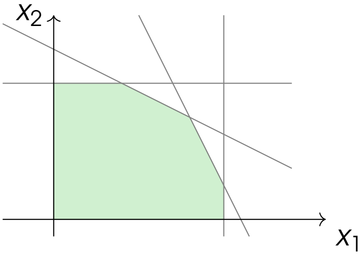
Polyhedra

Definition

A polyhedron $P \subseteq \mathbb{R}^n$ is a set of the form $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$ and some $b \in \mathbb{R}^m$.

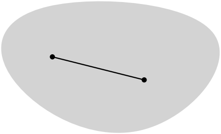
$$A = \begin{pmatrix} 3 & 6 \\ 8 & 4 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 30 \\ 44 \\ 5 \\ 4 \\ 0 \\ 0 \end{pmatrix}:$$

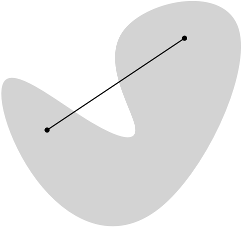


Convex sets

Definition

A set $K \subseteq \mathbb{R}^n$ is convex if for each $u, v \in K$ and $\lambda \in [0, 1]$ the point $\lambda u + (1 - \lambda)v$ is also contained in K.





Halfspaces

Definition

A halfspace is a set of the form

$$\{x \in \mathbb{R}^n \colon a^T x \leq \beta\}.$$

A hyperplane is a set of the form

$$\{x \in \mathbb{R}^n \colon a^T x = \beta\}.$$

Halfspaces are convex

Lemma

A half-space is convex.

Intersections of convex sets

Lemma

Let I be an index set and $C_i \subseteq \mathbb{R}^n$ be convex sets for each $i \in I$, then $\cap_{i \in I} C_i$ is a convex set.

Corollary

A polyhedron is a convex set.

Valid inequalities

Definition

$a^T x \leq \beta$ is valid for $K \subseteq \mathbb{R}^n$ if for each $x^* \in K$:

$$a^T x^* \leq \beta$$

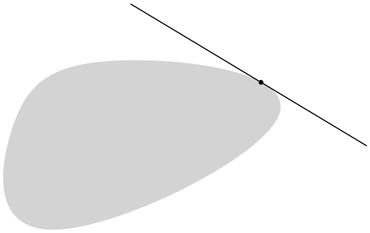
If in addition $(a^T x = \beta) \cap K \neq \emptyset$, then $a^T x \leq \beta$ is a supporting inequality and $a^T x = \beta$ is a supporting hyperplane

Extreme points

Definition

Let $K \subseteq \mathbb{R}^n$ be convex. $x^* \in K$ is extreme point or vertex of K if there exists a valid inequality $a^T x \leq \beta$ of K such that

$$\{x^*\} = K \cap \{x \in \mathbb{R}^n \colon a^T x = \beta\}.$$



Vertices of polyhedra – algebraic characterization

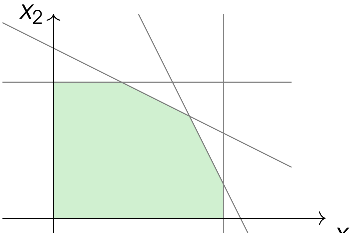
Theorem

Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be a polyhedron. $x^* \in P$ is extreme point iff there is sub-system $A'x \leq b'$ of $Ax \leq b$ s.t.

i) *x^* satisfies all inequalities of $A'x \leq b'$ with equality.*

ii) *A' has n rows and A' is non-singular.*

$$A = \begin{pmatrix} 3 & 6 \\ 8 & 4 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 30 \\ 44 \\ 5 \\ 4 \\ 0 \\ 0 \end{pmatrix}:$$



Optimal solutions and vertices

Theorem

If a linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ *is feasible and bounded and if* $\mathrm{rank}(A) = n$*, then the linear program has an optimal solution that is an extreme point.*

Bounded LP has optimal solution

Corollary

A linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ *which is feasible and bounded has an optimal solution.*

A first (inefficient) algorithm

Given: $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ with $\mathrm{rank}(A) = n$.

- Initialize $M = \emptyset$
- Enumerate all sets of n row-vectors of A that are basis of $\mathbb{R}^n$
  - Solve $A'x = b'$ for corresponding sub-system $A'x \leq b'$ of $Ax \leq b$.
  - If for solution x^*: $Ax^* \leq b$ then
    $M = M + x^*$
- Output element of M with largest objective function value

A first (inefficient) algorithm

Theorem

If LP is bounded then algorithm above computes optimal solution.

We will see ...

... we can do much better.

Linear, affine, conic and convex hulls

Let $X \subseteq \mathbb{R}^n$:

$$\begin{aligned}
\mathrm{lin.\,hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_0, \\
&\qquad x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\} \\
\mathrm{affine.\,hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_+, \\
&\qquad x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\} \\
\mathrm{cone}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_0, \\
&\qquad x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}_{\geqslant 0}\} \\
\mathrm{conv}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_+, \\
&\qquad x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}_{\geqslant 0}\}
\end{aligned}$$

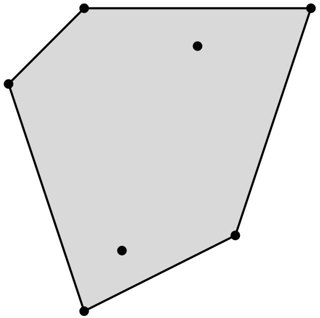


Figure: The convex hull of 7 points in $\mathbb{R}^2$.

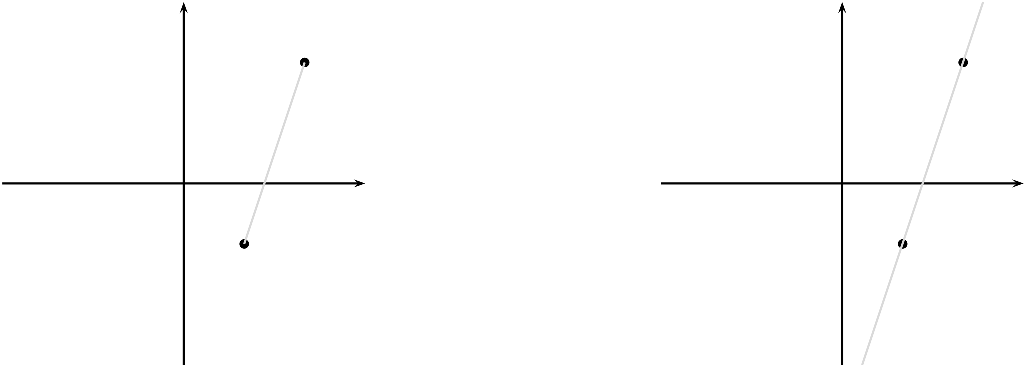


Figure: Two points with their convex hull on the left and their affine hull on the right.

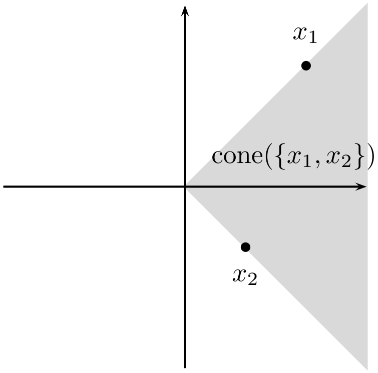


Figure: Two points with their conic hull

Bounded continuous functions

Theorem

Let $X \subseteq \mathbb{R}^n$ be compact and $f : X \to \mathbb{R}$ be continuous. Then f is bounded and there exist points $x_1, x_2 \in X$ with $f(x_1) = \sup\{f(x) \colon x \in X\}$ and $f(x_2) = \inf\{f(x) \colon x \in X\}$.

Linear and affine hulls

Theorem

Let $X \subseteq \mathbb{R}^n$ *and* $x_0 \in X$*. One has*

$$\mathrm{affine.\,hull}(X) = x_0 + \mathrm{lin.\,hull}(X - x_0),$$

where for $u \in \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$, $u + V$ denotes the set $u + V = \{u + v \mid v \in V\}$.

Convex hull is convex

Theorem

Let $X \subseteq \mathbb{R}^n$ be a set of points. The convex hull, $\mathrm{conv}(X)$, of X is convex.

Convex hull is minimal

Theorem

Let $X \subseteq \mathbb{R}^n$ be a set of points. Each convex set K containing X also contains $\mathrm{conv}(X)$.

Corollary

$$\operatorname{conv}(X) = \bigcap_{\substack{K \supseteq X \\ K \text{ convex}}} K\,.$$

Cones

Definition

A set $C \subseteq \mathbb{R}^n$ is a cone, if it is convex and for each $c \in C$ and each $\lambda \in \mathbb{R}_{\geqslant 0}$ one has $\lambda \cdot c \in C$.

Analogous theorems for cones

Theorem

For any $X \subseteq \mathbb{R}^n$, the set $\mathrm{cone}(X)$ is a cone.

Theorem

Let $X \subseteq \mathbb{R}^n$ be a set of points. Each cone containing X also contains $\mathrm{cone}(X)$.

$$\mathrm{cone}(X) = \bigcap_{\substack{C \supseteq X \\ C \text{ is a cone}}} C.$$

Carathéodory's Theorem

Theorem

Let $X \subseteq \mathbb{R}^n$*, then for each* $x \in \mathrm{cone}(X)$ *there exists a set* $\widetilde{X} \subseteq X$ *of cardinality at most* n *such that* $x \in \mathrm{cone}(\widetilde{X})$*. The vectors in* $\widetilde{X}$ *are linearly independent.*

Separation theorem

Theorem

Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $x^* \in \mathbb{R}^n \setminus K$, then there exists an inequality $a^T x \leq \beta$ such that $a^T y < \beta$ holds for all $y \in K$ and $a^T x^* > \beta$.

Farkas' Lemma – Version 1

Theorem (Farkas' lemma)

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax = b$, $x \geqslant 0$ has a solution if and only if for all $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geqslant 0$ one has $\lambda^T b \geqslant 0$.

Farkas' Lemma – Version 2

Theorem (Farkas' lemma)

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leq b$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.

Exercise!