Review: Duality

Dual of the Dual

Corollary

If the dual linear program has an optimal solution, then so does the primal linear program and the objective values coincide.

Table of possibilities

Further example

$$\begin{aligned} \max \quad & c^T x \\ Bx &= b \\ Cx &\leq d. \end{aligned}$$

(Primal)

$$\begin{aligned} \min\ b^T y_1 + d^T y_2 \\ B^T y_1 + C^T y_2 &= c \\ y_2 &\geqslant 0. \end{aligned}$$

(Dual)

Zero sum games

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 3 & 2 & 4 \\ -3 & 0 & 1 \end{pmatrix}$$

Row player: Chooses row i

Column player: Chooses column j

Rock – paper – scissors

Deterministic strategies

$$\max_i \min_j$$

$$\max_j \min_i$$

Mixed strategies

Definition

Let $A \in \mathbb{R}^{m \times n}$ define a two-player matrix game. A mixed strategy for the row-player is a vector $x \in \mathbb{R}^m_{\geqslant 0}$ with $\sum_{i=1}^m x_i = 1$. A mixed strategy for the column player is a vector $y \in \mathbb{R}^n_{\geqslant 0}$ with $\sum_{j=1}^n y_i = 1$.

$$E[\text{Payoff}] = x^T A y. \quad (5)$$

Example: Mixed strategy rock – paper – scissors

Weak duality

Lemma

Let $A \in \mathbb{R}^{m \times n}$*, then*

$$\max_{x \in X} \min_{y \in Y} x^T A y \leq \min_{y \in Y} \max_{x \in X} x^T A y,$$

where X *and* Y *denote the set of mixed row and column-strategies respectively.*

Minimax-Theorem

Theorem (von Neumann (1928))

$$\max_{x \in X} \min_{y \in Y} x^T A y = \min_{y \in Y} \max_{x \in X} x^T A y,$$

where X and Y denote the set of mixed row and column-strategies respectively.

Duality via Farkas' lemma

Theorem (Second variant of Farkas' lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The system $Ax \leq b$ has a solution if and only if for all $\lambda \geqslant 0$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.

Duality via Farkas' lemma

Algorithms and running time analysis

Consider the following algorithm to compute the product of two $n \times n$ matrices $A, B \in \mathbb{Q}^{n \times n}$:

for $i = 1, \ldots, n$
  for $j = 1, \ldots, n$
    $c_{ij} := 0$
    for $k = 1, \ldots, n$
      $c_{ij} := c_{ij} + a_{ik} \cdot a_{kj}$

O-notation

Definition

Let $T, f : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be functions. We say

- $T(n) = O(f(n))$, if there exist positive constants $n_o \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ with
$$T(n) \leq c \cdot f(n) \text{ for all } n \geqslant n_0.$$
- $T(n) = \Omega(f(n))$, if there exist constants $n_o \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ with
$$T(n) \geqslant c \cdot f(n) \text{ for all } n \geqslant n_0.$$
- $T(n) = \Theta(f(n))$ if
$$T(n) = O(f(n)) \text{ and } T(n) = \Omega(f(n)).$$

Example

Example

The function $T(n) = 2n^2 + 3n + 1$ is in $O(n^2)$, since for all $n \geqslant 1$ one has $2n^2 + 3n + 1 \leq 6n^2$. Here $n_0 = 1$ and $c = 6$. Similarly $T(n) = \Omega(n^2)$, since for each $n \geqslant 1$ one has $2n^2 + 3n + 1 \geqslant n^2$. Thus $T(n)$ is in $\Theta(n^2)$.

Efficient algorithm, first definition

An algorithm runs in polynomial time, if there exists a constant k such that the algorithm runs in time $O(n^k)$, where n is the length of the input of the algorithm.