Review: The simplex algorithm

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Finding an initial feasible basis

Linear program has equivalent form

$$\max\{c^T x : Ax \leq b,\ x \geqslant 0\}. \qquad (4)$$

Split $Ax \leq b$ as

$A_1 x \leq b_1$ and $A_2 x \leq b_2$

with $b_1 \geqslant 0$ and $b_2 < 0$.

Duality

$$\max\{c^T x : x \in \mathbb{R}^n,\ Ax \leq b\},$$

(Primal)

$$\min\{b^T y : y \in \mathbb{R}^m,\ A^T y = c,\ y \geqslant 0\}.$$

(Dual)

Duality

Weak Duality

Theorem (Weak duality)

If x^* and y^* are primal and dual feasible solutions respectively, then $c^T x^* \leq b^T y^*$.

Strong duality

Theorem

If the primal linear program is feasible and bounded, then so is the dual linear program. Furthermore in this case, both linear programs have an optimal solution and the optimal values coincide.

Dual of the Dual

$$\min\{b^T y : y \in \mathbb{R}^m, A^T y = c, y \geqslant 0\}$$

Dual of the Dual

Corollary

If the dual linear program has an optimal solution, then so does the primal linear program and the objective values coincide.

Table of possibilities

Further example

$$\begin{aligned} \max \quad & c^T x \\ Bx &= b \\ Cx &\leq d. \end{aligned}$$

(Primal)

$$\begin{aligned} \min\ b^T y_1 + d^T y_2 & \\ B^T y_1 + C^T y_2 &= c \\ y_2 &\geqslant 0. \end{aligned}$$

(Dual)

Zero sum games

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 3 & 2 & 4 \\ -3 & 0 & 1 \end{pmatrix}$$

Row player: Chooses row i

Column player: Chooses column j

Rock – paper – scissors

Deterministic strategies

$\max_i \min_j$

$\max_j \min_i$

Mixed strategies

Definition

Let $A \in \mathbb{R}^{m \times n}$ define a two-player matrix game. A mixed strategy for the row-player is a vector $x \in \mathbb{R}^m_{\geqslant 0}$ with $\sum_{i=1}^m x_i = 1$. A mixed strategy for the column player is a vector $y \in \mathbb{R}^n_{\geqslant 0}$ with $\sum_{j=1}^n y_i = 1$.

$$E[\text{Payoff}] = x^T A y. \tag{5}$$

Example: Mixed strategy rock – paper – scissors

Weak duality

Lemma

Let $A \in \mathbb{R}^{m \times n}$*, then*

$$\max_{x \in X} \min_{y \in Y} x^T A y \leq \min_{y \in Y} \max_{x \in X} x^T A y,$$

where X *and* Y *denote the set of mixed row and column-strategies respectively.*

Minimax-Theorem

Theorem (von Neumann (1928))

$$\max_{x \in X} \min_{y \in Y} x^T A y = \min_{y \in Y} \max_{x \in X} x^T A y,$$

where X and Y denote the set of mixed row and column-strategies respectively.