One iteration of the simplex algorithm

Start with feasible basis $B \subseteq \{1, \dots, m\}$.

$A_B \in \mathbb{R}^{n \times n}$ invertible.

Suppose: A_B^{-1} available.

$\lambda_B^T \cdot A_B = c^T$ $\quad \exists i \in B, (\lambda_B)_i < 0$

$\lambda_B^T = c^T \cdot A_B^{-1}$ $\boxed{O(n^2)}$

```
while $B$ is not optimal
$O(n^2)$      Let $i \in B$ be index with $\lambda_i < 0$
$O(n^2)$      Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0, j \in B \setminus \{i\}$ and $a_i^T d = -1$
$O(m \cdot n)$  Determine $K = \{k \colon 1 \leq k \leq m, a_k^T d > 0\}$   $A \cdot d$
              if $K = \emptyset$
                  assert LP unbounded
              else
$O(m \cdot n)$      Let $k \in \underbrace{K}$ index where $\min_{k \in K}(b_k - a_k^T x^*)/\underbrace{a_k^T d}_{O[m \cdot n]}$ is attained
                  update $B := B \setminus \{i\} \cup \{k\}$
```

$$A_B \cdot d = \begin{pmatrix} 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$$

$d = A_B^{-1}(-e_i)$ $\boxed{O(n)}$

$\underbrace{A \cdot x^* - b}_{O[m \cdot n]}$

ALLTOGETHER: $m \geq n \Rightarrow O(m \cdot n)$ arithmetic operations.

Updating the inverse: Computing $A_{B'}^{-1}$

$B' = \overset{i \uparrow \; \downarrow j}{B}$

- We have A_B^{-1}

- $$i \rightarrow \underbrace{A_{B'} \cdot A_B^{-1}}_{\uparrow} = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 0 & \cdots & & \\ \overset{0}{a_1} & a_2 & \cdots & & & a_n \\ 0 & 0 & 0 & 1 & 0 & \cdots\ 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \cdots \\ & & & & & \ddots\ 1 \end{pmatrix} \begin{array}{l} \rightarrow \times a_1 \\ \downarrow - \\ \leftarrow \\ \\ \\ \\ \end{array}$$

← assuming index i in position 3 of B, $B = \{b_1, b_2, i, b_3, \dots b_n\}$

perform same elementary row-operation on A_B^{-1}

⇝ $A_{B'}^{-1}$

Update of A_B^{-1} to $A_{B'}^{-1}$ requires $O(n^2)$ arithmetic operations.

Still need to find out:

sizes of rational numbers of A_B^{-1} for a basis B are polynomial in n and the largest size of component of A

Recall: Hadamard bound: $C \in \mathbb{R}^{n\times n}$, then $|\det(C)| \le \prod_{i=1}^{n} \|c_i\|_2$

$C=(c_1,\dots,c_n)$. **In our case:** assume $A \in \mathbb{Z}^{m\times n}$,

We have to estimate size(A_B^{-1}) for a basis $B \subseteq \{1,\dots,m\}$.

$$\det(A_B) \le \prod_{i=1}^{n} \underbrace{(\sqrt{n}\cdot B)}$$

B is upper bound on abs. value of entry in A.

$$= n^{n/2}\cdot B^{n}$$

$$A_B^{-1} = \operatorname{Adj}(A) \Big/ \det(A_B)$$

each entry is $(n-1)\times(n-1)$ minor.

$$\text{size}(A_B) = O(n(\log(n)+\text{size}(B)))$$

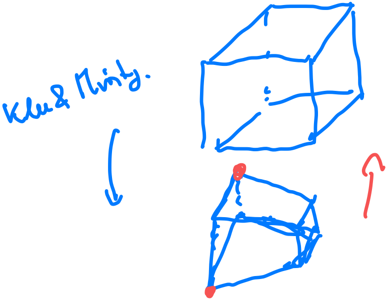
Similarly, we can bound numerators.

One iteration of the simplex algorithm

Theorem

One iteration of the simplex algorithm requires a total number of $O(m \cdot n)$ *operations on rational numbers whose size is polynomial in the input size.*

Open problem: Is there a pivoting rule (choice of entering and leaving element) that guarantees a polynomial number (in m) iterations through WHILE Loop?



$0 \le x_i$ $i=1,\dots,n$ Dim $= n$

$x_i \le 1$ $i=1,\dots,n$ #ineq. $m = 2n$

# of vertices: 2^n

Forces smallest index rule to visit all vertices.

Smale's ~~18~~ problems for the next century

David Hilbert 1900 : 23 open problems for the century.

Problem 18: Can linear programming be solved in strongly polynomial time?

Subquestion: Is there a pivot rule for simplex alg. guaranteeing $p(m)$ iterations, where $p(x) \in \mathbb{Z}[x]$ is fixed polynomial.

Later: $\exists$ algorithm that runs in $p(m, \text{size}(B))$ steps

where B is largest absolute value of number in the input.

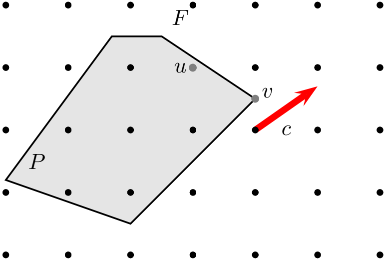
Integer Programming

$OPT_{LP} > OPT_{IP}$

$$\max c^T x$$
$$Ax \leq b$$
$$\underline{x \in \mathbb{Z}^n}$$

LP: $x \in \mathbb{Q}^n$

with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$
and $c \in \mathbb{Z}^n$.



Complexity of integer programming

Theorem

The integer feasibility problem is NP-complete.

Proof: Satisfyability (SAT)   Varibles: $x_1, \ldots, x_n$ (TRUE/FALSE)

m Clauses. $C = \{ x_1, \overline{x}_3, x_7, x_8 \}$

Set of literals

literal is either a varible x_i

or a negated varible $\overline{x}_i$

Determine: Does there exist assignment

$\phi : \{x_1, \ldots, x_n\} \to \{True, False\}$.

satisfying all clauses.

A Clause is satisfied by ϕ
if at least one literal is satisfied.

x_i satisfied if $\phi(x_i) = TRUE$

$\overline{x}_i$ —"— if $\phi(x_i) = FALSE$

AND

Example: $\{x_1, \overline{x}_5, x_2\} \wedge \{x_5, \overline{x}_2\}$ satisfies all clauses.

ϕ: $x_1 \to T$, $x_5 \to T$. / rest arbitrary

Example: **Var:** x_1, x_2, x_3 **FORMULA:** $\{x_1, \overline{x_2}, x_3\} \wedge \{\bar{x}_1, x_2\} \wedge \{\bar{x}_2, \bar{x}_3\}$

Idea: Integer program with same variables.

$$0 \le x_1 \le 1, \quad 0 \le x_2 \le 1, \quad 0 \le x_3 \le 1, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{Z}^3$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

0/1 0/1 0/1

Models FALSE/TRUE

$$x_1 + (1 - x_2) + x_3 \geq 1 \qquad (x_1 - x_2 + x_3 \geq 0)$$

$$1 - x_1 + x_2 \geq 1 \qquad (-x_1 + x_2 \geq 0)$$

$$-x_2 - x_3 \geq -1$$

This INTEGER PROGRAM HAS feas solution $\Longleftrightarrow$ FORMULA IS SATISFYABLE

- Satisfyability is NP- complete
- Tranlet any instance of SAT to INTEGER PROGRAMMING Feasibility problem.

- MESSAGE IP is hard

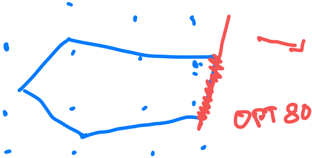
The LP-relaxation

IP: $\max c^T.x$, $Ax \leq b$, $x \in \mathbb{Z}^n$ → LP-RELAXATION: $\max c^T.x$, $Ax \leq b$, $x \in \mathbb{R}^n$

Theorem

If x^* is an optimal solution of *linear programming relaxation* $\max\{c^Tx : Ax \leq b, x \in \mathbb{R}^n\}$*, then* $c^Tx^* \geqslant c^Tx_I$*, for each integer feasible solution* x_I.

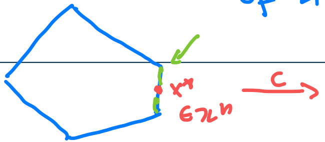
$OPT_{LP} \geqslant OPT_{IP}$



OPT SOL. OF LP-relaxation.

Def: Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $c^T.x \leq \delta$ valid for P. $F = P \cap \{x \in \mathbb{R}^n : c^T.x = \delta\}$ is FACE OF P

The LP-relaxation



FACE OF OPT. SOLUTIONS of LP relaxation

$\max c^T \cdot x$

$Ax \leq b$

Theorem

If x^* is integral optimal solution of the linear programming relaxation $\max\{c^T x \colon Ax \leq b, x \in \mathbb{R}^n\}$, then x^* is also an optimal solution of the integer programming problem $\max\{c^T x \colon Ax \leq b,\ x \in \mathbb{Z}^n\}$

Definition: A polyhedron $P \subseteq \mathbb{R}^n$ is integral if each non-empty face of P contains an integer point.

$(\forall F \neq \emptyset \text{ face of } P : \quad F \cap \mathbb{Z}^n \neq \emptyset)$

Undirected graphs – Matchings

$V = \{s, t, u, v\}$

$E = \{ \{s,u\}, \{s,v\}, \{v,t\}, \{u,t\}, \{u,v\} \}$

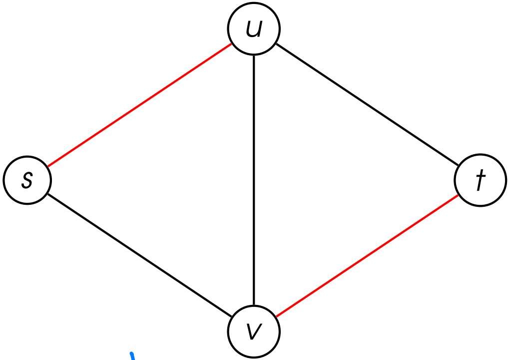
Definition

An undirected graph is a tuple $G = (V, E)$

V is finite set of vertices

$E \subseteq \binom{V}{2}$ is set of edges of G.

Matching: $M \subseteq E$ such that for all $e_1 \neq e_2 \in M$ one has $e_1 \cap e_2 = \emptyset$.

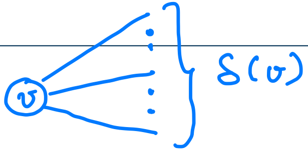


$M = \{su, tv\}$

$w : E \to \mathbb{R}$ edge weights.

$w(M) = \sum_{e \in M} w(e)$

$\delta(v)$



For a vertex $v \in V$, the set $\delta(v) = \{e \in E : v \in e\}$ denotes the incident edges to v.

Var: $x_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise} \end{cases}$

IP to solve max-weight Matching problem:

$$\max \sum_{e \in E} w(e) \cdot x(e)$$

$$\forall v \in V: \sum_{e \in \delta(v)} x(e) \leq 1$$

$$\forall e \in E: 0 \leq x_e \leq 1$$

$$x \in \mathbb{Z}^{|E|}$$

Max-weight matching as IP

$$\max \sum_{e \in E} w(e)x(e)$$

$$v \in V: \sum_{e \in \delta(v)} x(e) \leq 1$$

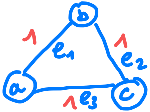
$$e \in E: x(e) \geqslant 0$$

$$x \in \mathbb{Z}^{|E|}.$$

Example

WEIGHS.

G :



MAX WEIGHT MATCHING ~ 1

IP:

max $x(e_1)+x(e_2)+x(e_3)$

s.t:

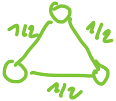
$x(e_1)+x(e_3) \le 1$

$x(e_2)+x(e_3) \le 1$

$x(e_1)+x(e_2) \le 1$

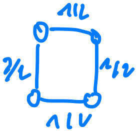
$0 \le x$

$\boxed{x \in \mathbb{Z}^3}$



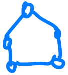
LP-RELAX ~ 1.5

NOT INTEGRAL POLYHEDRON.



LP ~ 2

IP ~ 2



IP ~ 2

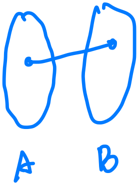
LP

Example

Definition: $G=(V,E)$ is bipartite if

$V = A \dot{\cup} B$ and $\forall e \in E$: $|e \cap A| = 1$

and $|e \cap B| = 1$



Next week:

$LP = IP$ if G is bipartite.

Integral polyhedra

Definition

A rational polyhedron is called integral if each nonempty face of P contains an integer vector.

Simplex on polyhedra whose vertices are integral

Lemma

Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ *be an integral polyhedron with* $A \in \mathbb{R}^{m\times n}$ *full-column rank. If the linear program*

$$\max\{c^T x \colon x \in \mathbb{R}^n, Ax \leq b\} \tag{7}$$

is feasible and bounded, then the simplex method computes an optimal integral solution to the linear program.

The study of integral polyhedra

Lemma

Let $A \in \mathbb{Z}^{n \times n}$ *be an integral and invertible matrix. One has* $A^{-1}b \in \mathbb{Z}^n$ *for each* $b \in \mathbb{Z}^n$ *if and only if* $\det(A) = \pm 1$.

Total unimodularity

Definition

An integral matrix $A \in \{0, \pm 1\}^{m \times n}$ is called totally unimodular if each of its square sub-matrices has determinant $0, \pm 1$.

Example: Node-edge incidence matrix of bipartite graph

Hoffman-Kruskal Theorem

Theorem

Let $A \in \mathbb{Z}^{m \times n}$ *be an integral matrix. The polyhedron* $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\, x \geqslant 0\}$ *is integral for each integral* $b \in \mathbb{Z}^m$ *if and only if* A *is totally unimodular.*