Introduction to Partial Differential Equations
— Exam —

General Information

The exam will be held on **Wednesday, January 29th**, starting at **08:15**, in the room **MA A1 10**.

The exam consists of a **30-minute oral examination** at the blackboard.

Each student will select **two questions**: one from each part of the course (A: **classical solutions**, B: **weak solutions**).

Students will then have an additional **30 minutes to prepare** their answers without external material or support **before** the oral examination begins.

Each student needs to arrive on time, that is, **30 minutes before their scheduled oral exam**. For example: **for the first examinee, the preparation time will start at 08:15**, and the oral examination itself will begin at 08:45, and so on.

The **tentative schedule** is as follows:

Beginning of Examination Day	08:15
3444xx	08:45–09:15
3418xx	09:15–09:45
3568xx	09:45–10:15
3953xx	10:25–10:55
Short Break	10:55–11:00
3559xx	11:00–11:30
3472xx	11:30–12:00
3256xx	12:00–12:30
Lunch Break	12:30–13:30
3616xx	14:00–14:30
3473xx	14:30–15:00
3257xx	15:00–15:30
3129xx	15:30–16:00
Short Break	16:00–16:10
3442xx	16:10–16:40
3106xx	16:40–17:10
3485xx	17:10–17:40

The students will be examined in alphabetical order. Exchanges in the order are possible with mutual agreement. Closer to the exam date, each student will receive an email containing some final information, including the rough assignment of their time slot (being punctual or early is advised).

Each student must bring a **CAMIPRO card or an ID**.

Paper and pen will be provided for the preparation.

NB: The teachers are available via email (write to `nicola.denitti@epfl.ch`) or during office hours to be scheduled, to answer questions in preparation for the exam throughout December and January.

List of topics

The list of topics for the examination roughly covers the whole syllabus, as developed in the lecture notes and exercise sheets.

Part A

1. Definition of the Laplace operator. Definition of harmonic, sub-harmonic, and super-harmonic functions.
2. Some examples of harmonic/sub-harmonic/super-harmonic functions. Chain rule and product rule for the Laplace operator.
3. Harmonicity is invariant under reflections and rotations (Problem 4 in Exercise Sheet 1).
4. Harmonic functions satisfy the mean-value identity.
5. Sub/super-harmonic functions satisfy a mean-value inequality.
6. Mean-value property implies harmonicity.
7. Gradient estimates for harmonic functions (estimate of the gradient by $\|u\|_\infty$ or by $\|u\|_1$).
8. Liouville's theorem for harmonic functions: case u bounded and cases u^2 or $|\nabla u|^2$ integrable (for which see Exercise Sheet 2, Problem 4).
9. Bochner's identity and application to Liouville's theorem (Exercise Sheet 4, Problem 1).
10. Estimates of higher derivatives and analyticity of harmonic functions.
11. Strong maximum principle for harmonic functions (including alternative proof via Zaremba–Hopf–Oleĭnik's boundary point lemma).
12. Weak maximum principle for harmonic functions.
13. Uniqueness for the Laplace equation via maximum principle.
14. Harnack's inequality.
15. Harnack's convergence theorem (both the one in the lecture notes and the one in Exercise Sheet 4, Problem 2).
16. The uniform limit of a sequence of harmonic functions is harmonic (Exercise Sheet 2, Problem 1)
17. Solution of the Poisson equation in $\mathbb{R}^n$ (using the fundamental solution of the Laplace operator).
18. Green's representation formula.
19. Definition of Green's function.
20. Green's function of the Laplace operator in a ball.
21. Green's function of the Laplace operator in a half-plane (including Problem 4 in Exercise Sheet 6).
22. Green's representation formula and the Neumann problem (Problem 2 in Exercise Sheet 5).
23. Symmetry of Green's function (Problem 3 in Exercise Sheet 6).
24. Characterization of sub-harmonic functions.
25. Ascoli–Arzelà's theorem for harmonic functions (Problem 1 in Exercise Sheet 7; NB: the proof of the classic Ascoli–Arzelà theorem is omitted).

26. Perron's construction of an harmonic function in Ω via sub-solutions (sketch of the proof: highlight the core ideas and steps of the proof; but, for the sake of brevity, some details may be skipped).
27. The Newtonian potential and the solvability of the Dirichlet problem for the Poisson equation (including Problems 3–4 in Exercise Sheet 7).
28. Weak maximum principle for general linear elliptic operators.
29. Strong maximum principle for general linear elliptic operators (together with Hopf's boundary point lemma).
30. Gradient estimate for the solution of a uniformly elliptic equation (Problem 3 in Exercise Sheet 11).

Part B

1. Space of test functions; distributions; distribution associated to a locally integrable function; Dirac delta.
2. Convergence in distribution (definition and examples, including those in Problem 3 of Exercise Sheet 4).
3. Distributional derivatives: definitions; some examples (including those in Problem 4 of Exercise Sheet 4); some properties of distributions and their derivatives (in particular, the content of Problem 3 in Exercise Sheet 5).
4. Zero distributional derivative implies constant distribution.
5. Weyl's lemma (just one of the two proofs presented in the lecture notes suffices).
6. Fundamental solution of the Laplace operator.
7. Weak derivatives and definition of Sobolev spaces $W^{k,p}(\Omega)$ (together with the properties proven in Problems 3–4 of Exercise Sheet 8); some examples of Sobolev functions (in particular, those in Problem 1 in Exercise Sheet 8).
8. Properties of weak derivatives (including those in Problems 2 and 5 in Exercise Sheet 8).
9. Example of a function $u \in H^1\left(\mathbb{R}^2\right)$ but $u \notin L^\infty\left(\mathbb{R}^2\right)$.
10. Approximation of $W^{k,p}(\mathbb{R}^n)$ functions with $C_c^\infty(\mathbb{R}^n)$ functions.
11. Gagliardo–Nirenberg–Sobolev's inequality (omitting the proofs of the required lemmas).
12. Sobolev's embedding theorem.
13. Morrey's inequality.
14. Rellich–Kondrashov's compact embedding theorem (statement only, proof omitted). NB: When giving the statement from the lecture notes, one is free to skip the sentences in points (1) and (3) after "More generally..."
15. Poincaré's inequality (proof only for $p = 2$).
16. Poincaré–Wirtinger's inequality.
17. Existence and uniqueness of weak solutions via Riesz' representation theorem for the Dirichlet and Neumann problem for the Poisson equation (NB: the proof of Riesz' representation theorem is omitted from this course; a quick reminder of the statement suffices when applying it here and or in the points below).
18. Existence and uniqueness of weak solutions via Riesz' representation theorem for the Dirichlet problem for the Poisson equation with potential.

19. Solvability of the Neumann problem.
20. Generalized Weierstrass theorem (statement only, proof omitted).
21. Existence of solutions to the homogeneous Dirichlet problem for the Poisson equation via variational methods.
22. Existence of solutions to the inhomogeneous Dirichlet problem for the Poisson equation via variational methods.
23. Existence of solutions to the homogeneous Neumann problem for $-\Delta + \mathrm{I}$ via variational methods.
24. Strict positivity of the eigenvalues of the Laplacian and orthogonality in H_0^1 of the eigenfunctions.
25. Variational characterization of the first eigenvalue (sketch of the proof).
26. Existence of solutions for a semilinear problem (sketch of the main steps of the proof).
27. Derrick–Pohozaev's identity.
28. Non-existence of solutions for a critical semilinear problem (assuming, without proof, Derrick–Pohozaev's identity and the lemma on star-shaped domains).
29. Gårding's inequality and boundedness in H_0^1 of the bilinear form associated with a general uniformly elliptic operator.
30. Existence of weak solutions for a general uniformly elliptic equation via Lax–Milgam's theorem (the proof of Lax–Milgam's theorem is omitted).

Some examples of questions

We provide below a few examples of question pairs for the exam.

NB: These examples are intended for illustrative purposes only. In particular, they do not cover all examinable topics listed above, nor do they comprehensively represent all possible variations, formulations, or combinations of questions.

Exam A.

Question 1. Discuss the mean-value property for harmonic functions. In particular, prove the following statements:

- If $u \in C^2(\Omega)$ is an harmonic function, then

$$u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,\mathrm{d}y = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y)$$

  holds for any closed ball $\overline{B_r(x)} \subset \Omega$;

- If $u \in C^2(\Omega)$ satisfies the mean-value property

$$u(x) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y)$$

  for any closed ball $\overline{B_r(x)} \subset \Omega$, then u is harmonic in Ω.

Question 2. Prove the following version of *Poincaré's inequality*: Let Ω be an open subset of $\mathbb{R}^N$ bounded in one direction. Then there exists a constant $C_{\mathrm{Poi}} > 0$ such that

$$\|u\|_{L^2(\Omega)} \le C_{\mathrm{Poi}}\|\nabla u\|_{L^2(\Omega)}$$

holds for any $u \in H_0^1(\Omega)$.

Exam B.

Question 1. State and prove *Harnack's inequality.*
Question 2.

- State (the generalized) Weierstrass theorem.
- Given $f \in H^{-1}(\Omega)$, define a functional $J : H_0^1(\Omega) \to \mathbb{R}$ as follows:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \langle f, u\rangle. \tag{1}$$

  Assuming that we already know that J is well-defined and Fréchet differentiable, show that, if J attains a minimum at $u \in H_0^1(\Omega)$, then u is a weak solution of the homogeneous Dirichlet problem

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

- Use the Weierstrass theorem stated previously to show that J has one and only one minimum.

Exam C.

Question 1. State *Harnack's inequalty* (without proof) and use it to show the following version of *Harnack's convergence theorem*: Suppose Ω is connected and $\{u_m\}_{m\in\mathbb{N}}$ is a pointwise increasing sequence of harmonic functions on Ω. Then either $\{u_m\}_{m\in\mathbb{N}}$ converges uniformly on compact subsets of Ω to a function harmonic on Ω, or $u_m(x) \to \infty$ for every $x \in \Omega$. Additional comment. In the proof, you may assume (without giving the proof) the following result: If $(u_m)_{m\in\mathbb{N}}$ is a sequence of harmonic functions in Ω converging uniformly to a limit function u, then u is harmonic in Ω.

Question 2. Let Ω be an open, connected, bounded set with C^1 boundary. Let $f \in L^2(\Omega)$ and consider the following *homogeneous Neumann boundary value problem*:

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \partial_\nu u = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

- Write the weak formulation of (3).
- Prove that there exists a weak solution of (3) if and only if $\fint_\Omega f(x)\,\mathrm{d}x = 0$.
- Discuss the uniqueness of weak solutions of (3).

Exam D.

Question 1. State and give a sketch of the proof of *Zaremba–Hopf–Oleĭnik's lemma*. As an application, use it to prove the *strong maximum principle for harmonic functions*.

Question 2. Prove *Weyl's lemma*: Let $u \in \mathcal{D}'(\Omega)$ and suppose that $\langle \Delta u, \varphi\rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. Then $u \in C^\infty(\Omega)$ and is harmonic in Ω.

Exam E.

Question 1. Prove that, if $u \in C(\Omega)$, then the following statements are equivalent.

1. For all $x \in \Omega$ and $B_r(x) \subset\subset \Omega$,
$$u(x) \le \fint_{\partial B_r(x)} u(y)\,\mathrm{d}S(y).$$
2. For all $x \in \Omega$ and for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum in x, then $-\Delta\phi(x) \le 0$.

Question 2. Suppose that Ω is a bounded open set in $\mathbb{R}^n$ and $f \in H^{-1}(\Omega)$. Use Riesz' representation theorem to show that there exists one and only one weak solution $u \in H_0^1(\Omega)$ of the homogeneous Dirichlet problem

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Hint: For this, you will need to use (stating it without proof) Poincaré's inequality.

Exam F.

Question 1. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Given a harmonic function u, prove that u^2 and $|\nabla u|^2$ are sub-harmonic. As an application of this observation, prove the following *Liouville-type theorems*:

- If u is harmonic and $\int_{\mathbb{R}^n} u^2 < \infty$, then $u = 0$.
- If u is harmonic and $\int_{\mathbb{R}^n} |\nabla u|^2 < \infty$, then u is constant.

Question 2. State (without proof) Rellich–Kondrashov's compact embedding theorem. Next, as an application, use it to prove *Poincaré–Wirtinger's inequality*: Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with C^1 boundary, and let $p \in [1, \infty]$. Then there exists a constant $C > 0$ (depending only on p and Ω), such that

$$\left\| u - \fint_\Omega u \, \mathrm{d}x \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \qquad \text{for every } u \in W^{1,p}(\Omega).$$

Exam G.

Question 1. State and prove the *strong maximum principle* for harmonic functions. (In the proof, use the mean-value property for harmonic functions).

Question 2. State and prove *Gagliardo–Nirenberg–Sobolev's inequality.* (Omit the proof of the two lemmas needed in the proof due to Nirenberg & Gagliardo).