Optimization problems

Optimization problem is pair $(\mathscr{F}, f)$

- $\mathscr{F}$ is set of feasible solutions
- $f : \mathscr{F} \longrightarrow \mathbb{R}$ objective or cost function.

Task: Find $x \in \mathscr{F}$ s.t.

$$f(x) \geqslant f(y) \text{ holds for all } y \in \mathscr{F}$$

Such $x \in \mathscr{F}$ is optimal solution

We also write

$$\max\{c(x) : x \in \mathscr{F}\}.$$

Minimization

If minimize $f(x)$, i.e.

find $x \in \mathscr{F}$ with

$$f(x) \leq f(y) \text{ for all } y \in \mathscr{F}$$

This is optimization problem $(\mathscr{F}, -f)$ since

$$-f(x) \geqslant -f(y) \text{ for all } y \in \mathscr{F}$$

if and only if

$$f(x) \leq f(y) \text{ for all } y \in \mathscr{F}.$$

Example: Maximizing quadratic form over sphere

$A \in \mathbb{R}^{n\times n}$

$S^{(d-1)}$: 

Max. $x^T Ax$ over domain

$$S^{(d-1)} = \{x \in \mathbb{R}^d : \|x\| = 1\} \subseteq \mathbb{R}^d$$

$A \in \mathbb{R}^{d\times d}$ symmetric

$$x^T \cdot A \cdot x = x^T \underbrace{\left(\tfrac{1}{2}A + \tfrac{1}{2}A^T\right)}_{\text{symmetric.}} \cdot x$$

$\mathscr{F} = S^{(d-1)}$

$f(x) = x^T Ax$

$f : \mathscr{F} \to \mathbb{R}.$

$$\max \left\{ x^T Ax : x \in S^{(d-1)} \right\}.$$

x^* nrd is EV corresponding to λ_{max} of A

then x^* is an optimal solution of $(\mathscr{F}, f)$

Example: Minimizing quadratic form over sphere

Min $x^T Ax$ over domain

$$S^{(d-1)} = \{x \in \mathbb{R}^d \colon \|x\| = 1\} \subseteq \mathbb{R}^d$$

$A \in \mathbb{R}^{d\times d}$ symmetric

$\mathscr{F} = S^{(d-1)}$

$f(x) = x^T(-A)x$

$$\min \left\{ x^T Ax : x \in S^{(d-1)} \right\}.$$

$\sim \max \{ x^T(-A)x : x \in S^{(d-1)} \}$

Example: Shortest path in directed graph

Directed graph: Tuple $G = (V, A)$

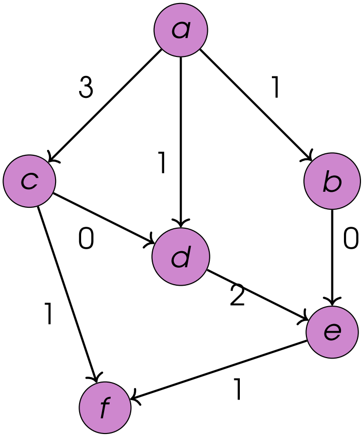
V: finite set of vertices

$A \subseteq V \times V$: arcs

$w : A \longrightarrow \mathbb{R}$: weights

Simple path: $P = v_1, v_2, \ldots, v_k$ s.t.
v_i distinct and
$(v_i, v_{i+1}) \in A$ $i \in \{1, \ldots, k-1\}$

length: $\ell(P) = \sum_{i=1}^{k-1} w(v_i, v_{i+1})$



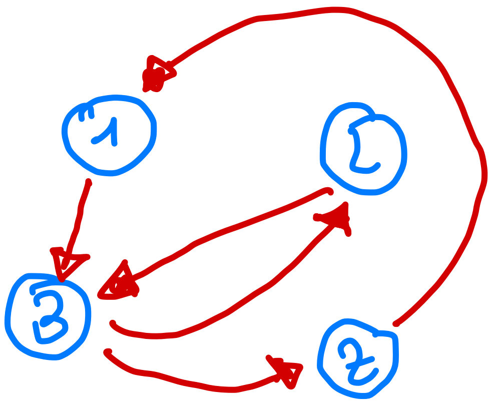
Directed Graph: $G = (V, A)$

V: finite set, the vertices of G.

$A \subseteq V \times V$, the arcs (edges) of G.

Example: $G = (V, A)$ $\quad V = \{1, 2, 3, z\}$

$A = \{(1,3), (2,3), (3,2), (3,z), (z,1)\}$



Example:

$3, z, 1$

Simple path: $v_1, \dots, v_k$ sequence of distinct vertices

$\boxed{v_i \neq v_j \text{ if } i \neq j.}$ s.th. $(v_i, v_{i+1}) \in A$

$\forall i \in \{1, \dots, k-1\}$

Shortest path

Given: $G = (V, A)$

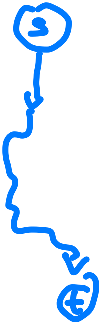
$w : A \longrightarrow \mathbb{R}$

$s, t \in V$

Shortest path problem:

$(\mathscr{F}, f)$

$\mathcal{F} = \{ P \subseteq V : P$ is simple path connecting s with $t \}$



$f(P) = -\ell(P)$

$|\mathcal{F}|$ can be exponential in $|V|$

There are graphs for which:

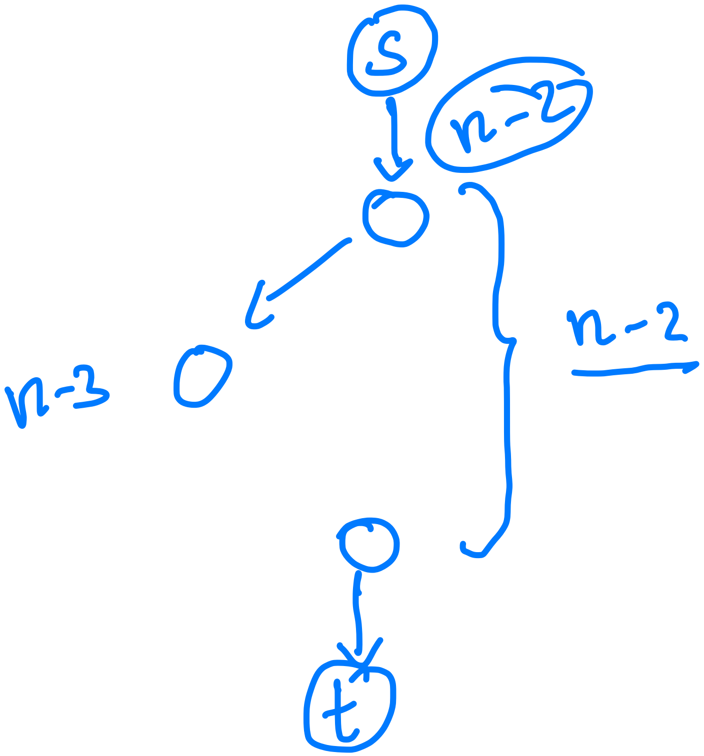
$|\mathcal{F}| \geq (n-2)! = (n-2)(n-1)\cdots(2)\cdot(1) \geq 2^{n-3}$

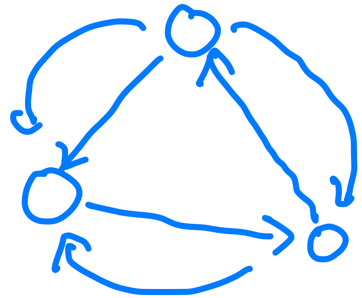
Example:

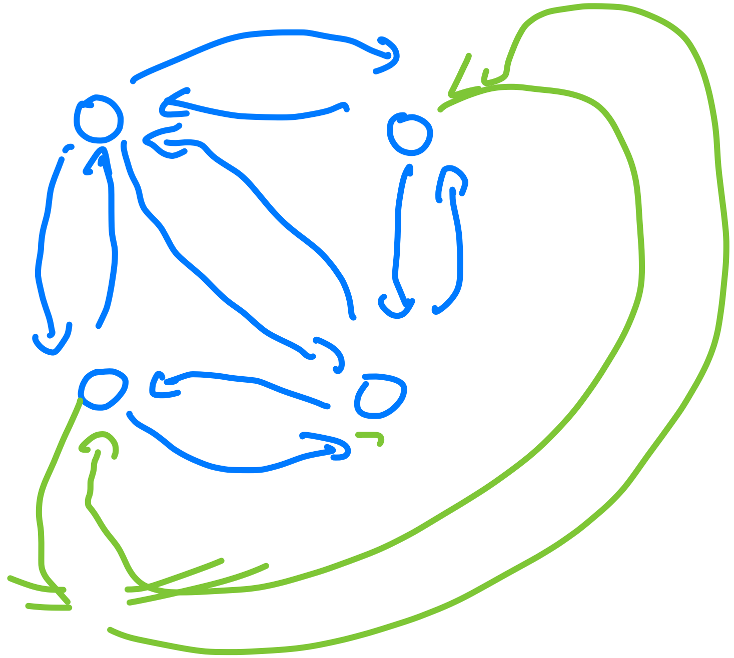
$G = (\{1, \dots, n\}, A)$

$A = \{(i,j) : i \neq j \in \{1, \dots, n\}\}$

$|A| \leq n^2$







$\mathcal{J} \geq (n-2)! \geq 2^{n-3}$ exponential in n.

Linear Programming

system of inequalities

$(\mathscr{F}, f)$

$\mathscr{F} = \{x \in \mathbb{R}^n : Ax \leq b\}$,

$A \in \mathbb{R}^{m \times n}$ $b \in \mathbb{R}^m$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ linear, i.e.,

$f(x) = c^T x$ for $c \in \mathbb{R}^n$

max x_1

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Example:

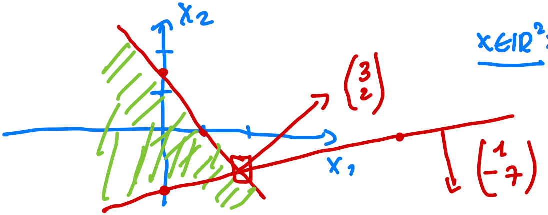
$A = \begin{pmatrix} 3 & 2 \\ 1 & -7 \\ \cancel{0} & \cancel{-1} \end{pmatrix}$ $b = \begin{pmatrix} 3 \\ 4 \\ \cancel{-2} \end{pmatrix}$

$\Downarrow$ feasible sol.

$x \in \mathbb{R}^2$:

$3 \cdot x_1 + 2 \cdot x_2 \leq 3$ (1)

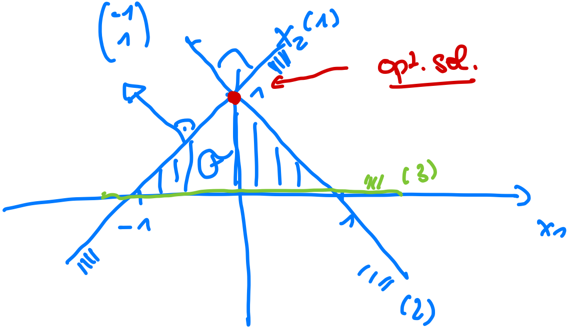
$x_1 - 7 x_2 \leq 4$ (2)



Linear programming: Example

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$



$$\mathscr{F} = \{x \in \mathbb{R}^2 : Ax \le b\}$$

$$f(x) = x_2$$

Why (0,1) optimal.

$-x_1 + x_2 \le 1$ (1)

$x_1 + x_2 \le 1$ (2) $\rightarrow$ $2x_2 \le 2$

$\iff$ $x_2 \le 1$ is satisfied by each $x \in \mathcal{F}$.

$x_2 \ge 0$ (3)

Linear programming – Focus of this course

- Geometry
- Efficient algorithms
- Duality
- Convexity

Certificates: An algorithmic paradigm

- Algorithms computes solution
- How can we be sure of correctness?
- Philosophy: Checking a certificate might be easy

Example: Linear systems solving

$$Ax = b \tag{1}$$

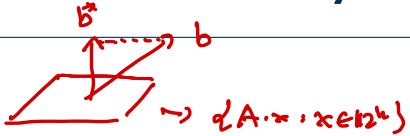
$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

If solvable, then solution $x^* \in \mathbb{R}^n$ serves as certificate.

row echelon form.

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \cdot \overbrace{\begin{pmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{b} \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -1 & -2 & -1 \end{array}\right)$$

Linear systems – Certificate for unsolvability



Theorem

(1) *has no solution if and only if there exists* $q \in \mathbb{R}^m$ *with*

$$q^T A = 0 \text{ and } q^T b \neq 0.$$

Proof: "$\Rightarrow$" if $q^T\cdot A = 0$ and $q^T\cdot b \neq 0$ suppose $x^* \in \mathbb{R}^n$ is solution:

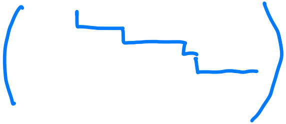
$$\left[\, A\cdot x^* = b \;\middle|\; \cdot\, q^T \,\right] \Rightarrow \underbrace{\underbrace{q^T\cdot A}_{=0}\cdot x^*}_{=0} = \underbrace{q^T\cdot b}_{\neq 0}$$

$\Rightarrow 0 \neq 0$ ↯

"$\Leftarrow$"

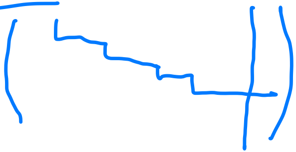
$$[A \mid b]$$

Gaussian elimination: $U \in \mathbb{R}^{n\times n}$ non-singular s.th

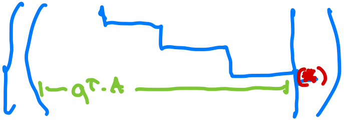
$U[A \mid b] =$ 

2 cases:

Case 1:



Case 2:

m 

$\Rightarrow$ row i

$Ax = b$ has solution x^*

We want: $q \in \mathbb{R}^m$ s.th

$q^T \cdot A = 0$ and $q^T \cdot b \neq 0$

$\boxed{q\,?}$

q^T: i-th row of U

$(*) = q^T \cdot b \neq 0$

Systems of linear inequalities – Deriving consequences

$$
\begin{aligned}
2x_1 + 3x_2 &\leq 5 \quad * \tfrac{1}{2} \\
4x_1 + 1x_2 &\leq 3 \quad * \tfrac{1}{2} \quad +
\end{aligned}
\qquad (2)
$$

$$\Rightarrow \quad 3x_1 + 2x_2 \leq 4$$

$$\boxed{\text{If } x^* \in \mathbb{R}^2 \text{ satisfies (2), then } x^* \text{ satisfies } \quad 3x_1 + 2x_2 \leq 4}$$

Derivation proves

Systems of linear inequalities – Deriving consequences

$a_1^T, \dots, a_m^T$ rows of A

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}^m_{\geqslant 0}$. If $x^* \in \mathbb{R}^n$ satisfies

$$Ax \leq b$$

the x^* satisfies

$$\lambda^T A x \leq \lambda^T b.$$

$$(\lambda^T \cdot A) \cdot x^*$$

$$= \underbrace{\underbrace{\underbrace{\lambda_1}_{\geq 0} a_1^T \cdot x^*}_{\leq \lambda_1 \cdot b_1} + \underbrace{\underbrace{\lambda_2}_{\geq 0} a_2^T \cdot x^*}_{\leq \lambda_2 \cdot b_2} + \dots + \underbrace{\lambda_m a_m^T \cdot x^*}_{\leq \lambda_m \cdot b_m}}_{= \lambda^T \cdot b}$$

One inequality derived from $Ax \leq b$ with $\lambda \in \mathbb{R}^m_{\geq 0}$

$Ax \leq b$: Certifying infeasibility

Theorem (Farkas' lemma)

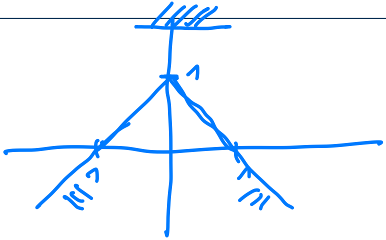
$Ax \leq b$ is infeasible if and only if there exists $\lambda \in \mathbb{R}^m_{\geqslant 0}$ such that

$$\lambda^T A = 0 \text{ and } \lambda^T b = -1.$$

Proof: "$\Leftarrow$" if x^* s.t. $A \cdot x \leq b$ then x^* satisfies.

$(\lambda^T A) x \leq (\lambda^T \cdot b) \iff 0^T \cdot x \leq -1 \Rightarrow$ Thus x^* does not exist.

Example: Infeasible system of inequalities



$$\overbrace{\begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}}^{A} x \leq \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$\Leftrightarrow x_2 \geq 2$

$$\lambda = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$\lambda \in \mathbb{R}^3_{\geq 0}$ s.t. $\lambda^T A = 0$ $\qquad \lambda^T \cdot A = (0,0)$

$\lambda^T \cdot b < 0$ $\qquad \lambda^T \cdot b = -2$

$\lambda = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$ sol. $\lambda^T A = 0$ $\quad \lambda^T \cdot b = -1$

Example: Production planning

Spring and Nebsi

Per 100ℓ:		A	B	Profit
	Spring	3ℓ	8ℓ	100 CHF
	Nebsi	6ℓ	4ℓ	125 CHF

$100\,\ell$

3ℓ A

6ℓ B

In stock: 30ℓ of A; 44ℓ of B

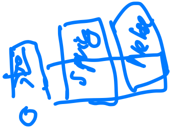
max $100 \cdot x_1 + 125 \cdot x_2$

Capacity of transport barrels:

A: $3 \cdot x_1 + 6 \cdot x_2 \le 30$

Spring: 500ℓ Nebsi: 400ℓ

B: $8 \cdot x_1 + 4 \cdot x_2 \le 44$



$x_1 \le 5\cancel{0}$

Production plan: (x_1, x_2)

$x_2 \le 4$

$x_1, x_2 \ge 0$

Example: Production planning

Spring and Nebsi

Per 100ℓ:

	A	B	Profit
Spring	3ℓ	8ℓ	100 CHF
Nebsi	6ℓ	4ℓ	125 CHF

In stock: 30ℓ of A; 44ℓ of B

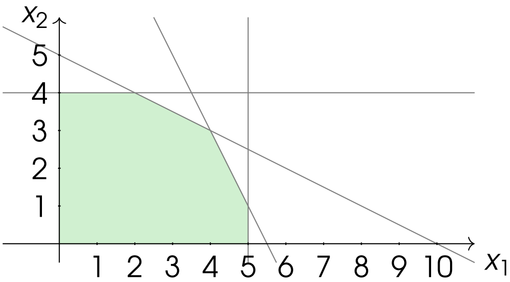
Capacity of transport barrels:

Spring: 500ℓ Nebsi: 400ℓ

Production plan: (x_1, x_2)

The linear program

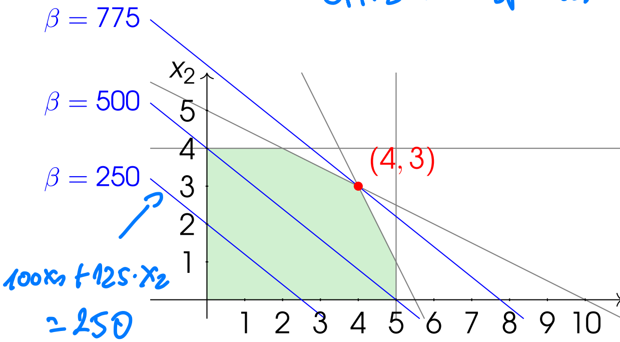
$$
\begin{array}{lrcl}
\text{max.} & 100 \cdot x_1 + 125 \cdot x_2 & & \\
\text{s.t.:} & 3 \cdot x_1 + 6 \cdot x_2 & \leq & 30 \\
 & 8 \cdot x_1 + 4 \cdot x_2 & \leq & 44 \\
 & x_1 & \leq & 5 \\
 & x_2 & \leq & 4 \\
 & x_1 & \geqslant & 0 \\
 & x_2 & \geqslant & 0
\end{array}
$$



The linear program

$$
\begin{array}{lrcl}
\text{max.} & 100 \cdot x_1 + 125 \cdot x_2 & & \\
\text{s.t.:} & 3 \cdot x_1 + 6 \cdot x_2 & \leq & 30 \\
 & 8 \cdot x_1 + 4 \cdot x_2 & \leq & 44 \\
 & x_1 & \leq & 5 \\
 & x_2 & \leq & 4 \\
 & x_1 & \geqslant & 0 \\
 & x_2 & \geqslant & 0
\end{array}
$$

OPT. sol. is special.



Proving optimality

$$\begin{aligned} 3 \cdot x_1 + 6 \cdot x_2 &\leq 30 \\ 8 \cdot x_1 + 4 \cdot x_2 &\leq 44. \end{aligned}$$

multiplied by $50/3$ and $25/4$ respectively:

$$\begin{aligned} 50 \cdot x_1 + 100 \cdot x_2 &\leq 500 \\ 50 \cdot x_1 + 25 \cdot x_2 &\leq 275 \end{aligned}$$