Exercise Sheet 3
Introduction to Partial Differential Equations (W. S. 2024/25)
EPFL, Mathematics section, Dr. Nicola De Nitti

- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am, via moodle.

Exercise 1. This exercise explores the link between holomorphic $\mathbb{C} \to \mathbb{C}$ functions[1] and harmonic functions on $\mathbb{R}^2$.

(i) Let $\tilde{D}$ be an open connected subset of $\mathbb{C}$, and let $f : \tilde{D} \to \mathbb{C}$ be holomorphic. Define $D = \left\{(x, y) \in \mathbb{R}^2 : x + iy \in \tilde{D}\right\}$. Show that the functions $u, v : D \to \mathbb{R}$,

$$u(x, y) = \mathrm{Re}(f(x + iy)), \quad v(x, y) = \mathrm{Im}(f(x + iy))$$

are harmonic in D.

(ii) Let $D \subset \mathbb{R}^2$ be a simply connected domain, and let u be (real-valued) harmonic in D. Define $\tilde{D} = \{x + iy \in \mathbb{C} : (x, y) \in D\}$. Show that there exists a second (real-valued) function v, harmonic in D, such that $f : \tilde{D} \to \mathbb{C}$, defined as

$$f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic in $\tilde{D}$.

(iii) Show that the v in Point (ii) is unique up to a constant.

Hints: For (i): use the Cauchy–Riemann equations. For (ii), you can use the following fact (of which we omit the proof): Let g be a holomorphic function on a simply connected domain $\tilde{D}$; then, there exists a holomorphic function G on $\tilde{D}$ such that $G' = g$.

[1] Short reminder (from Complex Analysis). Let D be an open set in $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is *holomorphic* if it is complex differentiable at every point of D. The existence of a complex derivative in a neighbourhood is a very strong condition: it implies that a holomorphic function is infinitely differentiable and analytic. The Cauchy–Riemann equations (named after Augustin-Louis Cauchy and Bernhard Riemann) provide a necessary and sufficient condition for a complex function $f(x + iy) = f(x, y) = u(x, y) + iv(x, y)$ of a single complex variable $z = x + iy$, (with $x, y \in \mathbb{R}$) to be complex differentiable. We have that f is complex differentiable at $z = x + iy$ if and only if u and v are real differentiable functions and the partial derivatives of u and v satisfy

$$\partial_x u(x, y) = \partial_y v(x, y) \qquad \text{and} \qquad \partial_y u(x, y) = -\partial_x v(x, y).$$

We note that the term *holomorphic* was introduced in [BB75, §15 fonctions holomorphes].

Solution:

(i) In view of the Cauchy-Riemann equations, we have

$$\partial_x u = \partial_y v, \quad \partial_x v = -\partial_y u.$$

Thus, we obtain

$$\Delta u = \partial_x(\partial_x u) + \partial_y(\partial_y u) = \partial_x(\partial_y v) - \partial_y(\partial_x v).$$

Since f is analytic, $v \in C^2$, and thus $\partial^2_{xy} v = \partial^2_{yx} v$. Hence, $\Delta u = 0$. Similarly, $\Delta v = 0$ can be checked in the same way.

(ii) Let $g : \tilde{D} \to \mathbb{C}$, with $g(x+iy) = \partial_x u(x,y) - i\partial_y u(x,y)$, which is holomorphic by construction since u is harmonic (and thus in $C^2(\Omega)$). Thus, using the proposition from the hint, there exists a holomorphic function $G : \tilde{D} \to \mathbb{C}$, $G(x+iy) = A(x,y) + iB(x,y)$, such that $g = G'$, i.e.,

$$\begin{aligned} G'(x+iy) &= \partial_x A(x,y) + i\partial_x B(x,y) \\ &= \partial_x A(x,y) - i\partial_y A(x,y) \\ &= \partial_x u(x,y) - i\partial_y u(x,y). \end{aligned}$$

Hence, $A(x,y) = u(x,y) + c$ for some real constant c. The desired v is $v = B$: indeed, $u + iv = G - c$ is holomorphic.

(iii) Suppose $f(x+iy) = u(x,y) + iv(x,y)$ and $\widetilde{f}(x+iy) = u(x,y) + i\widetilde{v}(x,y)$ are holomorphic. Then $g(x+iy) = i\widetilde{f}(x+iy) - if(x+iy) = v(x,y) - \widetilde{v}(x,y)$ is holomorphic. From the Cauchy-Riemann equations, we have

$$\partial_x v - \partial_x \widetilde{v} = 0, \quad \partial_y v - \partial_y \widetilde{v} = 0.$$

Hence, $c_1(y) + \widetilde{v}(x,y) = v(x,y) = c_2(x) + \widetilde{v}(x,y)$, and thus $\widetilde{v}(x,y) = v(x,y) + c$ for some constant c.

Exercise 2. Let Ω be open connected and u be harmonic in Ω. Show that if $|u|$ attains its maximum in Ω, then u is constant.

Solution: Since u is harmonic, we know (see Exercise 3 in the Exercise Sheet 2) that $v := u^2$ is sub-harmonic in Ω. By the strong maximum principle, if v attains its maximum in Ω (which happens if and only if $|u|$ does), then v is constant. The claim follows by observing that u is continuous and Ω is connected.

Exercise 3. Let $\Omega = \mathbb{R}^n \backslash \overline{B_1(0)}$. Let $u \in C^2(\bar{\Omega})$ be harmonic in Ω, and such that $\lim_{|x|\to\infty} u(x) = 0$. Show that

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Solution: Let $R > 1$ and $\Omega_R = B_R(0)\backslash\overline{B_1(0)}$. The maximum principle implies

$$\max_{\Omega_R} |u| = \max_{\partial\Omega_R} |u| = \max\left\{\max_{\partial\Omega} |u|, \max_{\partial B_R(0)} |u|\right\}$$

Taking the limit for $R \to \infty$ yields

$$\sup_{\bar{\Omega}} |u| = \max\left\{\max_{\partial\Omega} |u|, 0\right\} = \max_{\partial\Omega} |u|$$

and the supremum is then attained on $\partial\Omega \subset \bar{\Omega}$.

Exercise 4. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Let $b \in L^\infty(\Omega)^n$, and $c \in L^\infty(\Omega)$, with $c > 0$ in Ω. Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$-\Delta u + b \cdot \nabla u + cu = 0, \quad x \in \Omega \tag{1}$$

and $u = 0$ on $\partial\Omega$. Show that $u = 0$ in Ω.

Hint: Show that $\max_{\bar{\Omega}} u \leq 0$ and $\min_{\bar{\Omega}} u \geq 0$. Follow the lines of the alternative proof of the maximum principle.

Solution: Let the maximum (resp. minimum) of u over $\bar{\Omega}$ be attained at $x \in \Omega$ (if this is not the case, the hint holds trivially, as we know $u|_{\partial\Omega} = 0$). Since $u \in C^2(\Omega)$, we have the necessary conditions

$$\begin{cases} \nabla u(x) = 0, \\ \partial^2_{x_j} u(x) \leq 0 \quad (\text{resp. } \geq 0), \quad \text{for } j = 1, \dots, n. \end{cases}$$

Plugging everything into (1), we obtain

$$c(x)u(x) = \Delta u(x) - b \cdot \nabla u(x) = \sum_{j=1}^{n} \partial^2_{x_j} u(x) \leq 0 \quad (\text{resp. } \geq 0).$$

Since $c(x) > 0$, it follows that $u(x) \leq 0$ (resp. ≥ 0). Hence, $\max_{\bar{\Omega}} u \leq 0 \leq \min_{\bar{\Omega}} u$, and the result follows.

References

[BB75] C.A. Briot and J.-C. Bouquet. *Théorie des fonctions elliptiques.* Gauthier-Villars, 2nd edition, 1875.