Uniqueness of Positive Solutions of $\Delta u - u + u^p = 0$ *in* R^n

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Abstract

We establish the uniqueness of the positive, radially symmetric solution to the differential equation $\Delta u - u + u^p = 0$ (with $p > 1$) in a bounded or unbounded annular region in R^n for all $n \geqq 1$, with the Neumann boundary condition on the inner ball and the Dirichlet boundary condition on the outer ball (to be interpreted as decaying to zero in the case of an unbounded region). The regions we are interested in include, in particular, the cases of a ball, the exterior of a ball, and the whole space. For $p = 3$ and $n = 3$, this a well-known result of COFFMAN, which was later extended by MCLEOD & SERRIN to general n and all values of p below a certain bound depending on n. Our result shows that such a bound on p is not needed. The basic approach used in this work is that of COFFMAN, but several of the principal steps in the proof are carried out with the help of Sturm's oscillation theory for linear second-order differential equations. Elementary topological arguments are widely used in the study.

1. Introduction

The interesting semilinear elliptic differential equation

$$\Delta u + f(u) = 0, \qquad x \in R^n, \tag{1.1}$$

arises in many areas of applied mathematics including astrophysics, fluid mechanics, and population genetics. If a solution exists in the whole space R^n, satisfying

$$u(x) \to 0 \quad \text{as} \quad |x| \to \infty, \tag{1.2}$$

it is called a ground state. One natural question to ask is whether the ground state is unique or not. This is an extremely difficult problem to tackle in general. The classical work of GIDAS, NI, & NIRENBERG [4, 5] tells us that with some mild conditions on f, all ground states are radially symmetric. This allows us to shift our study to the ordinary differential equation

$$u'' + \frac{n-1}{r} u' + f(u) = 0, \qquad r > 0 \tag{1.3}$$

$$u'(0) = 0, \qquad u(r) \to 0 \quad \text{as} \quad x \to \infty. \tag{1.4}$$

Still this is a sufficiently difficult problem that few general results are known. One exception is the recent result of PELETIER & SERRIN [13], which provides uniqueness for those f that satisfy a starlike condition for large u and are essentially more negative than positive for small u. See also [6] for an improvement on this result.

COFFMAN [3] established uniqueness for the ground state of the equation with the particular choice of $n = 3$ and

$$f = u^3 - u. \tag{1.5}$$

The result for the problem on the whole of R^n is deduced from that of the corresponding problem on a finite interval $[0, b]$. The main part of the proof is a study of the zeros of the function

$$w(x) = \frac{\partial u(x)}{\partial \alpha}, \tag{1.6}$$

where u is considered as a function of both r and the parameter $\alpha = u(0)$. Very clever identities are used to show that $w(x)$ changes sign exactly once in $[0, b]$. The required conclusion then follows. COFFMAN attributed the approach, especially the use of special identities, to KOLODNER [8]. KOLODNER was concerned with a more general nonlinear eigenvalue problem (rather than just the uniqueness of the ground state) for some sublinear equation arising from the study of the rotation of a heavy string. COFFMAN remarked that the proof he gave in [3] did not extend to other choices of n or other powers of u in f since some of the clever identities no longer hold.

When f takes the more general form

$$f = u^p - u, \qquad p > 1, \tag{1.7}$$

there is the added complication of the non-existence of solutions if p exceeds or is equal to the critical value $(n+2)/(n-2)$; see, for instance, [1, 2].

Improving on COFFMAN'S method, MCLEOD & SERRIN [10] were able to establish a rather general result that includes COFFMAN'S. In particular, for the special f above, uniqueness holds for

$$p < \infty \qquad (1 \leqq n \leqq 2),$$

$$p \leqq \frac{n}{n-2} \qquad (2 < n \leqq 4),$$

$$p < \frac{8}{n} \qquad (4 < n < 8).$$

Slightly sharper results are also available for p close to 1.

In this paper the expected result for all values of p up to the critical exponent is confirmed. In addition, we also study the ground state solutions for a ball, an annulus, and the exterior of a ball. In the cases when the origin is not included, uniqueness is established for all $p > 0$.

The approach used in this paper is basically the same as that of COFFMAN. Instead of using KOLODNER-type identities we resort to STURM comparison techniques in the oscillation theory of linear second-order differential equations. The two methods are essentially equivalent, but the latter seems to make the proofs more transparent. In the present case, the main difficulty lies in the choice of the suitable comparison equation and in the proof that the equation has the correct oscillatory behavior. See the survey paper [9] for a more detailed explanation of our modification of COFFMAN's method and its use to obtain earlier uniqueness results.

For some recent success of STURM's comparison technique in the study of another property of the EMDEN-FOWLER equation, see [7].

In the spirit of COFFMAN's paper, the result is presented only for the particular choice of f given by (1.7). We refrain from stating the theorem for more general f (although it appears not hard to do so), in order to keep the ideas of the proof clear.

The related results of NI [11] and NI & NUSSBAUM [12] should be mentioned. They established uniqueness for the Dirichlet problem on a finite interval $[a, b]$, $0 < a < b < \infty$, for a class of f including the choice $f = u^p$. They also showed the possibility of non-uniqueness under perturbations of f by a different power of u.

The outline of the paper is as follows. In Section 2, the STURM comparison theorem and related lemmas that are needed in the sequel are stated.

In the next two sections we assume that the interval in question does not contain the origin. In Section 3, we give a first attempt to classify the solutions, which are regarded as continuous functions of two parameters, the initial height and the dimension of R^n. Simple topological arguments play a role. In Section 4, we introduce the functions w and $\theta = -ru'/u$, which are used to prove the main theorem.

In the last section we complete our study by letting the left endpoint of our interval approach the origin.

The whole proof is very involved, and I do not deny that a shorter proof might be found.

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2. Some Sturmian Theory

The form of Sturm's comparison theorem found in most textbooks is not strong enough for our purpose. The following formulation can be found, for instance, on p. 229 in INCE's classical book, *Ordinary Differential Equations*, Dover, 1956. By a solution we always mean a nontrivial solution.

Lemma 1 (Sturm). *U and V are, respectively, solutions of the following equations:*

$$U''(x) + f(x)\, U'(x) + g(x)\, U(x) = 0, \quad x \in (a, b), \tag{2.1}$$

$$V''(x) + f(x)\, V'(x) + G(x)\, V(x) = 0, \quad x \in (a, b), \tag{2.2}$$

where f, g, and G are continuous (actually, only local integrability is needed). Let (μ, ν) *be a subinterval in which* $V(x) \neq 0$ *and* $U(x) \neq 0$, *and in which the comparison condition*

$$G(x) \geqq g(x) \quad \textit{for all } x \in (\mu, \nu) \tag{2.3}$$

holds. Suppose further that

$$\frac{V'(\mu)}{V(\mu)} \leqq \frac{U'(\mu)}{U(\mu)}. \tag{2.4}$$

Then

$$\frac{V'(x)}{V(x)} \leqq \frac{U'(x)}{U(x)} \quad \textit{for all } x \in (\mu, \nu). \tag{2.5}$$

Equality in (2.5) *can occur only if* $U \equiv V$ *in* $[\mu, x]$. *If either* μ *or* ν *is a zero of U or V, then the fractions in* (2.4) *and* (2.5) *are interpreted as* ∞.

From this lemma follows the usual assertion that between any two consecutive zero of $U(x)$, there must be at least one zero of $V(x)$. In fact, $V(x)$ may have more than one zeros in that interval. It is only natural to say that $V(x)$ oscillates faster than $U(x)$.

Sometimes it is more convenient to replace the comparison conditions (2.1), (2.2) and (2.3) by the equivalent conditions

$$U''(x) + f(x)\, U'(x) + h(x)\, U(x) \geqq 0, \quad U(x) > 0 \quad \text{in } (\mu, \nu), \tag{2.1'}$$

$$V''(x) + f(x)\, V'(x) + h(x)\, V(x) \leqq 0, \quad V(x) > 0 \quad \text{in } (\mu, \nu). \tag{2.2'}$$

If either U or V is negative, the corresponding inequality has to be reversed.

A simple consequence of Sturm's theorem is the following.

Lemma 2.

(i) *If in addition to all the hypotheses of Lemma 1, we have*

$$0 < V(\mu) \leqq U(\mu), \tag{2.6}$$

then

$$V(x) \leqq U(x) \quad \text{for all } x \in (\mu, \nu). \tag{2.7}$$

(ii) *If in addition to all the hypotheses of Lemma 2,* $\lim_{x \to \nu} V(x) = \pm\infty$, *then* $\lim_{x \to \nu} U(x) = \pm\infty$.

Proof. The conclusion follows from an integration of (2.5). □

Suppose that equation (2.1) has solutions that do not vanish in a neighborhood of the point b. The largest neighborhood of b, (c, b), on which there exists a solution of (2.1) without zeros is called the disconjugacy neighborhood of b, or, in short, the disconjugacy interval of (2.1). It follows from Sturm's theorem that no non-trivial solution can have more than one zero in (c, b). On the other hand, unless $c = a$, any solution of (2.1) that has a zero before c must have another zero in (c, b). The following is another corollary of Sturm's theorem.

Lemma 3. *Consider the same two equations,* (2.1) *and* (2.2), *satisfying the comparison condition* (2.3). *In addition, we assume that* $U \not\equiv V$ *in any neighborhood of* b. *If there exists a solution* V *of* (2.2) *with a largest zero at the point* ϱ, *then the disconjugacy interval of* (2.1) *is a strict superset of* (ϱ, b).

We need another comparison lemma that handles equations with different second terms. It is not usually included in the classical Sturm theory. It can be proved easily using results from the theory of differential inequalities. See [9] for the proof of a related result.

Lemma 4. *Suppose instead of* (2.2), V *satisfies the differential equation*

$$V''(x) + F(x)\, V'(x) + G(x)\, V(x) = 0, \qquad x \in (a, b), \tag{2.8}$$

and the comparison condition

$$F(x) \geqq f(x) \geqq 0 \quad \textit{for all } x \in (\mu, \nu). \tag{2.9}$$

The conclusions of Lemma 1 or Lemma 2 still hold provided that either

$$U'(x) \geqq 0 \quad \textit{for all } x \in (\mu, \nu) \tag{2.10}$$

or

$$V'(x) \geqq 0 \quad \textit{for all } x \in (\mu, \nu). \tag{2.11}$$

Proof. Let us assume that (2.10) holds; the proof for the case in which (2.11) holds is similar. Define $r(x) = U'(x)/U(x) \geqq 0$ and $R(x) = V'(x)/V(x)$. They satisfy

$$r'(x) = -(f(x)\, r(x) + g(x) + r^2(x)) \geqq -(F(x)\, r(x) + G(x) + r^2(x)),$$

and

$$R'(x) = -(F(x)\, R(x) + G(x) + R^2(x)).$$

We also have the initial comparison condition $r(\mu) \geqq R(\mu)$. From the theory of differential inequalities, we can then conclude that $r(x) \geqq R(x)$ for all x in (μ, ν). □

The next lemma, though not a direct consequence of Sturm's theorem, is definitely motivated by it. It is well known that if U is a solution of a second-order ordinary differential equation such as (2.1), then at a zero of U, the derivative U' cannot vanish. If z satisfies a second-order ordinary differential inequality in such a way that z oscillates more than U, then intuition tells us that the tangent line to the graph of z is being bent more strongly towards the x-axis than that of U. Hence the derivative of z at a zero cannot vanish. A convenient reference for a proof of this fact has not been found. Reproduced here is the short proof that was given in [7].

Lemma 5. *Suppose a function $z(t)$ is positive (negative) in an interval (μ, ν), either $z(\mu) = 0$ or $z(\nu) = 0$, and it satisfies the inequality*

$$z''(t) + f(t)\, z'(t) + g(t)\, z(t) \leqq (\geqq)\, 0, \qquad in\ (\mu, \nu), \tag{2.12}$$

where f and g are any continuous functions. Then $z'(\mu) \neq 0$ $(z'(\nu) \neq 0)$.

Proof. We give the proof only for the case $z(t) \geqq 0$ and $z(\mu) = 0$; the other cases can be proved similarly. We may assume without loss of generality that $f(x) = 0$, since an equation of the more general form can be reduced to this particular case using a change of independent variable. Equation (2.12) now takes the form

$$z'' + g(x)\, z = -P(x), \qquad x > 0, \tag{2.13}$$

where $P(x) \geqq 0$. Let $z_1(x)$ and $z_2(x)$ be independent solutions of the homogeneous equation associated with differential equation (2.13), satisfying the initial conditions

$$z_1(\mu) = 0, \quad z_1'(\mu) = 1,$$

$$z_2(\mu) = 1, \quad z_2'(\mu) = 0.$$

Suppose that the conclusion of the lemma is false, namely, that $z'(\mu) = 0$. By the formula of variation of constants

$$z(x) = \int_\mu^x [z_1(s)\, z_2(x) - z_1(x)\, z_2(s)]\, P(s)\, ds. \tag{2.14}$$

Let $h(s, x) = [z_1(s)\, z_2(x) - z_1(x)\; z_2(s)]$. At $x = s$, $\partial h(s, x) \partial x$, being the Wronskian of $z_1(x)$ and $z_2(x)$, is -1. By continuity $\partial h(s, x)/\partial x$ is therefore negative in a neighborhood of $(0, 0)$. At $x = s$, $h(s, x) = 0$. Thus for $x > s$ and x sufficiently close to s, $h(s) < 0$. It then follows that the integrand in (2.14) is negative for $x > \mu$ but sufficiently close to μ. This then implies that $z(x)$ is negative, contradicting our assumption that it is positive in (μ, ν). □

If the inequality sign in (2.12) is strict at the point $x = \mu$, then the conclusion follows trivially from the facts that z has a minimum at μ so that $z''(\mu) \geqq 0$, contradicting (2.12). The proof for Lemma 5 is needed to handle the general case.

Next are established some asymptotic properties of the solutions of the linear equation

$$U'' + \frac{m}{x} U' + g(x) U = 0, \quad x > 0, \tag{2.15}$$

where $m > 0$ is a constant and g is continuous with $\lim_{x\to\infty} g(x) = -2k^2$, for some $k > 0$.

Lemma 6. *Let (c, ∞) be the disconjugacy interval of* (2.15). *Every solution of* (2.15) *with a zero in (c, ∞) is unbounded.*

Conversely, if the last zero of an unbounded solution of (2.15) *is ϱ, then ϱ is an interior point of the disconjugacy interval. In the other words, $\varrho > c$.*

Proof. We have to show that equation (2.15) is non-oscillatory, so that the disconjugacy interval is well defined. Let μ be so large that $g(x) \leqq -k^2$, and $m/x \leqq 2k$ for $x \in [\mu, \infty)$. We compare (2.15) with the equation

$$V'' + 2kV' - k^2 V = 0, \quad x \in [\mu, \infty), \tag{2.16}$$

using Lemma 4 in the form of Lemma 2. We can conclude that for the solutions U_1 and V_1 of (2.15) and (2.16), respectively, that satisfy the initial conditions

$$U_1(\mu) = V_1(\mu) = 1, \quad U_1'(\mu) = V_1'(\mu) = 1, \tag{2.17}$$

the inequality

$$V_1(x) \leqq U_1(x), \quad x > \mu, \tag{2.18}$$

holds. Since V_1 is non-oscillatory and unbounded, so is U_1. If all solutions of (2.15) are unbounded, then our lemma is true. Thus suppose there is a bounded solution U_2. Then U_1 and U_2 are linearly independent, and all other solutions are of the form $U = c_1 U_1 + c_2 U_2$, with constants c_1 and c_2. It follows that any solution that is not a multiple of U_2 is unbounded. Let us show that U_2 cannot have a zero in the disconjugacy interval, from which the assertion of the lemma follows. Suppose that U_2 does have a zero at $\varrho \in (c, \infty)$. Let U be the solution of (2.15) that satisfies the initial conditions, $U(\varrho + 1) = U_2(\varrho + 1)$ and $U'(\varrho) = U_2'(\varrho) - \varepsilon$ with $\varepsilon > 0$. By taking ε small enough, we ensure that U has a zero so close to ϱ that it falls in the disconjugacy interval. Thus U cannot have another zero beyond $\varrho + 1$. In other words, U remains positive in $[\varrho + 1, \infty)$. By Lemma 2, $U(x) \leqq U_2(x)$, for $x \geqq \varrho + 1$. It follows that U is bounded, but this contradicts the fact that U is not a multiple of U_2. □

Lemma 7. *If U is a solution of* (2.15) *such that $U'(x) < 0$ for all x sufficiently large, then $-U'(x)/U(x) \geqq k$ for all x sufficiently large.*

Proof. Let $R(x) = -U'(x)/U(x)$. It satisfies the Riccati equation

$$R' = R^2 - \frac{m}{x} R + g(x) < R^2 + g(x). \tag{2.16}$$

If $R(x) < k$ for some large x, for which $g(x)$ is close to its limit $-2k^2$, then R' will remain strictly negative, eventually causing R to change sign. This contradiction proves the lemma. □

3. Classification of Solutions

In this section and the next, we study our differential equation in an interval not containing the origin, in order to avoid having a singular term in the equation. Let $p > 1$ and $m \geqq 0$ be any constants, and (a, b) be a bounded or unbounded open subinterval of $[0, \infty)$, with $a > 0$. We are concerned with the following boundary value problems:

$$u''(r) + \frac{m}{r} u'(r) + u^p - u = 0, \quad u(r) > 0, \quad r \in (a, b), \tag{3.1}$$

$$u'(a) = 0, \tag{3.2}$$

$$u(b) = 0 \tag{3.3}$$

or

$$\lim_{r \to \infty} u(x) = 0 \quad \text{if } b = \infty. \tag{3.4}$$

These come from corresponding boundary value problems for the semilinear equation (1.1) on an annulus $a < |x| < b$, with the Neumann condition on the ball $|x| = a$ and the Dirichlet condition on the outer ball $|x| = b$. The constant m is one less than the dimension n of the Euclidean space R^n in which (1.1) holds. In this section, we take m to be any positive constant, not necessarily integral. Our main theorem asserts the uniqueness of the solution to any of the above boundary value problems. Since the question of existence has been answered in the affirmative, a unique solution always exists.

In this section we discuss the classification of solutions of (3.1). Some of the lemmas stated here are well known, but we prefer to give complete proofs. The proof of the main theorem is complicated enough that it is perhaps better to make the paper as self-contained as possible. Trying to adopt notations from other sources would only add to the confusion.

Following KOLODNER and COFFMAN, instead of considering directly the boundary value problems, we look at u as the solution of an initial value problem, (3.1), together with the initial conditions

$$u(a) = \alpha > 0, \quad u'(a) = 0. \tag{3.5}$$

The solution u now depends on α as a parameter. In addition, u depends on the constant m. Besides, we think of $u = u(r, \alpha, m)$ as defined for all values of $r > 0$ extending beyond the interval $[a, b]$.

In other words, we are shooting out a solution $u(r, \alpha, m)$ from the point $r = a$ and hope that with the correct choice of the initial height α, the solution will land on the right spot $r = b$. Rather than regarding b to be a preassigned fixed point, we let $b(\alpha, m)$ be the first point where $u(r, \alpha, m)$ intersects the r-axis, and study the dependence of b on α and m. The point b is not always defined. This happens, for instance, if for a fixed m, α is small enough; the solution $u(r, \alpha, m)$ will not cross the r-axis.

We divide the set of solutions into three subsets:

1. Solutions that eventually takes on some negative values. These are the ones for which $b(\alpha, m)$ are defined. Let N be the set of (α, m) for which the corresponding solutions belong to this class.
2. Solutions that remain positive and satisfy $\lim_{r\to\infty} u(r) = 0$. Let G (stands for "ground state") be the set of (α, m) for which the corresponding solutions belong to this class.
3. Solutions that remain non-negative but do not belong to case 2. Let P (stands for "positive") be the corresponding set of (α, m). The term "positive" is justified because, as Lemma 5 shows, no solutions of an ordinary differential equation can be tangent to the r-aixs.

The sets N, G, and P partition the "quadrant" $(0, \infty)\times[0, \infty)$ into three mutually disjoint subsets. Sometimes, for convenience, we say that a solution belongs to N, G, or P when we should have said that the corresponding (α, m) belongs to the appropriate set. Also whenever there is no danger of confusion, we suppress the explicit mention of the variables α and m in functions such as u and b.

From the classical theory of ordinary differential equations, we know that, in any compact r-interval, solutions of (3.1)–(3.5) depend continuously on the parameters α and m.

As is well known, we can obtain a lot of information about the solutions from the Energy or Lyapunov function

$$E(r) = \frac{u'^2(r)}{2} + \frac{u^{p+1}(r)}{p+1} - \frac{u^2(r)}{2}. \tag{3.6}$$

The inequality $E'(r) = -mu'^2/r \leqq 0$ implies that E decreases to a finite constant $E(\infty)$ as $r \to \infty$. It follows that all solutions must be bounded.

Lemma 8. *If* $E(a) = \frac{\alpha^{p+1}}{p+1} - \frac{\alpha^2}{2} \leqq 0$, *then* $(\alpha, m) \in P$ *for any* m. *In other words,* $(0, \gamma)\times[0, \infty) \subset P$, *where* $\gamma = \left(\frac{p+1}{2}\right)^{1/(p-1)}$.

Proof. Since E is decreasing, $E(r) \leqq E(a) \leqq 0$, for $r > a$. The solution cannot cut the r-axis; otherwise at a point of intersection $E(r) = u'^2/2 > 0$ contradicting our previous assertion. □

Lemma 9. *If* $u \in P$, *then it is oscillatory about the value 1 in the following sense. The sets of its local maxima* $\{a = r_0 < r_2 < \ldots\}$ *and local minima* $\{r_1 < r_3 < \ldots\}$ *interlace and, unless* $u \equiv 1$,

$$u(r_0) > u(r_2) > u(r_4) > \ldots > 1 \tag{3.7}$$

and

$$u(r_1) < u(r_3) < u(r_5) < \ldots < 1. \tag{3.8}$$

Proof. We claim that u cannot be monotone. Suppose this is not true. We know that u has a local maximum at $r = a$, because $u''(a) < 0$. Thus u can only be monotone decreasing. Let the limit $\lim_{r\to\infty} u(x)$ be denoted by $u(\infty)$. We have

$$\lim_{r\to\infty} u''(r) = \lim_{r\to\infty} \left(-\frac{m}{r} u' - u^p + u\right) = -u^p(\infty) + u(\infty).$$

This is incompatible with the monotonicity of u unless the limit is zero, which implies that $u(\infty) = 1$. Let $v = u - 1 > 0$. It satisfies the differential equation.

$$v'' + \frac{m}{r} v' + \left(\frac{u^p - u}{u - 1}\right) v = 0.$$

Note that the fraction inside the parentheses is larger than or equal to $p - 1$ (for $u > 1$). We use Sturm's comparison theorem to conclude that v oscillates faster than the solutions of

$$U'' + \frac{m}{r} U' + (p - 1) U = 0.$$

It is well known that all solutions U of this differential equation are oscillatory, but this contradicts the fact that the more oscillatory v remains positive for all r.

Hence we know that u has at least one local minimum. Let r_1 be the first of them. We cannot have $u(r_1) = 1$, for then $E(r_1)$ attains the lowest possiblie value that E can take. This means that $E(r) = E(r_1)$ for all $r > r_1$, and we have the case $u \equiv 1$ In all other cases, since r_1 is a local minimum, $u'(r_1) = 0$ and $u''(r_1) = u(r_1) - u^p(r_1) \geqq 0$. This can hold only if $u(r_1) < 1$. It follows that $E(r_1) < 0$. The same arguments as above show that u cannot be monotone increasing in (r_1, ∞). Repeating the arguments then gives two infinite sequences of critical points as asserted. The strict inequalities in (3.7) and (3.8) are due to the fact that $E' = -u'^2/2$ cannot vanish identically in any subinterval of (a, b). □

Lemma 9 has the following corollary

Lemma 10. *Suppose a solution does not belong to N. Then it belongs to G if and only if $E(\infty) = 0$, and it belongs to P if and only if $E(\infty) < 0$.*

Proof. If $u \in G$, then $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} u'(r) = 0$. Thus $E(\infty) = 0$. On the other hand, if $u \in P$, then $E(r_1) < 0$ because $u(r_1) < 1$. Thus $E(\infty) < E(r_1) < 0$. □

Lemma 11. *Let u belong to either N or G. Then $u'(r) < 0$ in $(a\ b(\alpha, m)]$ or (a, ∞), respectively.*

Proof. Suppose $u'(c)$ vanishes in an interior point $c \in (a, b)$. Then $u(c) \neq 1$, lest $u \equiv 1$. If $u(c) < 1$, then $E(c) < 0$, contradicting the assumption that u belongs to $N \cup G$. Finally, if $u(c) > 1$, then $u''(c) < 0$, so that c is a local maximum. Between the two local maxima $r = a$ and c there must be a local minimum, at which $u'' \geqq 0$, and this reduces to the previous case.

The derivative $u'(b)$ cannot vanish at the endpoint b, because this contradicts the uniqueness of initial value problem for (3.1); the trivial solution is the only solution of (3.1) that can vanish at b with a double zero. □

Lemma 12. *For* $(\alpha, m) \in N$, $b(\alpha, m)$ *is a continuous function of* α *and* m.

Proof. This is a simple consequence of the fact that a solution u can never be tangent to the r-axis at a zero. □

Lemma 13. *The sets* N *and* P *are open subsets of* $(0, \infty) \times [0, \infty)$.

Proof. Let (α_0, m_0) be a point in N. The corresponding solution u takes a negative value at some point $r = c$. By continuity, there is a neighborhood of (a_0, m_0) for which $u(c, \alpha, m) < 0$ for each (α, m) in the neighborhood. Thus all such solutions belong to N. Let (a_0, m_0) be a point in P. Then $E(c) < 0$ for some $r = c$. Continuity again yields a neighborhood for which $E(c) < 0$ for each of the solutions belonging to the neighborhood. It follows that $E(\infty) < E(c) < 0$ for each of these solutions, and so they must all belong to P. □

An immediate consequence of this lemma is that for each fixed m, the sets N_m and P_m (the intersections of N and P with the straight line $(., m)$) are open subsets of $(0, \infty)$. Each is therefore a union of countably many open intervals. The boundary points of N or P belong to G. The following lemma is a straightforward consequence of Lemmas 8 and 13.

Lemma 14. *For a fixed* m, *the boundary value problem* (3.1)–(3.4) *has a unique solution if and only if the sets* P_m *and* N_m *are both open intervals of the form* $(0, \alpha_m)$ *and* (α_m, ∞), *respectively, with one single common endpoint* $\alpha_m \in G_m$.

We call a point m regular if it satisfies the conditions of Lemma 14. For each regular m, there is therefore a unique $\alpha(m) = \alpha_m$ as asserted. This is, in fact, a continuous function of m within any interval of regularity. As a first step towards establishing that fact, we prove that give any fixed $\bar{m} > 0$, the set of points in G with $m \leqq \bar{m}$ is bounded.

Lemma 15. *For any given* $\bar{m} > 0$, *there exists an* $\bar{\alpha} > 0$ *such that* $G \subset (0, \bar{\alpha}) \times [0, \bar{m}]$.

Proof. It suffices to show that for some $\bar{\alpha}$, $[\bar{\alpha}, \infty) \times [0, \bar{m}] \subset N$. Consider the comparison equation

$$U'' + \frac{\bar{m}}{a} U' + U^p - U = 0 \quad \text{on } [a, \infty). \tag{3.9}$$

Since the coefficient of the second term is a constant, the equation can be explicitly solved, at least in theory. It is then not hard to see that there exists an $\bar{\alpha} > 0$, such that if $U(a) \geqq \bar{\alpha}$, $U'(a) = 0$, then $U(r)$ cuts the r-axis at some point, say b.

Using as new variables $S = U'^2(r)$ and $x = U(r)$, we can rewrite (3.9) as

$$\frac{dS}{dx} = \frac{2\bar{m}}{a}\sqrt{S} + f(x), \qquad S(\alpha) = 0, \tag{3.10}$$

where $f(x) = 2(x^p - x)$. Similarly, we can rewrite (3.1), using $s = u'^2(r)$ and $x = u(r)$, as

$$\frac{ds}{dx} = \frac{2m}{r}\sqrt{s} + f(x) \leqq \frac{2\bar{m}}{a}\sqrt{s} + f(x), \qquad s(\alpha) = 0. \tag{3.11}$$

An application of the theory of differential inequalities gives

$$S(x) \leqq s(x), \qquad \text{for all } 0 \leqq x \leqq \alpha. \tag{3.12}$$

In other words, $U'^2(r_1) \leqq u'^2(r_2)$, whenever $x = U(r_1) = u(r_2)$; see Figure 1. Since both u' and U' are negative, we have

$$0 > U'(r_1) \geqq u'(r_2). \tag{3.13}$$

This implies that u must cut the r-axis before b. It follows that $(\alpha, m) \in N$. ☐

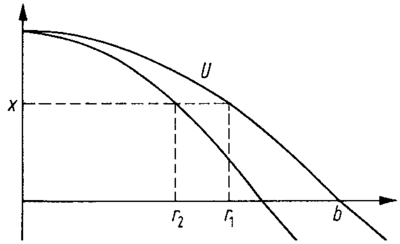

Fig. 1

Lemma 16. *If all points $m \in (m_1, m_2)$ are regular, then α_m is a continuous function of m. The set of limit points of the curve $\{\alpha(m) : m_1 < m < m_2\}$ that lie on each of the lines $m = m_1$ and $m = m_2$ are thus connected closed intervals. In case one of the endpoints is also regular, the corresponding limit set reduces to the single point $\alpha(m_1)$ or $\alpha(m_2)$.*

Proof. Let $m_0 \in (m_1, m_2)$. Suppose $\lim_{m \to m_0} \alpha(m)$ is not $\alpha(m_0)$. By compactness, there exists a sequence of points $\{m_{(i)}\} \subset (m_1, m_2)$, $m_{(i)} \to m_0$, but $\alpha_\infty = \lim_{i \to \infty} \alpha(m_{(i)}) \neq \alpha(m_0)$. Then α_∞ belongs either to N or to P, contradicting the fact that both are open sets. The other assertions are obvious. ☐

4. Main Result, $a > 0$

The main result is established through a careful study of the signs of the function values in $[a, b(\alpha, m))$ of the function

$$w(r, \alpha, m) = \frac{\partial u}{\partial \alpha}(r, \alpha, m). \tag{4.1}$$

We are interested only in those w that come from solutions in either N or G, and we are concerned only with their behavior in the interval $[a, b)$ (we interpret b to be ∞ in case $u \in G$). When there is no danger of confusion, we suppress the variables α and m in the function notation.

By differentiating (3.1) and (3.5), we derive the following initial value problem for w:

$$w'' + \frac{m}{r} w' + (pu^{p-1} - 1) w = 0, \tag{4.2}$$

$$w(a) = 1, \quad w'(a) = 0. \tag{4.3}$$

Equation (4.2) is a "linear" equation in w if we regard u as a known function. An immediate consequence is that w cannot be tangent to the r-axis. As r goes through a zero of w, w has to change sign.

Lemma 17. *For $u \in G \cup N$, w has to change sign at least once in $[a, b)$.*

Proof. Rewrite (3.1) as

$$(u-1)'' + \frac{m}{r}(u-1)' + \left(\frac{u^p - u}{u-1}\right)(u-1) = 0, \tag{4.4}$$

and view it as a "linear" equation in $(u - 1)$, with the expression inside the large parentheses as the coefficient of the last term. Since this coefficient is smaller than the corresponding one in (4.2), when $u \geqq 1$, we conclude from Sturm's comparison theorem that w oscillates faster than $(u - 1)$, so w must vanish before the first zero of $(u - 1)$, namely, before the point $\xi \in (a, b)$ at which $u(\xi) = 1$. □

Our ultimate aim is to show that w cannot change sign more than once in $[a, b)$. We say that (α, m) is admissible if this is true, more precisely, when

$$(\alpha, m) \in G \cup N, \quad \text{and} \quad w(r, \alpha, m) \text{ has exactly one zero in } [a, b). \tag{4.5}$$

If, in addition,

$$w(b, \alpha, m) < 0, \quad \text{for } (\alpha, m) \in N, \tag{4.6}$$

or

$$\lim_{r\to\infty} w(r, \alpha, m) = -\infty, \quad \text{for } (\alpha, m) \in G, \tag{4.7}$$

we say that (α, m) is strictly admissible. The following well known lemma connects the concepts of admissibility and regularity and is the key idea in the KOLODNER-COFFMAN method.

Lemma 18. *Let m be fixed. If $(\alpha, m) \in N$ is strictly admissible, then in a neighborhood of α, b is a strictly decreasing function of α.*

Proof. By assumption, $\frac{\partial u}{\partial \alpha}(b(\alpha), \alpha) = w(b(\alpha)) < 0$. Hence, for $\varepsilon > 0$ small enough, $u(b(\alpha), \hat{\alpha}) < 0$ for all $\hat{\alpha} \in (\alpha, \alpha + \varepsilon)$. By the Intermediate Value theorem, $u(r, \hat{\alpha})$ must have a zero in $(a, b(\alpha))$. Therefore, $b(\hat{\alpha}) < b(\alpha)$. It is not hard to see that this local monotonicity together with the continuity of b (Lemma 12) implies monotonicity in a neighborhood of α. □

The following lemma complements Lemma 18.

Lemma 19. *If for* $(\alpha, m) \in G$, $\lim_{r\to\infty} w(r) = -\infty\ (\infty)$, *in particular if* (α, m) *is strictly admissible, then there exists a right* (*left*) *neighborhood of* α *that belongs to* N.

Proof. Let τ be the largest zero of w in $[a, b)$. By Lemma 6, the disconjugacy interval (c, ∞) of equation (4.2) contains τ. Suppose first that w is negative after τ. Take a point $\zeta \in (c, \tau)$ and a point $\lambda \in (\tau, \infty)$. Since $w(\zeta) > 0$ and $w(\lambda) < 0$, for $\varepsilon > 0$ small enough, $u(\zeta, \hat{\alpha}) > w(\alpha)$, and $u(\lambda, \hat{\alpha}) < w(\alpha)$, for all $\hat{\alpha} \in (\alpha, \alpha + \varepsilon)$. In case w is positive beyond τ, the same is true for all $\hat{\alpha}$ in a left neighborhood $(\alpha - \varepsilon, \alpha)$ instead. So $\hat{u}(r) = u(r, \hat{\alpha})$ must intersect $w(r)$ at a point χ in (ζ, λ). This point is therefore within the disconjugacy interval (c, ∞). We claim that $\hat{u}(r)$ must intersect the r-axis in (χ, ∞). Suppose that this is not true. Then $u(r, \hat{\alpha})$ remains positive in (χ, ∞). It lies below the graph of w in a right neighborhood of the point χ. Let us suppose that it catches up with w at a point $\eta > \chi$. Then in (χ, η), the function $z(r) = u(r) - \hat{u}(r)$ is positive and satisfies the differential equation

$$z'' + \frac{m}{r} z' + \left(\frac{u^p - \hat{u}^p}{u - \hat{u}} - 1 \right) z = 0. \tag{4.8}$$

Observe that the coefficient of the last term is less than that in (4.2). Hence z oscillates less than the solutions of (4.2). But the disconjugacy interval of (4.2) being (c, ∞) means that the solutions of (4.2) cannot have two zeros in the interval $[\chi, \eta]$. It follows that the less oscillatory z cannot have two zeros in the same interval, contradicting the assumption that $z(\chi) = z(\eta) = 0$. Thus $\hat{u}$ remains below u throughout (χ, ∞). Then the coefficient of (4.8) is less than that of (4.2) in the whole interval (χ, ∞), implying that z is less oscillatory than the solutions of (4.2) in (χ, ∞). Since χ is in the disconjugacy interval of (4.2), by Lemma 6, a solution of (4.2) that vanishes at χ must be unbounded. By Lemma 2, the less oscillatory z must also be unbounded. This contradicts the fact that $\hat{u}(r) < u(r) \to 0$ as $r \to \infty$, unless $\hat{u}$ intersects the r-axis. Thus $u(r) = u(r, \bar{\alpha})$ belongs to N for all $\bar{\alpha}$ in $(\alpha, \alpha + \varepsilon)$ or $(\alpha - \varepsilon, \alpha)$ as asserted. □

Now if we can show that all points (α, m) in the union $G \cup N$ are strictly admissible, then we have uniqueness for the boundary value problems (3.1), (3.2), (3.3) and (3.1), (3.2), (3.4). Let us follow the development as α increases from 0. By Lemma 8, we first have solutions in P. As the first boundary point α_m of G is reached, we have a solution in G. By Lemma 19, a right neighborhood of α_m belongs to N. By Lemma 18, as α continues to increase from α_m, the point $b(\alpha)$ moves strictly towards the lefthand side without retracing any previous locations. Thus all $\alpha > \alpha_m$ belongs to N, and no two values of α can solve the boundary value problem (3.1), (3.2), (3.3) for the same b.

The desired claim (that all members of $G \cup N$ are strictly admissible), although valid, is by no means easy to verify.

We introduce the function

$$\theta(r) = -ru'(r)/u(r), \qquad r \in [a, b), \tag{4.9}$$

for a solution $u \in G \cup N$.

Lemma 20. *The function θ is continuous in $[a, b)$, $\theta(a) = 0$ and $\lim_{r \to b} \theta(r) = \infty$.*

Proof. The last assertion for the case $b = \infty$ is the only non-trivial one. By Lemma 7, $-ru'(r)/u(r) > kr$ for large r. The conclusion then follows. □

We draw a horizontal line of height $\beta \geqq 0$ above the r-axis. It can interact with the graph of θ in any one of the ways depicted in Figures 2–4.

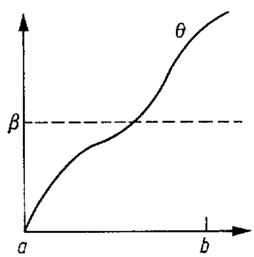

Fig. 2

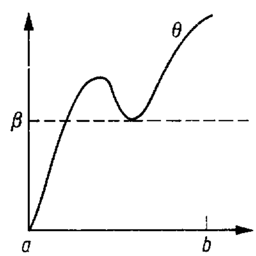

Fig. 3

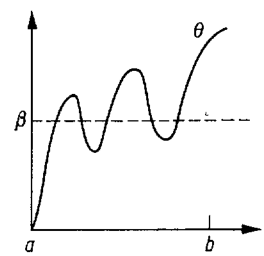

Fig. 4

With the given β, we define the function

$$v_\beta(r) = ru'(r) + \beta u(r), \qquad r \in [a, b]. \tag{4.10}$$

The next lemma relates the properties of v to the way θ interacts with the horizontal line. We omit the simple proof. The last assertion is nothing more than the elementary Inverse Function theorem.

Lemma 21. *$v(r) = (>, <) 0$ if and only if θ intersects (is below, is above) the straight line at r.*

v is tangent to the r-axis at r if and only if θ is tangent to the line at r.

The smallest zero of v (the first intersection point of θ with the straight line), denoted by $\varrho = \varrho_\beta$ is a non-decreasing function of β. If $\theta'(\varrho_\beta) \neq 0$, then the function ϱ is continuous at β.

It is easy to verify that v satisfies the differential equation

$$v'' + \frac{m}{r} v' + (p^{p-1} - 1)\, v = -2(u^p - u) + \beta(p - 1)\, u^p. \tag{4.11}$$

Let us define

$$\phi(u) = -2(u^p - u) + \beta(p-1)\, u^p. \tag{4.12}$$

The right-hand side of (4.11) is then the following composite function

$$\Phi(r) = \phi(u(r)).$$

Lemma 22. *For any* $\beta \geqq 0$, *there exists a unique point* $\sigma = \sigma_\beta \in [a, b)$ *such that*

$$\Phi(r) < 0 \text{ for } r < \sigma \quad \text{and} \quad \Phi(r) > 0 \text{ for } r > \sigma. \tag{4.13}$$

The point σ *is a continuous non-decreasing function of* β.

Proof. If $\beta(p-1) - 2 \geqq 0$, then $\sigma = a$. If $\beta(p-1) - 2 < 0$, $\phi(u)$ is a concave function of u. It therefore has exactly one zero u_0 distinct from 0. Then σ is the unique (since u is strictly decreasing) point for which $u(\sigma) = u_0$ in case $u_0 < \alpha$ and is a otherwise. The continuity of σ follows from the continuity of u_0 on β and the continuity of the inverse of u. Finally, the monotonicity of σ follows from the positivity of the term containing β in (4.12.) □

Let us follow the movement of ϱ and σ as β increases. For $\beta = 0$, $\varrho = 0$ and $\sigma > 0$. Thus at this starting moment, the point ϱ is positioned to the left of the point σ. As we increase β, ϱ moves to the right while σ moves to the left. For β large enough, ϱ will be very close to b while σ will be very close to a. Thus a switching of the positions of ϱ and σ is bound to occur. If it is not for the possibility of a discontinuous jump of ϱ, we can conclude at once that for some suitable value of β, the two points ϱ and σ meet head-on. Nevertheless, this assertion is still true.

Lemma 23. *There exists a* $\beta_0 > 0$ *at which* $\varrho = \sigma$.

Proof. We need to affirm the impossibility of a discontinuity of ϱ (by Lemma 21, we have only to show that $\theta'(\varrho) \neq 0$) before the meeting of the points takes place. Suppose, then, $\beta > 0$ is such that $\varrho_\beta \leqq \sigma_\beta$. By the definition of σ, $\Phi(r) \leqq 0$ for $r \leqq \sigma_\beta$. In particular, $\Phi(r) \leqq 0$ in the interval $[a, \varrho_\beta]$. Hence, in this interval, v is non-negative and satisfies the differential inequality

$$v'' + \frac{m}{r} v' + (pu^{p-1} - 1)\, v \leqq 0. \tag{4.14}$$

By Lemma 5, $v'(\varrho_\beta) \neq 0$. This is equivalent, by Lemma 21, so the fact $\theta'(\varrho_\beta) \neq 0$, as desired. □

What do θ and v and their properties have to do with our goal? We answer this question with the following lemma.

Lemma 24. *If* θ *intersects the straight line* $\theta = \beta_0$ *exactly once, then the corresponding* u, *or more precisely, the corresponding* (α, m) *is strictly admissible.*

Proof. The hypotheses imply that $v = v_{\beta_0}$ has exactly one zero in (a, b). Note that in case $b \neq \infty$, $v(b) = b_0 u'(b) < 0$. Thus v has exactly one zero in $[a, b]$. In (a, ϱ), v is positive and satisfies inequality (4.14), while w satisfies the equation (4.2). At a,

$$v'(a) = (1 + \beta_0)\, u'(a) + \beta_0 a u''(a) = \beta_0 a u''(a) < 0,$$

so that the comparison conditions for the initial point is verified. By Lemma 1, v oscillates faster than w, so that ϱ, the first zero of v is less than τ, the first zero of w. In the interval (τ, b), v is negative and satisfies the reverse differential inequality

$$v'' + \frac{m}{r} v' + (pu^{p-1} - 1)\, v \geqq 0. \tag{4.15}$$

Thus v again oscillates faster than w in (τ, b). By assumption v has no more zeros beyond τ, so w cannot have any zero beyond τ either. In case $b < \infty$, we conclude that $w(b) \neq 0$, and so (α, m) is strictly admissible. When $b = \infty$, using Lemma 3, we see that the disconjugacy interval for (4.2) is a proper superset of (τ, b). By Lemma 6, w is thus unbounded, and (α, m) is strictly admissible. □

Our next lemma tells us that we can even start out with less and yet end up with more than we expect.

Lemma 25. *Let ξ be the only point in (a, b) at which $u(\xi) = 1$. If $\theta(r) \geqq \beta_0$ for all $r \in [\varrho, \xi]$, then $\theta'(r) > 0$ for all $r \in [a, b)$. As a result the corresponding (α, m) is strictly admissible.*

Proof. We first prove that $\theta'(r) \geqq 0$ in $[\xi, b)$ in any case. Suppose this is not true. Then there exist local minima in (ξ, b), since $\lim_{r \to b} \theta(r) = \infty$. We draw a horizontal line of height β to touch the lowest of all such minima, say at the point $c \in (\xi, b)$. The corresponding v_β is then negative in (c, b) and has a double zero at $r = c$. In $[\xi, b)$, $\Phi(r) \geqq 0$, so v_β satisfies differential inequality (4.15). This contradicts Lemma 5.

Together with the hypotheses, we now have $\theta(r) \geqq \beta_0$ for all $r \in [\varrho, b)$. As already established in the proof of Lemma 23, $\theta'(r) > 0$ for $r \in [a, \varrho]$. Let us derive a contradiction by assuming that $\theta'(r) < 0$ for $r > \varrho$. Under this assumption, θ has local minima in (ϱ, b). Since $\theta(r) \geqq \beta_0$ in (r, b), such minima have height above β_0. Let us raise the horizontal line from height β_0 to a height β to touch the smallest of these local minima, say at a point $c > \varrho$. Then v_β is negative in (c, b), and has a double zero at c. Since $\beta \geqq \beta_0$, $\varrho_\beta \geqq \sigma_\beta$, implying that $\Phi(c) \geqq 0$. Again this contradicts Lemma 5. □

For convenience, we say that the point (α, m) is normal if the hypotheses of Lemma 25 is satisfied. In particular, if θ is monotone, or alternatively if $\theta' \geqq 0$, then (α, m) is normal. Such a θ must in fact be strictly montone. An important corollary of Lemma 25 is that normality is "contagious".

Lemma 26. *Suppose that $(\bar{\alpha}, \bar{m}) \in G \cup N$ is normal. There exists a neighborhood of $(\bar{\alpha}, \bar{m})$ within which all members of $G \cup N$ are normal.*

On the other hand, any limit point of a set of normal points is normal.

Proof. Let $\bar{\xi} = \xi(\bar{\alpha})$. Pick a point $\eta \in (\bar{\xi}, b(\bar{\alpha}))$. We first find a neighborhood of $(\bar{\alpha}, \bar{m})$ small enough that for all its members (α, m), $\xi(\alpha) < \eta < b(\alpha)$. This is possible because of the continuity of ξ and b on α. By Lemma 25, $\theta'(r, \bar{\alpha}) > 0$ for all $r \in [a, \eta]$. The continuity of θ on r, α, and m implies that given any r in this interval, there exists a neighborhood of r and a neighborhood of $(\bar{\alpha}, \bar{m})$ such that $\theta'(r, \alpha, m) > 0$ for all r and (α, m) in these neighborhoods. A compactness argument gives a neighborhood of (α, m) in which $\theta' > 0$ for all $r \in [\alpha, \eta]$. By Lemma 25, all members of $G \cup N$ in this neighborhood must be normal.

For any sequence of normal points, $\theta' > 0$ for all $r \in [a, b)$. After taking limit, $\theta' \geqq 0$ for each point in the domain of the limit function. Thus the limit point must be normal. □

Ths lemma has some useful repercussions. The first is the connectedness property of sets of normal points.

Lemma 27. *Let C be a connected subset of $G \cup N$. If one of its members is normal, then all it members are normal.*

Proof. The set of normal points in C is both relatively open and relatively closed in $G \cup N$. □

Lemma 28. *We fix an m. If $\bar{\alpha}$ is known to be normal, then there exists an $\alpha_0 \leqq \bar{\alpha}$ such that $\alpha_0 \in G_m$, $(\alpha_0, \infty) \subset N_m$, and all $\alpha \geqq \alpha_0$ are normal. For some $\varepsilon > 0$, the interval $(\alpha_0 - \varepsilon, \alpha_0)$ is disjoint from $G_m \cup N_m$.*

Proof. Let C be the largest connected component of $G_m \cup N_m$ containing $\bar{\alpha}$. Since C is a closed set, it has a least element α_0. By Lemma 27, all members of C, in particular α_0, are normal. By Lemma 24, each member of C, in particular α_0, is strictly admissible, and so by Lemma 19, a right neighborhood of α_0 belongs to N. By Lemma 18, the point $b(\alpha)$ is decreasing in α. Thus as we increase α from α_0, we remain in N. It follows that $(\alpha_0, \infty) \subset N$. The last assertion in the lemma is a consequence of the fact that a member of a connected component cannot be a limit point of other components. □

As required in the hypotheses of Lemma 27, we need to get hold of some normal points before we can start things rolling. In fact, we have many normal points.

Lemma 29. *For any $m \in [0, 1]$, all members of $G \cup N$ are normal.*

Proof. In the interval $[a, \xi]$,

$$(-ru'(r))' = -ru''(r) - u'(r) = -(1-m)\,u'(r) + r(u^p(r) - u(r)) \geqq 0. \quad (4.16)$$

Thus $-ru'(r)$ is non-decreasing in r in $[a, \xi]$. Since u is decreasing, the quotient $\theta(r) = -ru'(r)/u(r)$ is non-decreasing in $[a, \xi]$. Thus the solution is normal. □

This lemma yields our main result for the values $m \in [0, 1]$. McLeod & Serrin of course have obtained this part for the case $[a, b) = [0, \infty)$ using a different method. We are now ready to make some progress concerning other values of m.

Lemma 30. *If all points $m \in [0, \infty)$ are regular, then our main result holds, namely, that all the boundary value problems we are interested in have unique solutions.*

Proof. By Lemma 16, if all m are regular, the curve of the function $\alpha(m)$ is continuous on $[0, \infty)$. The set $G \cup N$ coincides with the set $\{(\alpha, m) : \alpha \geqq \alpha(m)\}$ of points on or above the curve. This is a large, connected piece. By Lemmas 27 and 29, every member of this set is normal. By Lemma 24, they are all strictly admissible, and our main result follows. □

To establish our main result, we assume that the hypothesis of this lemma is not true and derive a contradiction. Thus let us suppose that there is a largest connected interval of regularity $[0, \overline{m})$, $\overline{m} < \infty$.

Let us first dispose of the possibility that $\overline{m}$ is regular. Suppose it is. Then by Lemma 16, the curve of $\alpha(m)$ is continuous up to the endpoint $\overline{m}$. There exists a sequence of irregular points $m_i > \overline{m}$ $(i = 1, 2, .)$, such that $\lim_{i \to \infty} m_i = \overline{m}$. For each i, let α_i be the smallest member of G on the line $m = m_i$. Just as in the proof of Lemma 16, it can be shown that the sequence α_i cannot has a subsequence converging to a point in G or N. It follows that $\lim_{i \to \infty} \alpha_i = \alpha(\overline{m})$. Some of these α_i must enter into the normality neighborhood of $(\alpha(\overline{m}), \overline{m})$ constructed in Lemma 16. Such α_i are therefore normal. By Lemma 28, all $\alpha > \alpha_i$ on the line $m = m_i$ must be in N. This contradicts the irregularity of m_i with α_i being the smallest but not the unique member of G. It remains to show that the other possibility, that $\overline{m}$ is irregular, is also void. It takes several more lemmas. First of all, let us see how the sets G_m, N_m, and P_m look like on the line $L : m = \overline{m}$. The curve of $\alpha(m)$ for $m < \overline{m}$ must be continuous up to the endpoint $\overline{m}$, lest by Lemma 16, the limit point set of the curve on L is a non-degenerate closed interval, a possibility excluded by Lemma 18. Let $\alpha_0 = \lim_{m \to \overline{m}} \alpha(m)$. By Lemma 26, α_0 must be normal and, since it belongs to G, must coincide with the α_0 found in Lemma 28. Hence the part of L above α_0 belongs to $N_{\overline{m}}$. The set under the graph of $\alpha(m)$, $m < \overline{m}$, shown as the shaded area in Figure 5, belongs to G. The part of L below α_0, shown as the dotted line in the figure, cannot contain any members of the open set N, lest every neighborhood of such a member will intrude into the shaded area and therefore cannot be made up of points in N alone.

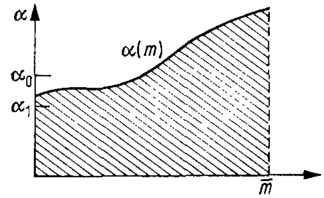

Fig. 5

Lemma 31. *Let α_0 be the largest member of $G_{\bar{m}}^-$ as described above, and α_1 be the next member of G smaller than α_0. Let ψ denote the largest zero of the function $w(r, \alpha_1)$. Then for all $\alpha < \alpha_0$ and $\alpha \neq \alpha_1$, $u(\psi, \alpha) < u(\psi, \alpha_1)$.*

Proof. By Lemma 28, α_0 cannot be a limit point of smaller members of G. Thus α_1 exists. The existence of a largest zero of w (i.e., w is non-oscillatory near ∞) follows from the fact that w satisfies differential equation (4.2), which is of the form considered in Lemma 6. To simplify the notations, we use u to represent the solution corresponding to the parameter α and u_1 that corresponding to α_1. Suppose for some α the conclusion of the lemma is not true; in other words, we have instead

$$u(\psi) \geqq u_1(\psi). \tag{4.17}$$

We claim that the two solutions u and u_1 cannot intersect at a point in (ψ, ∞). The arguments we use are similar to those in the proof of Lemma 19. Suppose the solutions do intersect, for the first time at $r = \mu > \psi$. In view of (4.17), u is below u_1 in a right neighborhood of the point μ. Suppose the former catches up with the latter at some point beyond μ. Let ν be the next point where the two solutions intersect again. In the interval $[\mu, \nu]$, $u \leqq u_1$ and the function $z = u_1 - u$ satisfies the differential equation

$$z'' + \frac{m}{r} z' + \left(\frac{u_1^p - u^p}{u_1 - u} - 1 \right) z = 0. \tag{4.18}$$

Notice that the coefficient in the last term is smaller than that in equation (4.2) for $w(r, \alpha_1)$. Therefore, w oscillates more than z. Since z is zero at the endpoints of $[\mu, \nu]$, w must have at least one zero in the interval, contradicting the choice of ψ as the last zero of w. Thus u must remain below u_1 in (μ, ∞). In this interval, the coefficient of the last term of (4.19) is smaller than that of (4.2). Hence the disconjugacy interval of (4.19) is larger than that of (4.2), which is (ψ, ∞). Since z has a zero within the disconjugacy interval, by Lemma 6, z is unbounded, contradicting the fact that u is being trapped between u_1 and the r-axis.

Thus we have

$$u(r) > u_1(r), \quad \text{for all } r > \psi. \tag{4.19}$$

In the interval $[\psi, \infty)$, the same function z as defined above satisfies (4.18), but this time the coefficient in the last term is larger than that in (4.2). Since z does not vanish in $[\psi, \infty)$, the disconjugacy interval of (4.18), and hence also that of the less oscillatory (4.2), is a proper superset of $[\psi, \infty)$. Hence, by Lemma 6,

$$\lim_{r \to \infty} w(r, \alpha_1) = \pm \infty. \tag{4.20}$$

By Lemma 19, a right or left neighborhood of α belongs to N, contradicting the fact that no point below α_0 can be in N. □

Lemma 32. *The solutions* $u_0 = u(r, \alpha_0)$ *and* $u_1 = u(r, \alpha_1)$ *cannot intersect more than once.*

Proof. The technique used in the proof of Lemmas 19 and 31 applies. The function $z = u_0 - u_1$ satisfies the differential equation

$$z'' + \frac{m}{r} z' + \left(\frac{u_0^p - u_1^p}{u_0 - u_1} - 1 \right) z = 0, \tag{4.21}$$

which oscillates faster than (3.1) for $u(r, \alpha_0)$ but slower than (4.2) for $w(r, \alpha_0)$, as long as $u_0 > u_1$. Thus z must have a zero in $[a, \infty)$, after the only zero τ of $w(r, \alpha_0)$. Suppose z has more than one zero, contrary to the conclusion of the Lemma. Then after the second zero, τ_2, (4.21) again oscillates more slowly than the equation for w. Since the disconjugacy interval of the equation for w must be at least $[\tau, \infty)$, that of the "less oscillatory" (4.21) must be at least $[\tau_2, \infty)$. By Lemma 6, z is therefore unbounded, an obvious contradiction. □

Lemma 33. *For all* $\alpha \in (\alpha_1, \alpha_0)$, *the solutions* $u = u(r, \alpha)$ *and* $u_1 = u(r, \alpha_1)$ *cannot intersect more than once in* $[a, \psi]$.

Proof. As we vary the parameter α from α_0 towards α_1, we have a continuous deformation of the solution curve u over the closed interval $[a, \psi]$. At the left endpoint $r = a$, the curve u stays clear above that of u_1, while at the right endpoint $r - \psi$, the former stays clear below the latter. We start out with one single point of intersection, when $\alpha = \alpha_0$. The number can increase only if at some point α, the curve of u bulges up or down somewhere to touch the curve of u_1. But this is impossible because the function $z = u_1 - u$ satisfies a "linear" second-order differential equation, namel,y (4.19), and so cannot have a double zero. □

Lemma 34. *The point* $(\alpha_1, \overline{m})$ *is admissible.*

Proof. Since ψ has been chosen to be the last zero of $w(r, \alpha_1)$, we need to show that w has no other zeros before ψ, in order to satisfy the definition of admissibility.

Let us first show that there cannot be more than one zero before ψ. Suppose this is not the case. Then for a point μ between the first and second zeros, $w(\mu) = \partial u(u)/\partial \alpha < 0$. Likewise for a point ν between the second and third (which may be ψ) zeros $w(\nu) = \partial u(\nu)/\partial \alpha > 0$. We can choose an $\alpha > \alpha_1$ sufficiently close to

α_1, that $u(\mu) > u_1(\mu)$ but $u(\nu) < u_1(\nu)$. By the Intermediate Value Theorem, u must intersect u_1 at least once in (a, μ) and another time in (ν, ϕ), contradicting Lemma 33.

Next we show that w cannot have exactly one zero before ψ. Suppose it does; then for c between the two zeros of w, $w(c) < 0$. Hence, for $\alpha < \alpha_0$ but close to α_0, $u(c) > u_1(c)$. It follows that u must intersect u_1 at least twice, once in (a, c) and once in (c, ψ). A continuity argument as in the proof of Lemma 33 shows that all solutions u with $\alpha < \alpha_1$ must intersect u_1 at least twice. But this contradicts the obvious fact that the solution $u(r, 1) \equiv 1$ intersects u_1 only once. □

The last lemma we need turns out to be the most surprising.

Lemma 35. *Admissibility implies normality.*

Proof. Suppose the point in question is not normal. Then the graph of θ intersects the line at height β_0 more than once in $[\varrho, \xi]$, as shown in Figure 6.

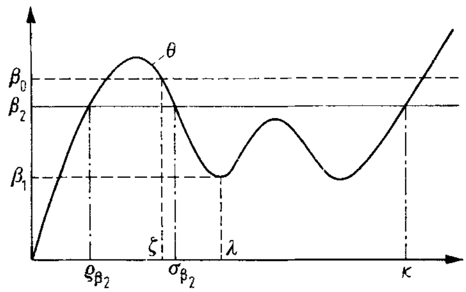

Fig. 6

Let ζ be the next point of intersection after ϱ. Using Lemma 5 as in the proof of Lemma 25, we see that θ cannot be tangent to the straight line at ζ. We now lower the horizontal line to a height β_1 when it touches the first point λ to the right of ζ at which $\theta'(\lambda) = 0$. Our useful Lemma 5 shows that $\sigma_{\beta_1} > \lambda$. In the interval $[\zeta, \lambda]$ the function θ is strictly monotone, so that the inverse function is continuous. In other words, the point of intersection of the graph with the line $\theta = \beta$ is a continuous function of β. As β varies from β_1 to β_0 a switching of positions of the point of intersection with the curve in $[\zeta, \lambda]$ and the point σ has taken place. Thus, by continuity, there is a β_2 at which σ coincides with the point of intersection in $[\zeta, \lambda]$. Let us focus on the line $\theta = \beta_2$. In the interval (ϱ, σ), $\Phi(r) < 0$ but v is negative so that equation (4.11) oscillates less than equation (4.2). This imply that w must have a zero in between the two zeros, ϱ and σ, of v. Since θ tend to ∞ near b, the graph of θ must intersect the line one more time after σ, say at $\varkappa$. In the interval $(\sigma, \varkappa)$, $\Phi(r) > 0$ but v is positive so that again (4.11) oscillates less than equation (4.2). As a result, w must have another zero between the two zeros, σ and $\varkappa$ of v. This contradicts the admissibility of the solution. □

We can now complete the proof of our main theorem by observing that Lemmas 34 and 35 imply that the point α_1 is normal. But by Lemma 28, the point $\alpha_0 > \alpha_1$ will be in $N_{\bar{m}}^-$, obviously contradicting the definition of α_0.

5. Main Result, $a \geqq 0$

The fact that there is a critical exponent for boundary value problems on finite intervals $[0, b]$ is a reminder of the presence of the singular term mu'/r in the differential equation. The singularity is, however, more benign than it first appears. For any fixed m, the solution u still depends continuously on the parameters α, at least in any compact subinterval of $[0, b)$. From now on we fixed an m that is less than the critical exponent, so we no longer think of u as depending on m. Instead, we affix the initial point a, at which (3.5) is imposed, to the parameter list. Now $u = u(r, \alpha, a)$ is continuous in a at each fixed point on the r-axis except the origin.

Instead of considering the (α, m) plane as before, we now have the (α, a) quadrant $(0, \infty) \times [0, \infty)$. We define the same sets N, G, and P as before but with a in place of m. The two sets N and P are still open. Since for each $a > 0$, uniqueness holds for all the boundary value problems in question, each vertical line through an $a > 0$ contains exactly one point $(\alpha(a), a)$ in G. The function $\alpha(a)$ gives a continuous curve that must converge to a single limit point $\alpha(0)$ on the line $a = 0$. All the lemmas on admissibility and normality holds with m replaced by a. Thus all the arguments in the last section can be repeated. In particular, the point $\alpha(0)$ is normal since it is the limit point of the curve $\alpha(a)$, which consists of normal points. The half line $(\alpha(0), \infty)$ coincides with N. We must verify that the other half line $(0, \alpha(0))$ cannot contain members of G. That is done by use of lemmas analogous to Lemmas 31 through 35.

We summarize our results in one main theorem.

Theorem. *Under any one of the following conditions*:

1. $a > 0, \; m \geqq 0, \; p > 1, \; a < b \leqq \infty,$
2. $a = 0, \; 0 \leqq m \leqq 1, \; p > 1, \; 0 < b \leqq \infty,$
3. $a = 0, \; m > 1, \; 1 < p < \dfrac{m+3}{m-1}, \; 0 < b \leqq \infty,$

here is exactly one positive solution to the boundary value problem

$$u''(r) + \frac{m}{r} u'(r) + u^p - u = 0, \qquad u(r) > 0, \qquad r \in (a, b), \tag{5.1}$$

$$u'(a) = 0, \tag{5.2}$$

$$u(b) = 0 \tag{5.3}$$

or

$$\lim_{r \to \infty} u(x) = 0 \qquad \text{if } b = \infty. \tag{5.4}$$

For a fixed endpoint a, $\alpha = u(a)$, the value of the solution at a, is a strictly decreasing function of the other endpoint b. Let α_0 be the initial height of the solution of the boundary value problem when $b = \infty$. No solutions with initial height below α_0 can intersect the r-axis.

Equivalently, under one of the three conditions listed above, there exists a unique positive radially symmetric solution of the reaction-diffusion equation

$$\Delta u + u^p - u = 0, \quad a < |x| < b, \tag{5.5}$$

with the Neumann boundary condition at $|x| = a$ *if* $a \neq 0$, *and the Dirichlet boundary condition at* $|x| = b$ (*or* $u(x) \to \infty$ *as* $|x| \to \infty$ *if* $b = \infty$).

Although we have made no attempt to seek the most general nonlinearity that our method can handle, it is obvious that the concavity of the function $u^p - u$ plays a crucial part. It is interesting to see if that alone is sufficient.

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