One iteration of the simplex algorithm

Start with feasible basis B

`while` B is not optimal

Let $i \in B$ be index with $\lambda_i < 0$
Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0, j \in B \setminus \{i\}$ and $a_i^T d = -1$
Determine $K = \{k \colon 1 \leq k \leq m,\ a_k^T d > 0\}$

`if` $K = \emptyset$

assert LP unbounded

`else`

Let $k \in K$ index where $\min\limits_{k \in K} (b_k - a_k^T x^*)/a_k^T d$ is attained

update $B := B \setminus \{i\} \cup \{k\}$

One iteration of the simplex algorithm

Theorem

One iteration of the simplex algorithm requires a total number of $O(m \cdot n)$ operations on rational numbers whose size is polynomial in the input size.

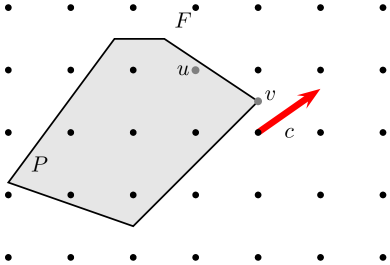
Smale’s 18 problems for the next century

Problem 18: Can linear programming be solved in strongly polynomial time?

Integer Programming

$$\max c^T x$$
$$Ax \leq b$$
$$x \in \mathbb{Z}^n$$

with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$.



Complexity of integer programming

Theorem

The integer feasibility problem is NP-complete.

The LP-relaxation

Theorem

If x^* is an optimal solution of linear programming relaxation $\max\{c^T x : Ax \leq b, x \in \mathbb{R}^n\}$, then $c^T x^* \geqslant c^T x_I$, for each integer feasible solution x_I.

The LP-relaxation

Theorem

If x^* is integral optimal solution of the linear programming relaxation $\max\{c^T x : Ax \leq b, x \in \mathbb{R}^n\}$, then x^* is also an optimal solution of the integer programming problem $\max\{c^T x : Ax \leq b,\ x \in \mathbb{Z}^n\}$

Undirected graphs – Matchings

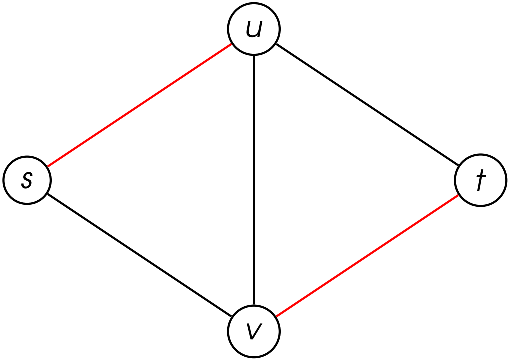
Definition

An undirected graph is a tuple $G = (V, E)$

V is finite set of vertices

$E \subseteq \binom{V}{2}$ is set of edges of G.

Matching: $M \subseteq E$ such that for all $e_1 \neq e_2 \in M$ one has $e_1 \cap e_2 = \emptyset$.



$\delta(v)$

For a vertex $v \in V$, the set $\delta(v) = \{e \in E \colon v \in e\}$ denotes the incident edges to v.

Max-weight matching as IP

$$
\begin{aligned}
\max \sum_{e \in E} w(e)x(e) \\
v \in V : \sum_{e \in \delta(v)} x(e) \leq 1 \\
e \in E : x(e) \geqslant 0 \\
x \in \mathbb{Z}^{|E|}.
\end{aligned}
$$

Example

Integral polyhedra

Definition

A rational polyhedron is called integral if each nonempty face of P contains an integer vector.

Simplex on polyhedra whose vertices are integral

Lemma

Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be an integral polyhedron with $A \in \mathbb{R}^{m \times n}$ full-column rank. If the linear program

$$\max\{c^T x \colon x \in \mathbb{R}^n, Ax \leq b\} \tag{7}$$

is feasible and bounded, then the simplex method computes an optimal integral solution to the linear program.

The study of integral polyhedra

Lemma

Let $A \in \mathbb{Z}^{n \times n}$ be an integral and invertible matrix. One has $A^{-1}b \in \mathbb{Z}^n$ for each $b \in \mathbb{Z}^n$ if and only if $\det(A) = \pm 1$.

Total unimodularity

Definition

An integral matrix $A \in \{0, \pm 1\}^{m \times n}$ is called totally unimodular if each of its square sub-matrices has determinant $0, \pm 1$.

Example: Node-edge incidence matrix of bipartite graph

Hoffman-Kruskal Theorem

Theorem

Let $A \in \mathbb{Z}^{m \times n}$ *be an integral matrix. The polyhedron* $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geqslant 0\}$ *is integral for each integral* $b \in \mathbb{Z}^m$ *if and only if* A *is totally unimodular.*