we obtain $yz\beta(y, z) = yzd(y, z) + y^2 e(y, z) + z^2 f(y, z)$, so $p = x^2 a + xyb + xzc + yz\beta(y, z)$. Hence $p \in I$.

Hence $I = (x, y) \cap (x, z) \cap (x, y, z)^2$ is a primary decomposition of I.

□

Exercise 7. Let $T \subseteq R$ be a multiplicative subset of a ring R and let $\{I_i\}_{1 \leq i \leq n}$ be finitely many ideals in R. By extension and contraction of ideals we shall mean extension and contraction via the natural morphism $R \to T^{-1}R$. Prove the following:

(1) $(\bigcap_i I_i)^{ec} = \bigcap_i I_i^{ec}$
(2) $(\bigcap_i I_i)^e = \bigcap_i I_i^e$

(3) Show that $T^{-1}(R/I) \cong T^{-1}R\big/I^e$ as R-modules. Use this to endow $T^{-1}(R/I)$ with a ring structure, so that it becomes in fact an isomorphism of rings.

(4) If I is primary, and $u \notin \sqrt{I}$, then $(I : u) = I$

(5) For an ideal I of a ring R admitting a finite primary decomposition, let $I = \bigcap_i I_i$ be such a primary decomposition, and show the following
  (i) $I^e = \bigcap_{T \cap I_i = \varnothing} I_i^e$,
  (ii) $I^{ec} = \bigcap_{T \cap I_i = \varnothing} I_i$

(6) From now on, let $R = F[x, y]$ for a field F, $I_1 = (x)$, $I_2 = \mathfrak{m}^s$ where $\mathfrak{m} = (x, y)$ and $s > 1$ is some integer, $I_3 = (x, y - 1)^2$, and $\mathfrak{p} \subseteq R$ a prime ideal for which we set $T = R \setminus \mathfrak{p}$. Show that
  (i) if $\mathfrak{p} = (x)$, then $T^{-1}\left(R/I_1 \cap I_2 \cap I_3\right) \cong F(y)$.
  (ii) if $\mathfrak{p} = (x, y)$, then $T^{-1}\left(R/I_1 \cap I_2 \cap I_3\right) \cong T^{-1}R\big/I_1^e \cap I_2^e$
  (iii) if $\mathfrak{p} = (x, y)$, compute the smallest integer n such that $\left(\frac{x}{1}\right)^n \in T^{-1}\left(R/I_1 \cap I_2 \cap I_3\right)$ is zero.

Proof. (1) We have

$$(\bigcap_i I_i)^{ec} \overset{\text{Prop 9.3.8}}{=} \bigcup_{u \in T}\left((\bigcap_i I_i) : u\right) \overset{\text{Prop 10.3.19}}{=} \bigcup_{u \in T}\left(\bigcap_i (I_i : u)\right).$$

Now we would like to swap the $\bigcup$ and the $\bigcap$. To this end, note that if $(u_i)_i$ is a sequence of elements of T, and $u := \prod_i u_i$, then

$$\bigcap_i (I_i : u_i) \subseteq \bigcap_i (I_i : u).$$

Hence

$$\bigcap_i \bigcup_{u \in T} (I_i : u) \subseteq \bigcup_{u \in T} \bigcap_i (I_i : u),$$

and as the reverse inclusion is elementary set theory we have

$$(\bigcap_i I_i)^{ec} = \bigcup_{u \in T}\left(\bigcap_i (I_i : u)\right) = \bigcap_i \bigcup_{u \in T} (I_i : u) \overset{\text{Prop 9.3.8}}{=} \bigcap_i I_i^{ec}.$$

(2) By Prop 9.3.8.(1), two ideals of $S^{-1}R$ are equal if and only if their contractions are equal. From the previous point we have

$$(\bigcap_i I_i)^{ec} \overset{(1)}{=} \bigcap_i I_i^{ec} = \left(\bigcap_i I_i^e\right)^c$$

where for the last equality we used that contraction (i.e. taking preimage) commutes with intersections. Hence it follows that $(\bigcap_i I_i)^e = \bigcap_i I_i^e$.

(3) The structure of $T^{-1}R\big/I^e$ as an R-module is given by $r \cdot (r'/t + I^e) = (rr')/t + I^e$. We have a natural morphism of R-modules $R \to T^{-1}R\big/I^e$ given by mapping $r \in R$ to $r/1 + I^e \in T^{-1}R\big/I^e$. This morphism has I in its kernel, so we obtain a morphism of R-modules $R/I \to T^{-1}R\big/I^e$. Notice that $T^{-1}R\big/I^e$ is T-invertible (see the solution

of Exercise 5 on Exercise Sheet 11), and thus by the universal property of localization of a module we obtain an R-module homomorphism $\phi : T^{-1}(R/I) \to T^{-1}R/I^e$ given by mapping $(r+I)/t$ to $r/t + I^e$. This is clearly surjective, so to prove injectivity suppose that $(r+I)/t$ is mapped to 0. Then $r/t \in I^e$, and thus by the proof of point (2) of Proposition 9.3.8 there exist $r' \in I$ and $t' \in T$ such that $r/t = r'/t'$. Hence there exists $t'' \in T$ such that $t''(rt' - r't) = 0$. Hence we have $t''t'(r+I) = 0$ inside R/I, and thus $(r+I)/t = 0$ inside $T^{-1}(R/I)$. Hence our map $\phi : T^{-1}(R/I) \to T^{-1}R/I^e$ is also injective. This endows $T^{-1}(R/I)$ with a natural ring structure by the formula

$$\frac{r+I}{t} \cdot \frac{r'+I}{t'} := \phi^{-1}\left(\phi\left(\frac{r+I}{t}\right)\phi\left(\frac{r'+I}{t'}\right)\right) = \phi^{-1}\left(\frac{rr'}{tt'} + I^e\right) = \frac{rr'+I}{tt'}.$$

With this ring structure, ϕ is tautologically a ring morphism.

(4) $t \in (I : u) \Rightarrow tu \in I \Rightarrow t \in I$, where in the last implication we used that no power of u is in I.

(5) Let $I = \cap I_i$ be such a primary decomposition.
   (i) From point (2) we have $I^e = \bigcap I_i^e$, but for I_i intersecting T non-trivially we have $I_i^e = S^{-1}R$. Hence $I^e = \bigcap_{T\cap I_i = \varnothing} I_i^e$.
   (ii) Since $(S^{-1}R)^c = R$ it follows from taking the contraction of the identity of point (2) that $I^{ec} = \bigcap_{T\cap I_i=\varnothing} I_i^{ec}$. Now for an ideal I_i with $T \cap I_i = \varnothing$, notice that as T is multiplicatively closed we also have $T \cap \sqrt{I_i} = \varnothing$. Hence it follows that

$$I_i^{ec} \overset{\text{Prop.9.3.8}}{=} \bigcup_{u\in T}(I_i : u) \overset{(4)}{=} \bigcup_{u\in T} I_i = I_i.$$

   So $I^{ec} = \bigcap_{T\cap I_i=\varnothing} I_i$.

(6) Note that I_i is primary for all i, as I_1 is prime, and $\sqrt{I_2} = (x,y)$ and $\sqrt{I_3} = (x, y-1)$ are maximal. Let $I = I_1 \cap I_2 \cap I_3$. We start with the following lemma.

Lemma 1. *Let R be a ring, $T \subseteq R$ a multiplicative subset and $I \subseteq R$ an ideal. Let $\tilde{T} := \{t + I \mid t \in T\} \subseteq R/I$. Then $T^{-1}(R/I) \cong \tilde{T}^{-1}(R/I)$ as rings, where the ring structure on $T^{-1}(R/I)$ is given by point (3).*

Proof. It is straightforward to see that the localisation map of R-modules $R/I \to T^{-1}(R/I)$ is a ring morphism for the ring structure on $T^{-1}(R/I)$ given by point (3). Furthermore, $t + I \in \tilde{T}$ is mapped to $(t+I)/1$, which is a unit with inverse $(1+I)/t$. Hence by the universal property of localisation there exists a ring morphism $\tilde{T}^{-1}(R/I) \to T^{-1}(R/I)$ mapping $(r+I)/(t+I)$ to $(r+I)/t$. This is clearly surjective. To prove that it is injective, let $(r+I)/(t+I)$ be in the kernel, i.e. there exists $t' \in T$ such that $t'(r+I) = 0$. But then $(t'+I)(r+I) = 0$, so $(r+I)/(t+I) = 0$ as well. Hence $\tilde{T}^{-1}(R/I) \to T^{-1}(R/I)$ is an isomorphism. □

   (i) By the previous point we have $I^e = \bigcap_{I_i\subseteq\mathfrak{p}} I_i^e$. As I_1 is the only ideal contained in $\mathfrak{p}$ we hence have $I^e = I_1^e = (x)^e$. Therefore, by point (3) we have

$$T^{-1}(R/I) \overset{(3)}{\cong} T^{-1}R/I^e = T^{-1}R/I_1^e \overset{(3)}{\cong} T^{-1}(R/(x)) \overset{\text{Lemma 1}}{\cong} \tilde{T}^{-1}(R/(x))$$

Now notice that $\tilde{T} = \{p + (x) \mid p \notin (x)\} = \{p(y) + (x) \mid p(y) \in F[y] \setminus \{0\}\}$. So under the identification $F[x,y] \big/ (x) = F[y]$ we have $\tilde{T} = F[y] \setminus \{0\}$ and thus $\tilde{T}^{-1}(R \big/ I_1) \cong F(y)$.

(ii) Note that I_3 is not contained in $\mathfrak{p}$, while I_1 and I_2 are. Hence by points (5) and (3) we have

$$T^{-1}(R \big/ I) \overset{(3)}{\cong} T^{-1}R \big/ I^e \overset{(5)}{=} T^{-1}R \big/ I_1^e \cap I_2^e.$$

(iii) Under the isomorphism of the previous point, $(x+I)/1$ is mapped to $x/1 + I_1^e \cap I_2^e$. So we need to compute the smalles integer $n > 0$ such that $x^n/1 \in I_1^e$ and $x^n/1 \in I_2^e$. Or equivalently, the smallest integer $n > 0$ such that $x^n \in I_1^{ec}$ and $x^n \in I_2^{ec}$. But by the argument in point (5).(ii) we have $I_1^{ec} = I_1$ and $I_2^{ec} = I_2$, so we need to find the smallest integer n with $x^n \in I_1$ and $x^n \in I_2$. Clearly $n = s$ works, and if $x^n \in I_2$ we must have $n \geq s$ as every non-zero element of I_2 has degree at least s. Hence $n = s$ is the minimal integer with the searched property.

□