The simplex algorithm

$$\max c^T x$$
$$Ax \leq b$$

$A \in \mathbb{R}^{m \times n}$

$\mathrm{rank}(A) = n$

Notation

Let $B \subseteq \{1, \ldots, m\}$

$A_B \in \mathbb{R}^{|B| \times n}$ rows indexed by B

$b_B \in \mathbb{R}^{|B|}$ components indexed by B

Example: For $A = \begin{pmatrix} 3 & 2 \\ 7 & 1 \\ 8 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$

and $B = \{2, 3\}$, one has

$$A_B = \begin{pmatrix} 7 & 1 \\ 8 & 4 \end{pmatrix} \text{ and } b_B = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

Feasible basis

Definition

An index set $B \subseteq \{1, \ldots, m\}$ is a basis if $|B| = n$ and A_B is non-singular. If in addition $x^* = A_B^{-1} b_B$ is feasible, then B is called a feasible basis.

Feasible basis vs extreme point

Optimal basis

Definition

A basis B is called optimal if it is feasible and the unique $\lambda \in \mathbb{R}^m$ with

$$\lambda^T A = c^T \quad \text{and } \lambda_i = 0,\ i \notin B \tag{3}$$

satisfies $\lambda \geqslant 0$.

Optimal basis vs. optimal solution

Theorem

If B is an optimal basis, then $x^* = A_B^{-1} b_B$ is an optimal solution of the linear program.

Moving to an improving vertex

$d \in \mathbb{R}^n$ unique solution to

$$a_j^T d = \begin{cases} 0 & \text{for } j \in B \setminus \{i\} \\ -1 & \text{if } j = i. \end{cases} \qquad c^T d =$$

How far can we move?

The simplex algorithm

Start with feasible basis B

`while` B is not optimal

Let $i \in B$ be index with $\lambda_i < 0$
Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0$, $j \in B \setminus \{i\}$ and $a_i^T d = -1$
Determine $K = \{k \colon 1 \leq k \leq m,\ a_k^T d > 0\}$

`if` $K = \emptyset$
assert LP unbounded
`else`
Let $k \in K$ index where $\min_{k \in K} (b_k - a_k^T x^*)/a_k^T d$ is attained
update $B := B \setminus \{i\} \cup \{k\}$

```
from sympy import *
A = Matrix([[1, 2, 2],
            [2, 1, 2],
            [2, 2, 1],
            [-1, 0, 0],
            [0, -1, 0],
            [0, 0, -1]])

b = Matrix([10,14,11,0,0,0])
c = Matrix([6,14,13])
r = Matrix([0,-1,0])

B = [0,1,2]

A_B = A[B,:]
b_B = b[B,:]

x =  A_B.solve(b_B)
l = A_B.transpose().solve(c)
d = A_B.transpose().solve(r)
```

Example

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; b = \begin{pmatrix} 10 \\ 14 \\ 11 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 6 \\ 14 \\ 13 \end{pmatrix}$$

starting basis

$$B = \{1, 2, 3\}.$$

$$A_B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \; b_B = \begin{pmatrix} 10 \\ 14 \\ 11 \end{pmatrix} \text{ and } x_B^* = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}.$$

$$\lambda_B = \begin{pmatrix} \frac{36}{5} \\ -\frac{4}{5} \\ \frac{1}{5} \end{pmatrix}, \; d = \begin{pmatrix} -\frac{2}{5} \\ \frac{3}{5} \\ -\frac{2}{5} \end{pmatrix}.$$

$$A \cdot d = \begin{pmatrix} 0 \\ -1 \\ 0 \\ \frac{2}{5} \\ -\frac{3}{5} \\ \frac{2}{5} \end{pmatrix}, \; b - Ax^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 3 \end{pmatrix}.$$

Non-degenerate LPs

Definition

The LP $\max\{c^T x \colon x \in \mathbb{R}^n, Ax \leq b\}$ is non-degenerate if number of zero-components in $Ax - b \in \mathbb{R}^m$ is at most n for each $x \in \mathbb{R}^n$.

Termination non-degenerate case

Theorem

If the linear program is non-degenerate, then the simplex algorithm terminates.

Smallest index rule

Start with feasible basis B

`while` B is not optimal

Compute $\lambda \in \mathbb{R}^n$ such that $\lambda^T A_B = c^T$
Let $i^* \in B$ be the smallest index with $\lambda_i < 0$
Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0, j \in B \setminus \{i^*\}$ and $a_{i^*}^T d = -1$
Determine $K = \{k \colon 1 \leq k \leq m, \, a_k^T d > 0\}$

`if` $K = \emptyset$

assert LP unbounded

`else`

Let $k^* \in K$ the smallest index where $\min_{k \in K} (b_k - a_k^T x^*)/a_k^T d$ is attained

update $B := B \setminus \{i^*\} \cup \{k^*\}$

Termination

Theorem

The simplex algorithm with the smallest index rule terminates.